Spatial Patterns in Nature

An Entry-Level Introduction to Their Emergence and Dynamics

SIAM DS23, Minitutorial MT1-2

Robbin Bastiaansen, Peter van Heijster, Frits Veerman

Minitutorial overview and slides:

bastiaansen.github.io/MTpatterns/patternMT.html

Peter van Heijster, Chair of Applied Mathematics, Biometris, Wageningen University & Research

Peter is an **applied analyst** and his research focusses on **nonlinear dynamics**, and in particular on understanding **pattern formation**. The aim of his research is to get a better understanding of the pattern formation processes in **paradigmatic mathematical models** (often with *scale separation*) and to apply the new insights to more **biologically-realistic models** from the Life Sciences and Mathematical Biology and Ecology.

WAGENINGEN
UNIVERSITY & RESEARCH

Robbin Bastiaansen

Assistant Professor

Mathematical Institute
Utrecht University
&
Institute for Marine and Atmospheric Research Utrecht (IMAU)
Utrecht University

Robbin is an applied mathematician and his research focusses on mathematics of and for climate, by the use of techniques and insights from nonlinear dynamical systems theory. The aim of his research is to get a better fundamental insight in climate and ecosystem responses due to forcings, and to develop and improve estimation and projection methodologies.

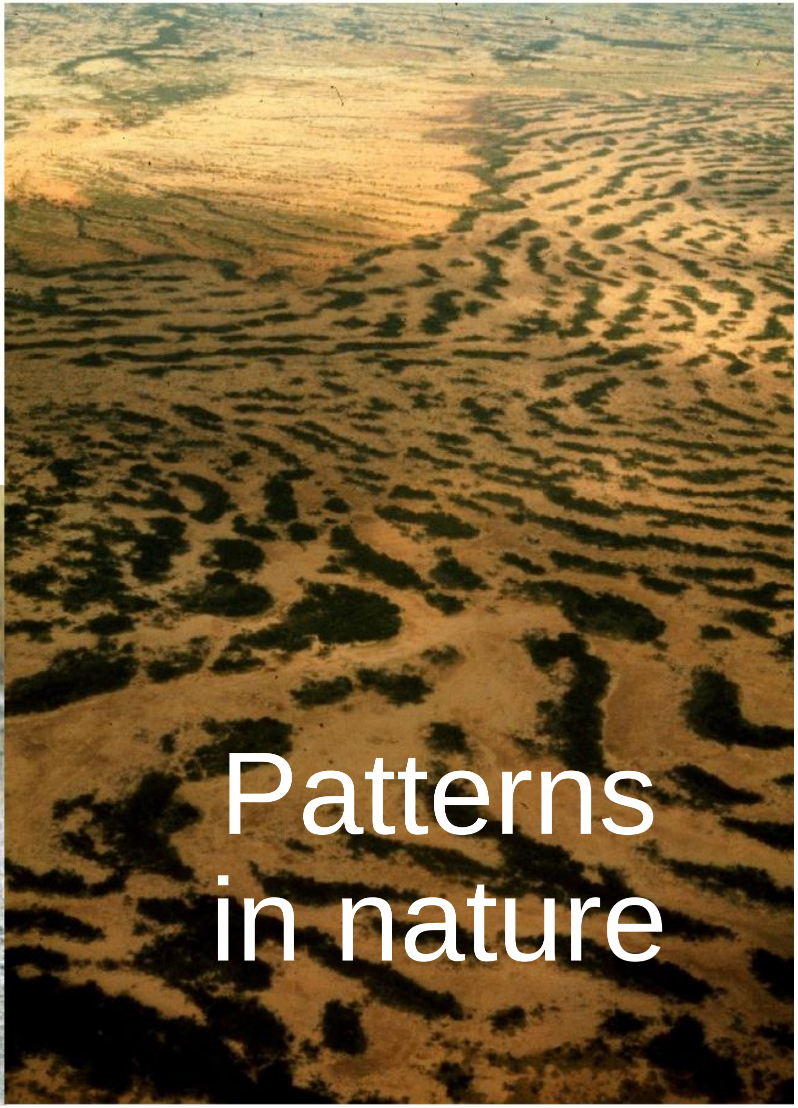
Frits Veerman *(Mathematical Institute, Leiden University, The Netherlands)*

develops analytical tools to investigate and predict phenomena such as pattern formation in spatially extended, nonlinear, dynamical systems, with a focus on applications in biology and ecology

Universiteit Leiden

Minitutorial setup

- Introduction
- Multistability and patterns
- Explicit construction of front solutions
  - Existence
  - Stability
- Dynamics of existing structures
- Summary & Outlook



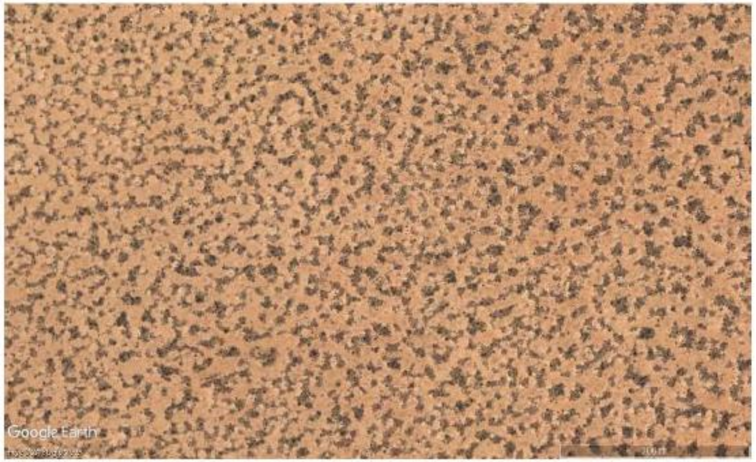
Patterns in nature

Dryland eco-systems

(a) Bands in Somalia

(b) Gaps in Niger

(c) Spots in Zambia



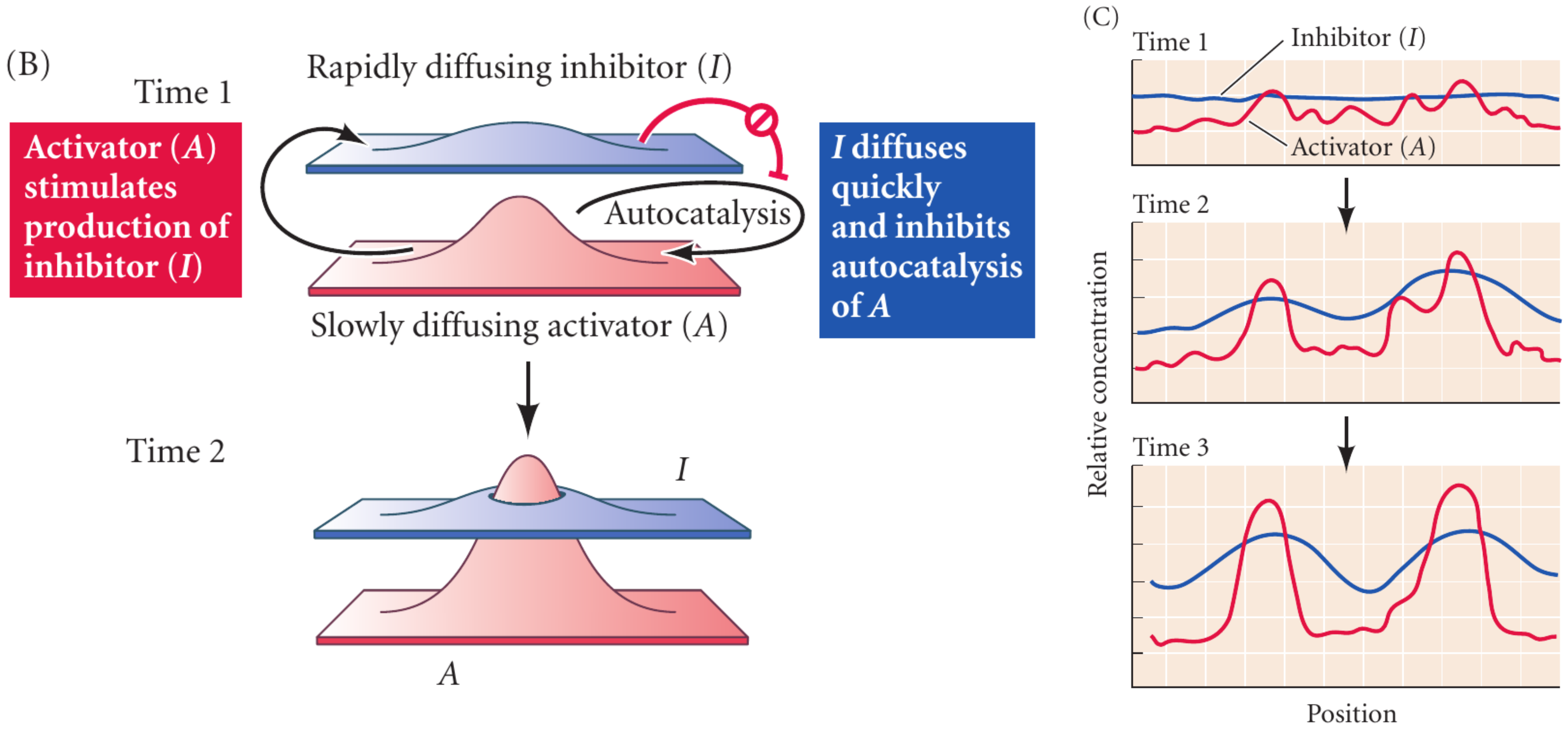
(d) Maze in Sudan

Patterns in developmental biology

Questions / research topics

- How and when do these patterns form?
- What are the underlying mechanisms behind pattern formation?
- When are initial conditions and/or external factors important?
- Can we predict the pattern wavelength?
- Are observed patterns stationary or transient?
- How about pattern stability/robustness?

Turing pattern formation



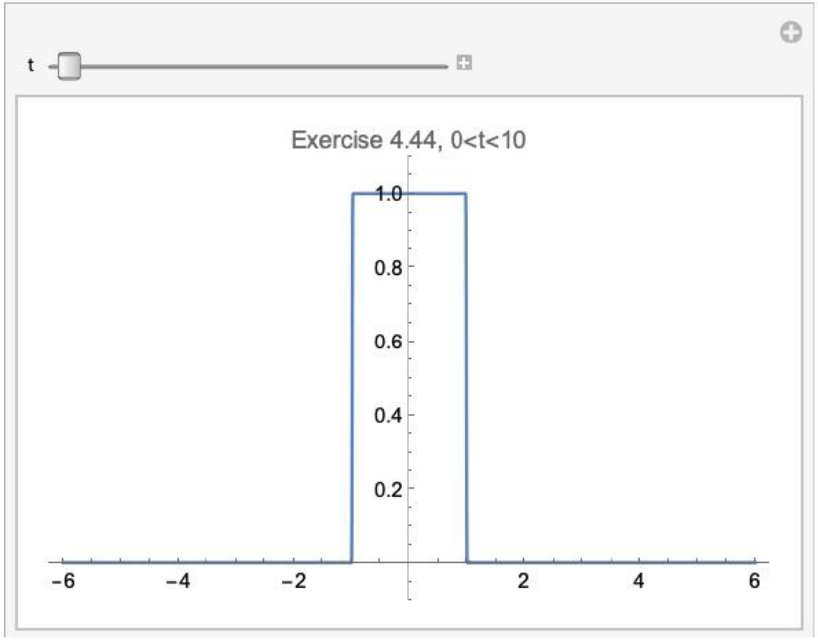
Introduction: *Turing* patterns

[wikipedia]

- Turing 1952: Stable uniform state in a kinetic system (ODE) can become unstable when you add diffusion (PDE).
- Diffusion driven pattern formation (*nowadays: Turing patterns*).
- Counter intuitive: Diffusion was/is thought of having a stabilising effect.

Heat equation:

$U_t = U_{xx}$



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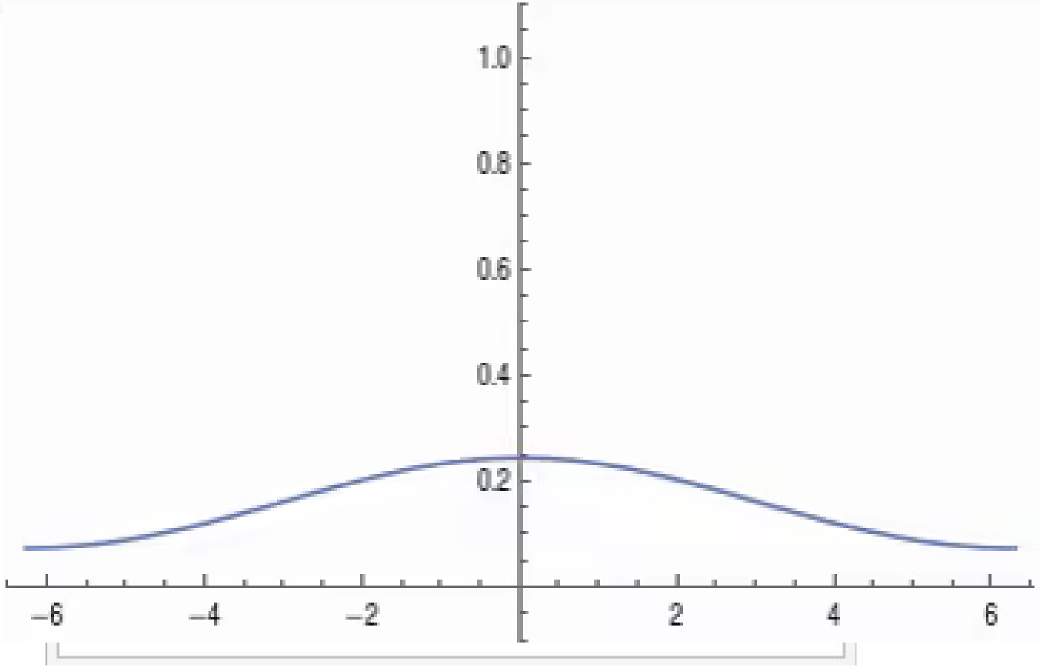

Introduction: *Turing* patterns

Stable uniform state in a kinetic system (ODE) can become unstable when you add diffusion (PDE)

Kinetic system (ODE):

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \gamma \begin{pmatrix} F(u,v) \\ G(u,v) \end{pmatrix}, \qquad \mathbf{w} = \begin{pmatrix} u \\ v \end{pmatrix}$$

Want: (0,0) to be stable fixed point, so F(0,0)=G(0,0)=0 and

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linearisation:

$$\mathbf{w}_t = \gamma \mathbf{A}\mathbf{w}\,, \quad \mathbf{A} = \begin{pmatrix} F_u & F_v \\ G_u & G_v \end{pmatrix}_{(0,0)}$$

Characteristic polynomial gives (substitute $e^{\lambda t}$) — eigenvalues of the Jacobian:

$$\begin{aligned} \overline{\lambda}_1 + \overline{\lambda}_2 &= \operatorname{tr}(\mathbf{A}) = F_u + G_v \\ \overline{\lambda}_1 \overline{\lambda}_2 &= \det(\mathbf{A}) = F_u G_v + F_v G_u \end{aligned}$$

So, for (0,0) to be stable fixed point we need:

$$F_u + G_v < 0 \qquad \& \qquad F_u G_v > F_v G_u$$

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Question: Can (0,0) transform into an unstable fixed point?

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Question: Can (0,0) transform into an unstable fixed point?

Linearization:

$$\mathbf{w}_t = \gamma \mathbf{A} \mathbf{w} + \mathbf{D}\, \mathbf{w}_{xx}, \ \ \mathbf{D} = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}$$

Characteristic polynomial (substitute $e^{\lambda t + ik}$):

(k: wave number)

$$0 = |\gamma \mathbf{A} - \lambda \mathbf{I} - \mathbf{D} k^2|$$

So:

$$\begin{aligned} \lambda_1 + \lambda_2 &= \gamma(F_u + G_v) - k^2(1+d) \\ \lambda_1 \lambda_2 &= (\gamma F_u - k^2)(\gamma G_v - d k^2) - \gamma^2 (F_v G_u) \end{aligned}$$

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Therefore, the sum of the eigenvalues of the PDE

$$\lambda_1 + \lambda_2 \quad = \quad \gamma(F_u + G_v) - k^2(1 + d)$$

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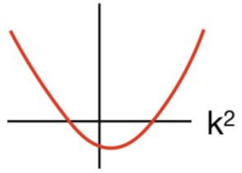
$$\begin{aligned} \lambda_1 \lambda_2 &= (\gamma F_u - k^2)(\gamma G_v - d k^2) - \gamma^2 (F_v G_u) \\ &= \gamma^2 (F_u G_v - F_v G_u) - \gamma k^2 (d F_u + G_v) + d k^4 < 0 \end{aligned}$$



This gives:

$$d F_u + G_v > 0 \quad (d \neq 1!!) \qquad \& \qquad (d F_u + G_v)^2 > 4 d (F_u G_v - F_v G_u) \qquad (**)$$

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So, the four conditions for Turing Instability are

$$\begin{aligned} F_u + G_v &< 0 \\ F_u G_v &> F_v G_u \\ dF_u + G_v &> 0 \\ (dF_u + G_v)^2 &> 4d(F_u G_v - F_v G_u) \end{aligned}$$

and since we have five "unknowns" we can realise this!

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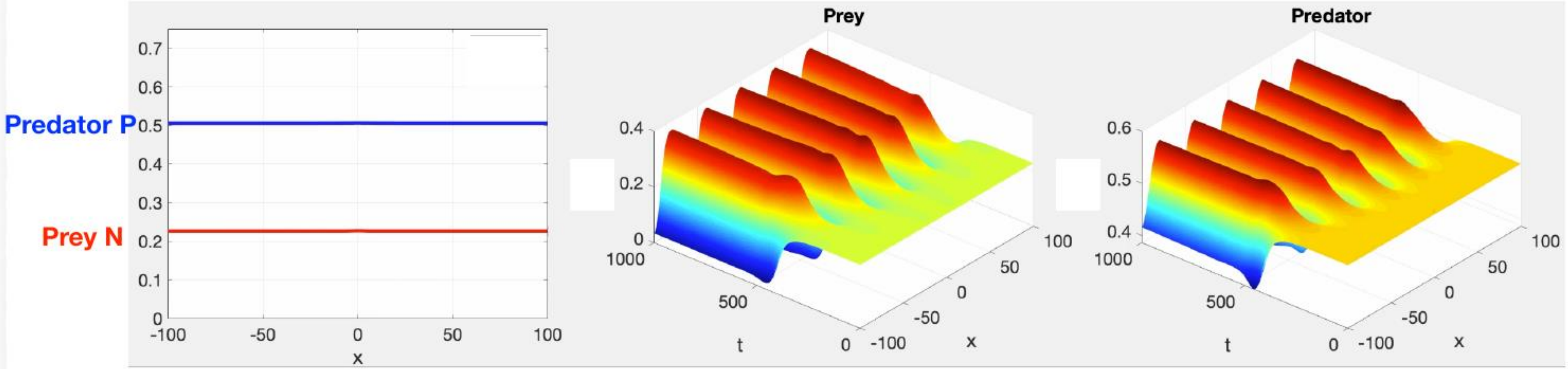
DIFFUSION CAN HAVE A DESTABILISING EFFECT!!

Introduction: *Turing* patterns

Example: Diffusive Holling–Tanner predator-prey model with an alternative food source for the predator

$$N_t = rN\left(1 - \frac{N}{K}\right) - \frac{qNP}{N+a} + D_1 N_{xx},$$
$$P_t = sP\left(1 - \frac{P}{hN+c}\right) + D_2 P_{xx}.$$

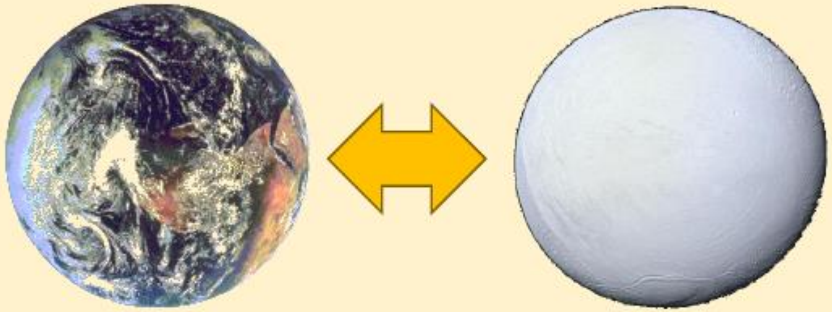


[Arancibia-Ibarra et al., 2021]

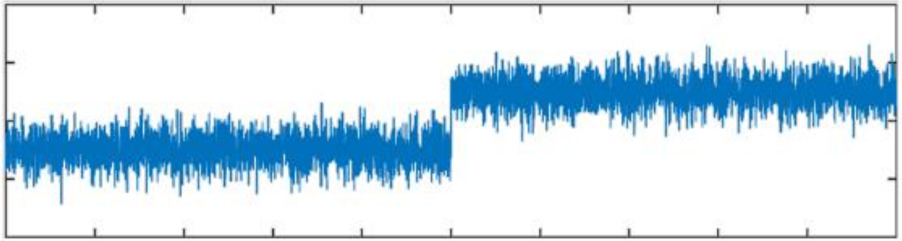

Patterns, spatial heterogeneity
and tipping

Tipping Points

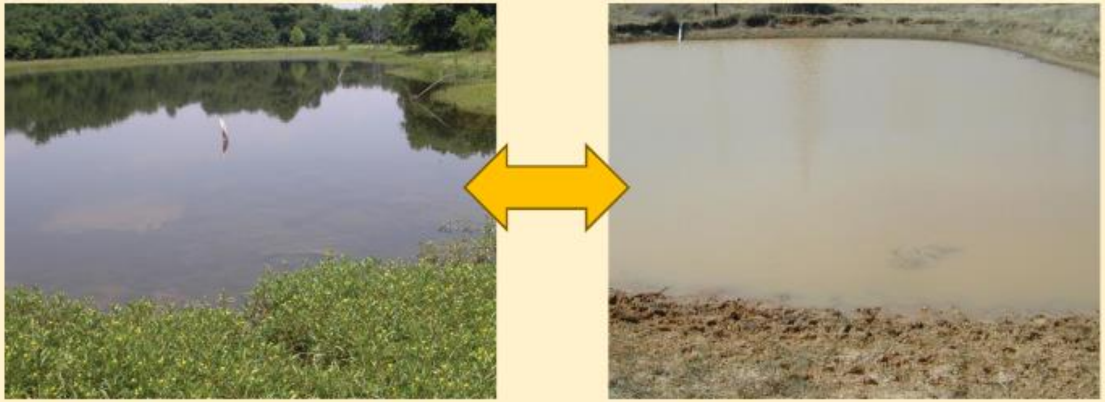
IPCC AR6 (2021) : “a critical threshold beyond which a system reorganizes, often abruptly and/or irreversibly”



Planetary transitions



Ecosystem shifts



Tipping Points

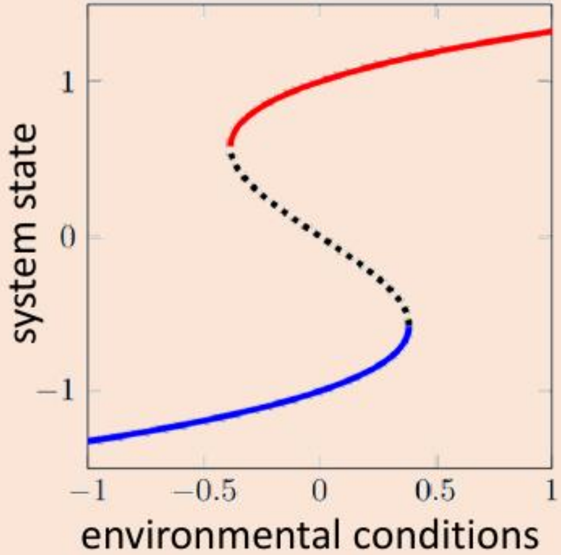
IPCC AR6 (2021) : "a critical threshold beyond which a system reorganizes, often abruptly and/or irreversibly"

source: McKay et al, 2022

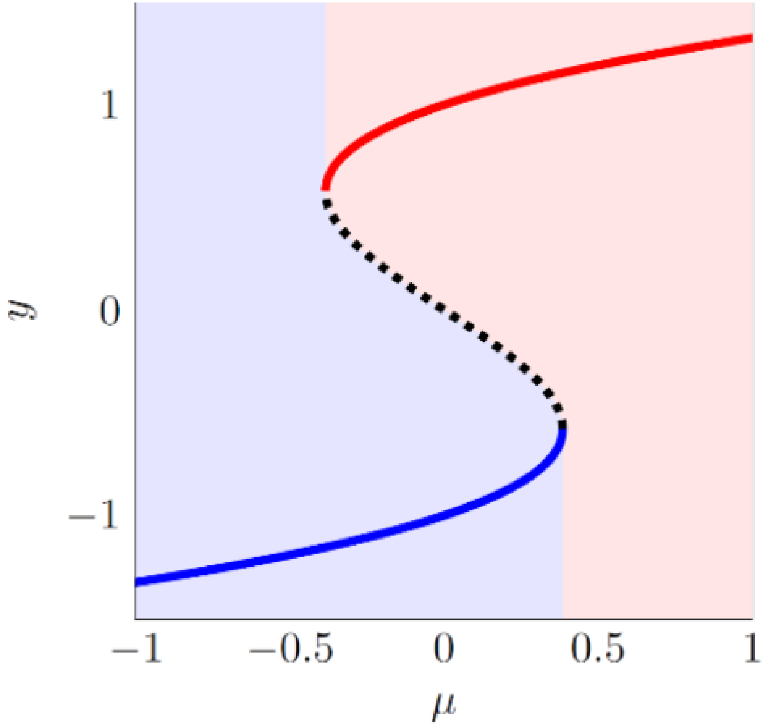
Mathematics

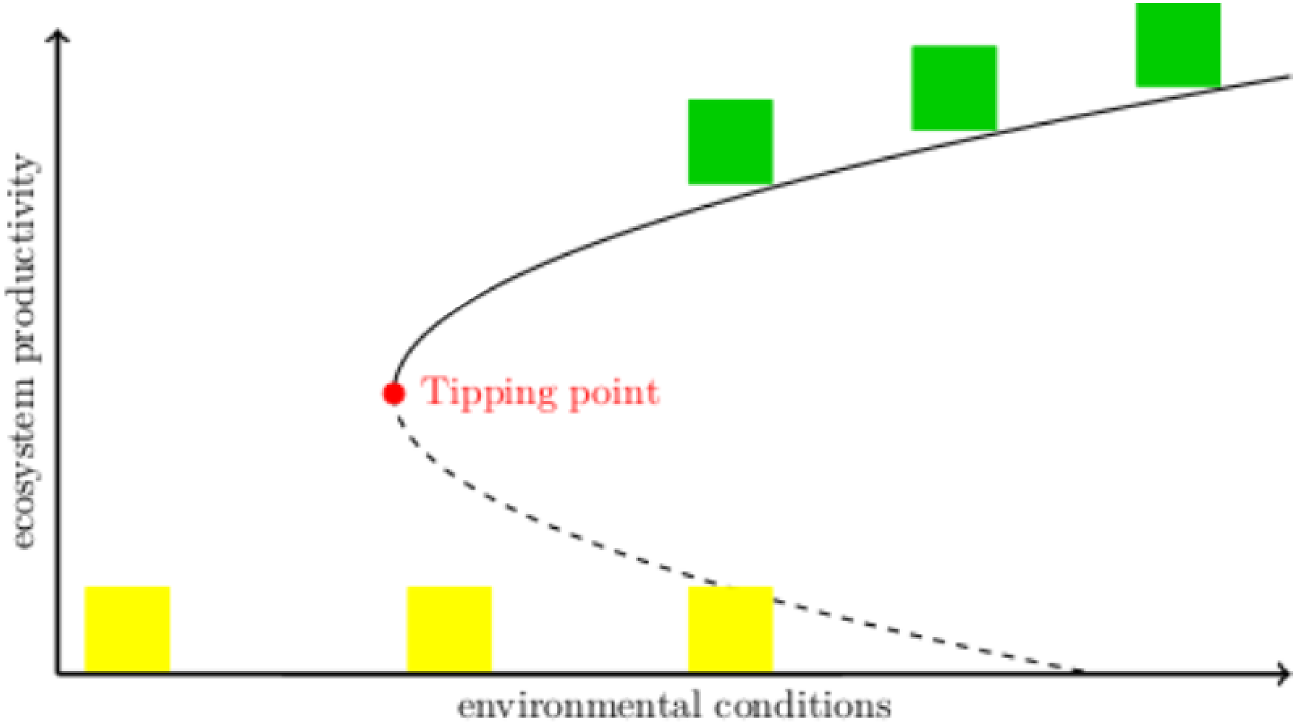
Tipping points ↔ Bifurcations

$$\frac{dy}{dt} = f(y, \mu)$$



Classic Theory of Tipping





Canonical example:

$$\frac{dy}{dt} = y(1-y^2) + \mu$$

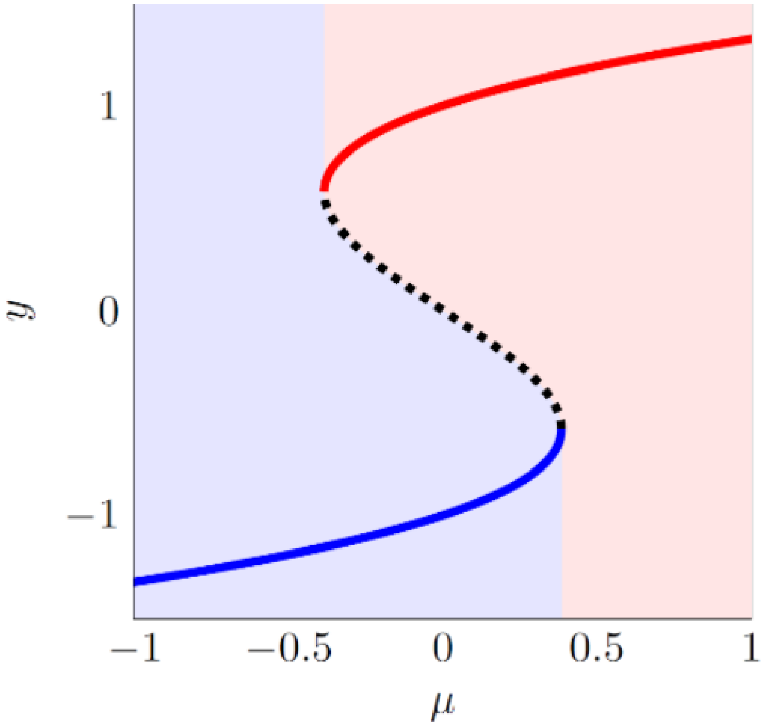
$$\frac{d\vec{y}}{dt} = f(\vec{y};\mu)$$

$$\begin{cases} \frac{du}{dt} = f(u,v;\mu) \\ \frac{dv}{dt} = g(u,v;\mu) \end{cases}$$

Tipping in ODEs (1)

Canonical example:

Concrete example: Global Energy Balance Model



Classic Literature
[Holling, 1973]
[Noy-Meier, 1975]
[May, 1977]

Tipping
[Ashwin et al, 2012]

Bifurcation-Tipping : Basin disappears

Noise-Tipping : Forced outside Basin

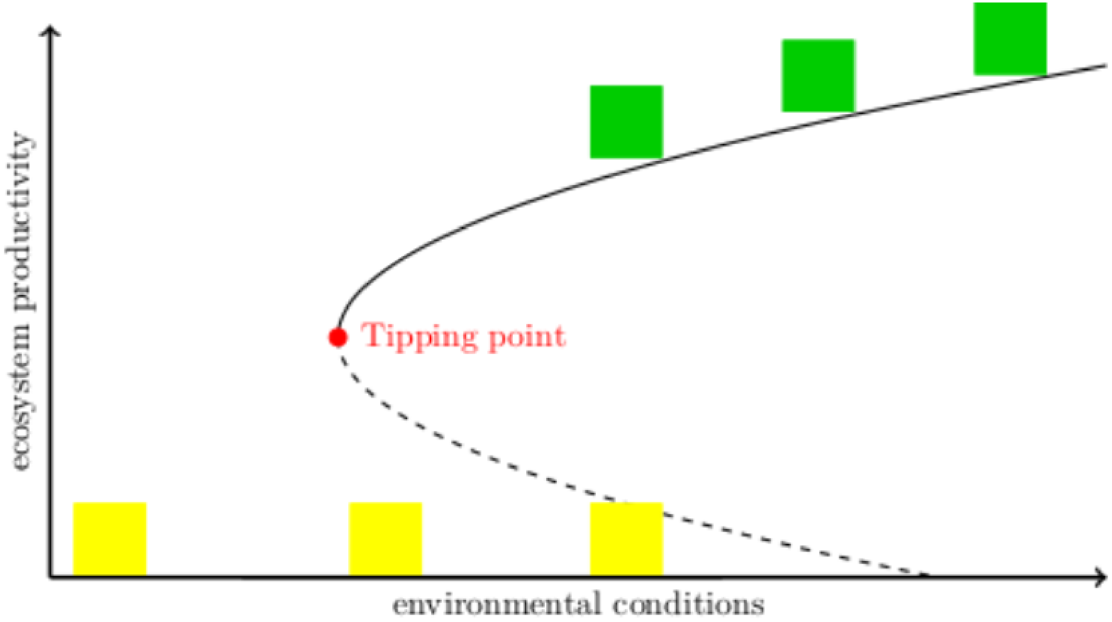
Rate-Tipping : *(more complicated)*

Tipping in ODEs (2)

Two components:

includes common models:

- Predetor-Prey
- Activator-Inhibitor



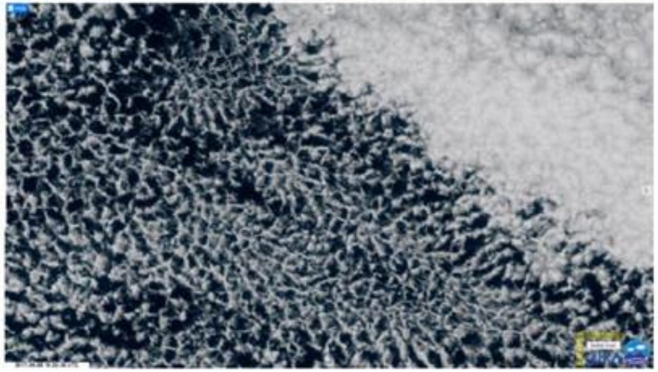
Examples of tipping in ODEs include:

- Forest-Savanna bistability
- Deep ocean exchange
- Cloud formation
- Ice sheet melting
- Turbidity in shallow lakes

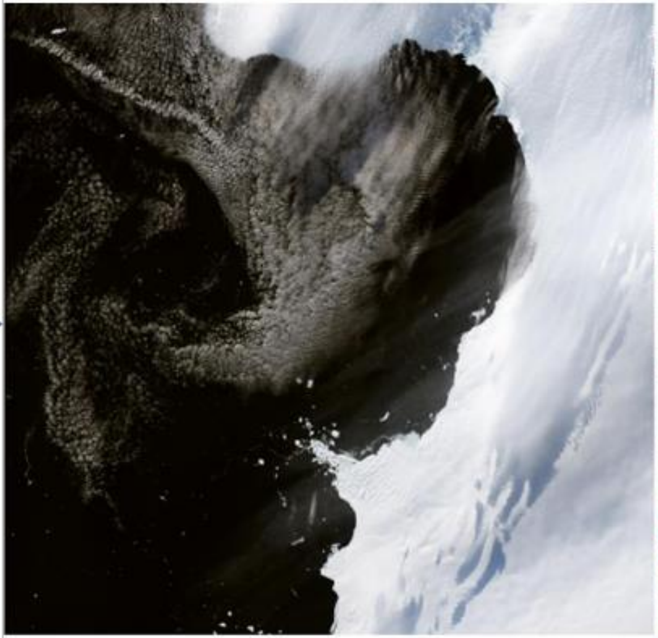
Reality is not always spatially-uniform!

tropical forest
& savanna
ecosystems
[Google Earth]

types of
stratocumulus
clouds
[RAMMB/CIRA SLIDER]



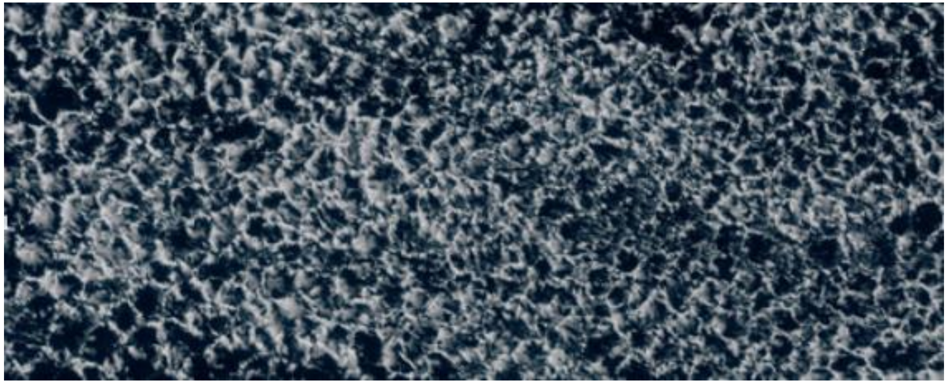
sea-ice & water
at Eltanin Bay
[NASA's Earth observatory]



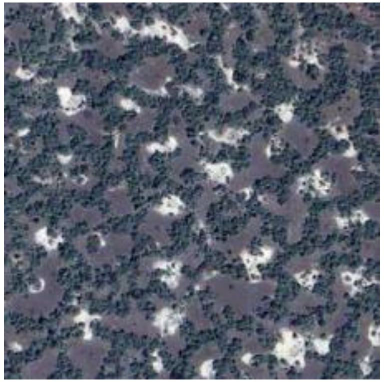
algae bloom
in Lake St. Clair
[NASA's Earth observatory]

Examples of spatial patterning – regular patterns

mussel beds

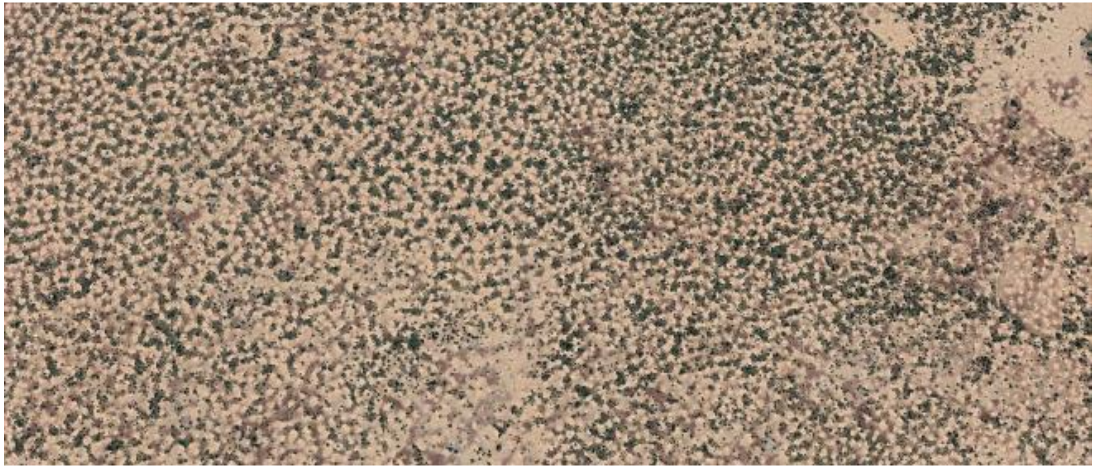


clouds



savannas

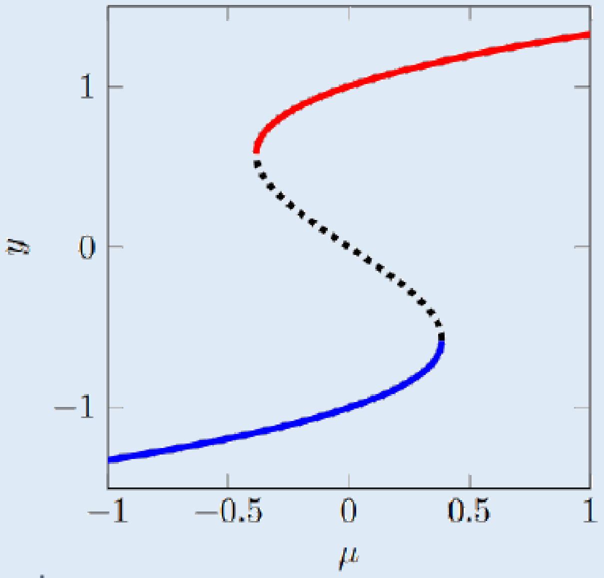
melt ponds



drylands

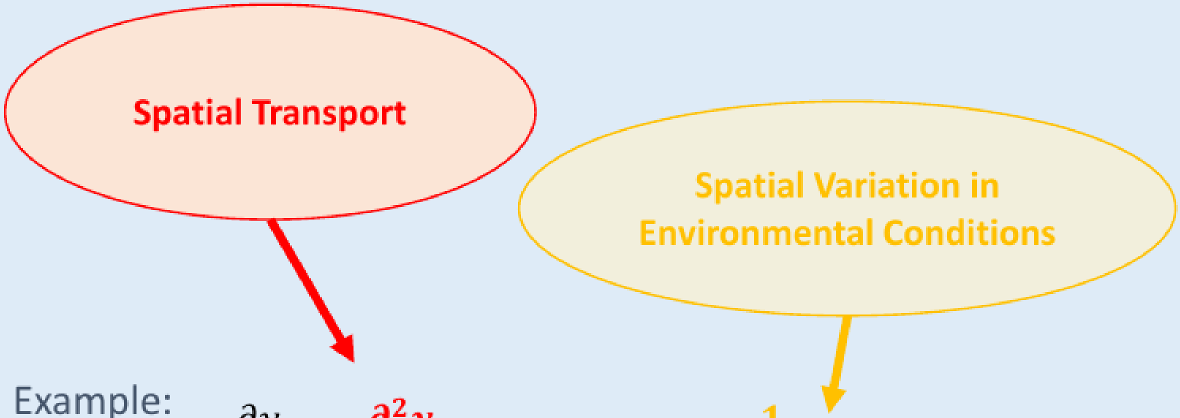
A spatially heterogeneous world

Classic Tipping



Example: $$\frac{dy}{dt} = y(1 - y^2) + \mu$$

Tipping in Spatially Heterogeneous Systems



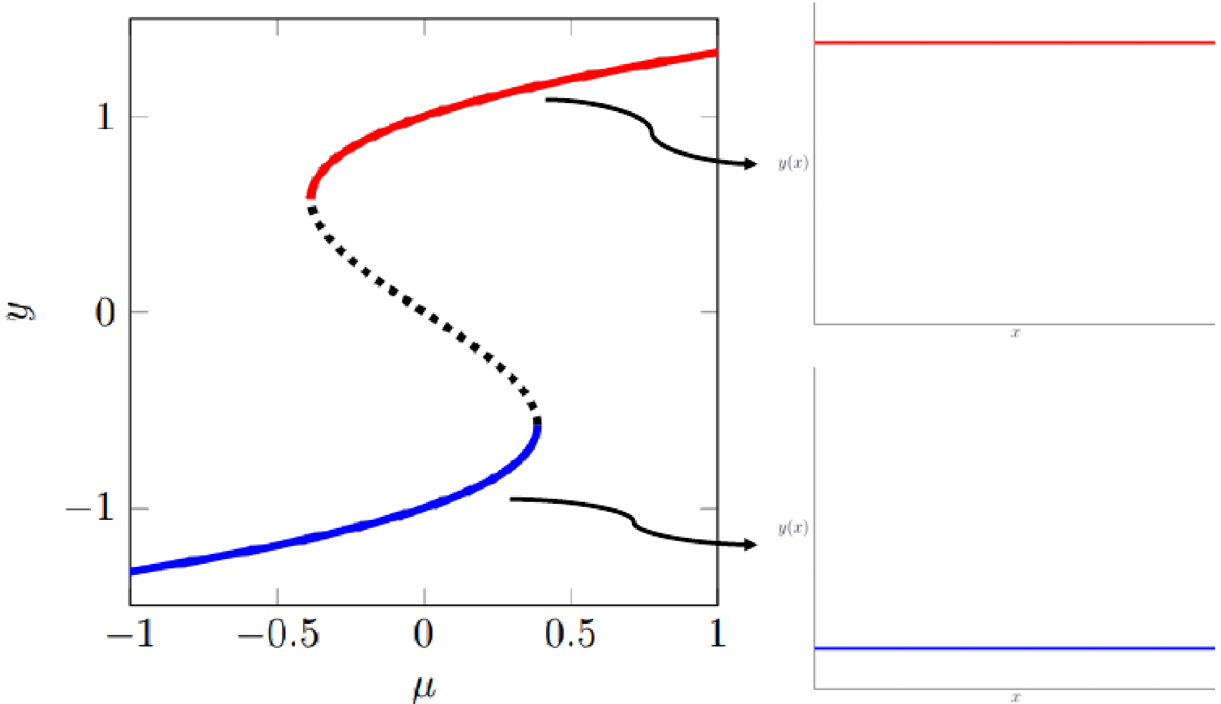
Example: $$\frac{\partial y}{\partial t} = D\frac{\partial^2 y}{\partial x^2} + y(1 - y^2) + \mu + \frac{1}{2}\cos(\pi x)$$

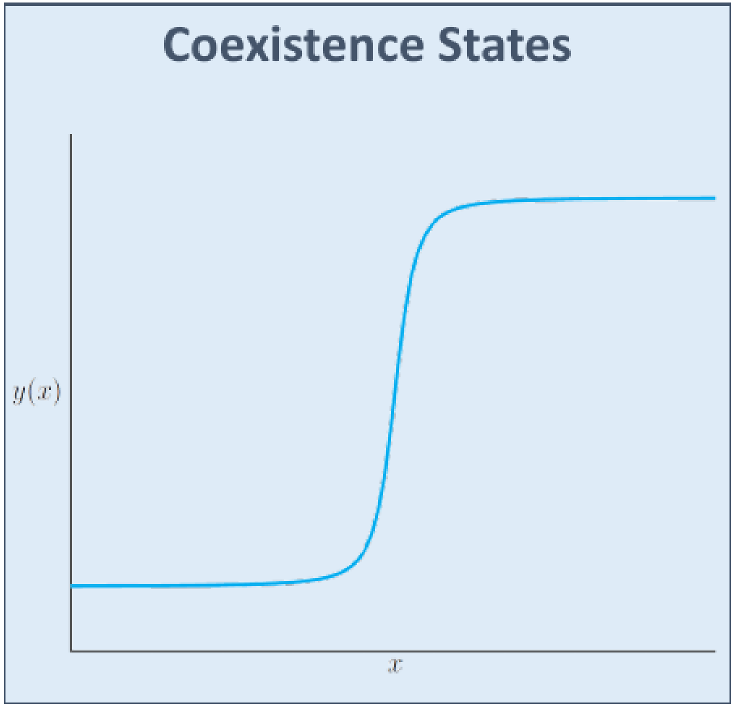
Stationary front solutions in bistable PDEs with coefficients that vary in space

Coexistence states

Bistable (Allen-Cahn/Nagumo) equation:

$$\frac{\partial y}{\partial t} = y(1 - y^2) + \mu + D\frac{\partial^2 y}{\partial x^2}$$



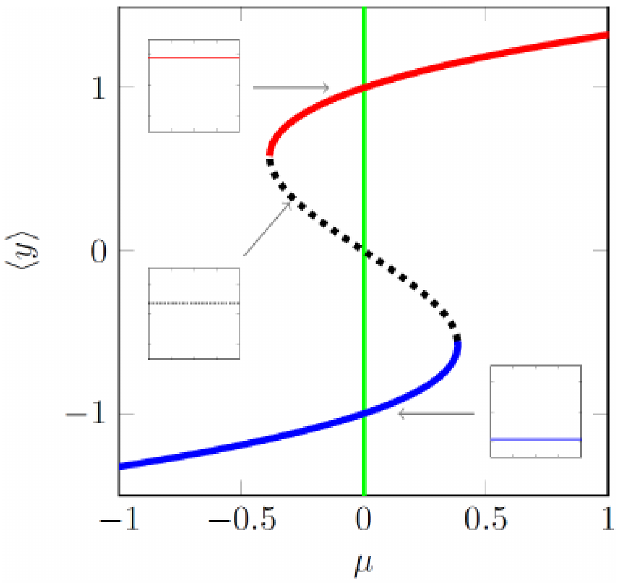


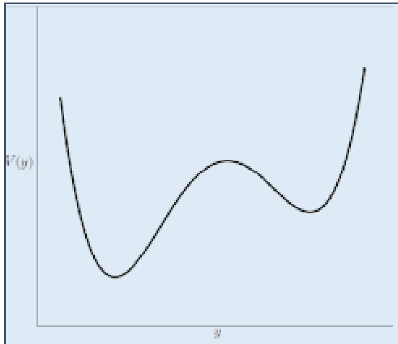
Front Dynamics

$$\frac{\partial y}{\partial t} = D\,\frac{\partial^2 y}{\partial x^2} + f(y;\mu)$$

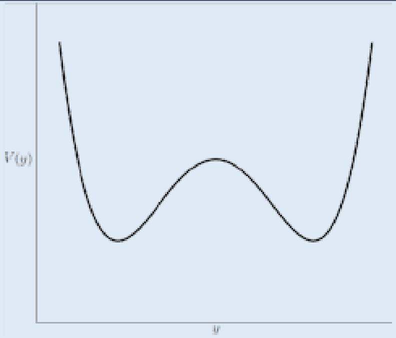
Potential function $V(y;\mu)$:

$$\frac{\partial V}{\partial y}(y;\mu) = -f(y;\mu)$$

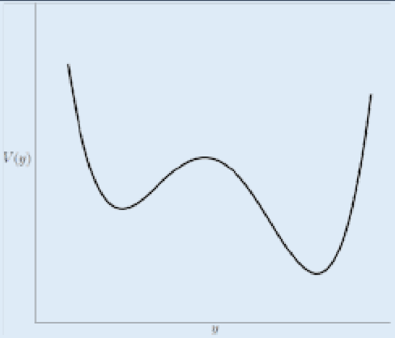




moves right

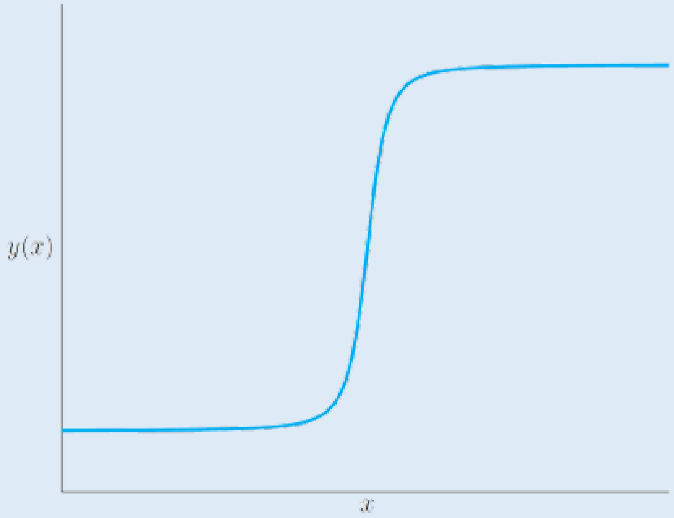


stationary



moves left

Maxwell Point $\mu_{maxwell}$

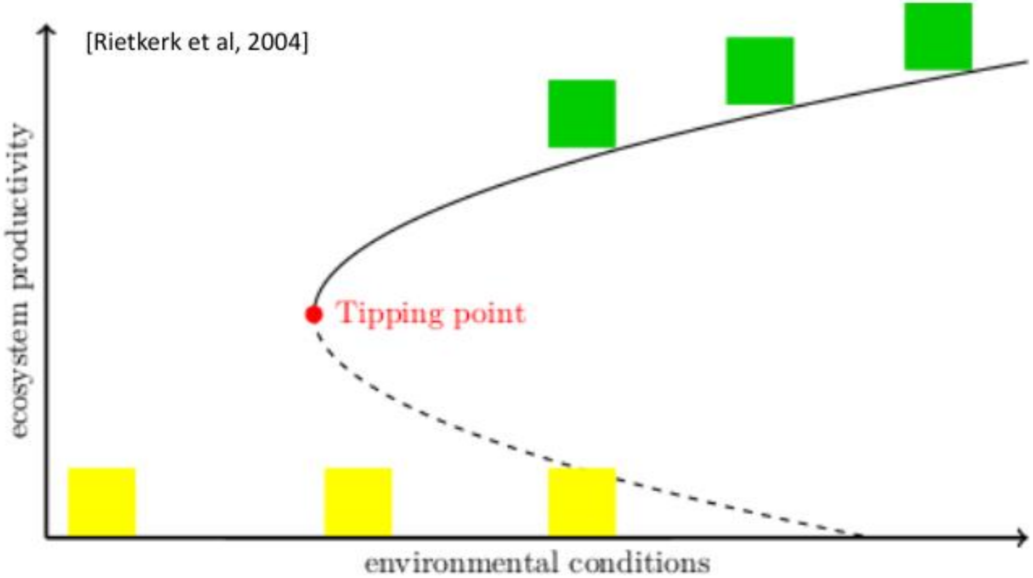


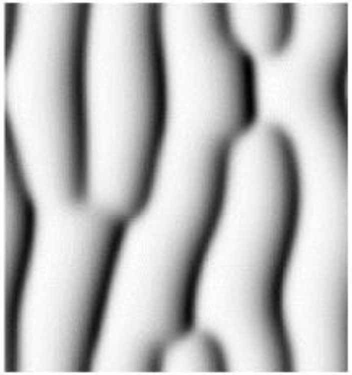
Patterns in models

Add spatial transport:

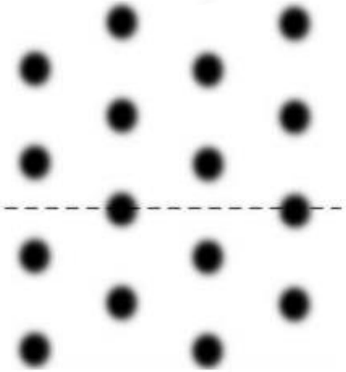
Reaction-Diffusion equations:

$$\begin{cases}\frac{du}{dt} = f(u,v) + D_u \,\Delta u \\ \frac{dv}{dt} = g(u,v) + D_v \,\Delta v\end{cases}$$

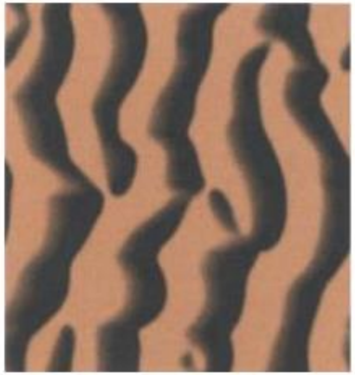




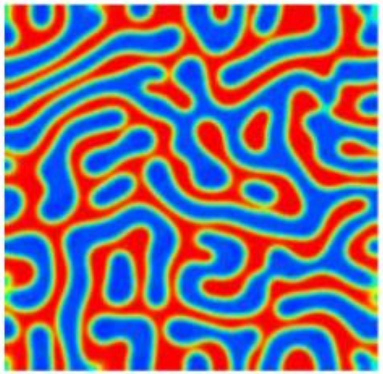
[Klausmeier, 1999]



[Gilad et al. 2004]



[Rietkerk et al. 2002]



[Liu et al. 2013]

Turing patterns

$$\begin{cases} \frac{du}{dt} = f(u,v) + D_u \Delta u \\ \frac{dv}{dt} = g(u,v) + D_v \Delta v \end{cases}$$

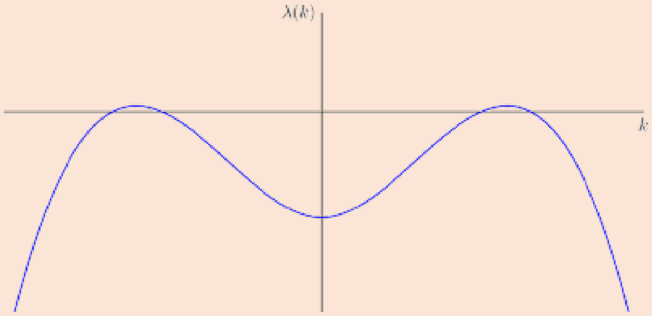
Turing bifurcation

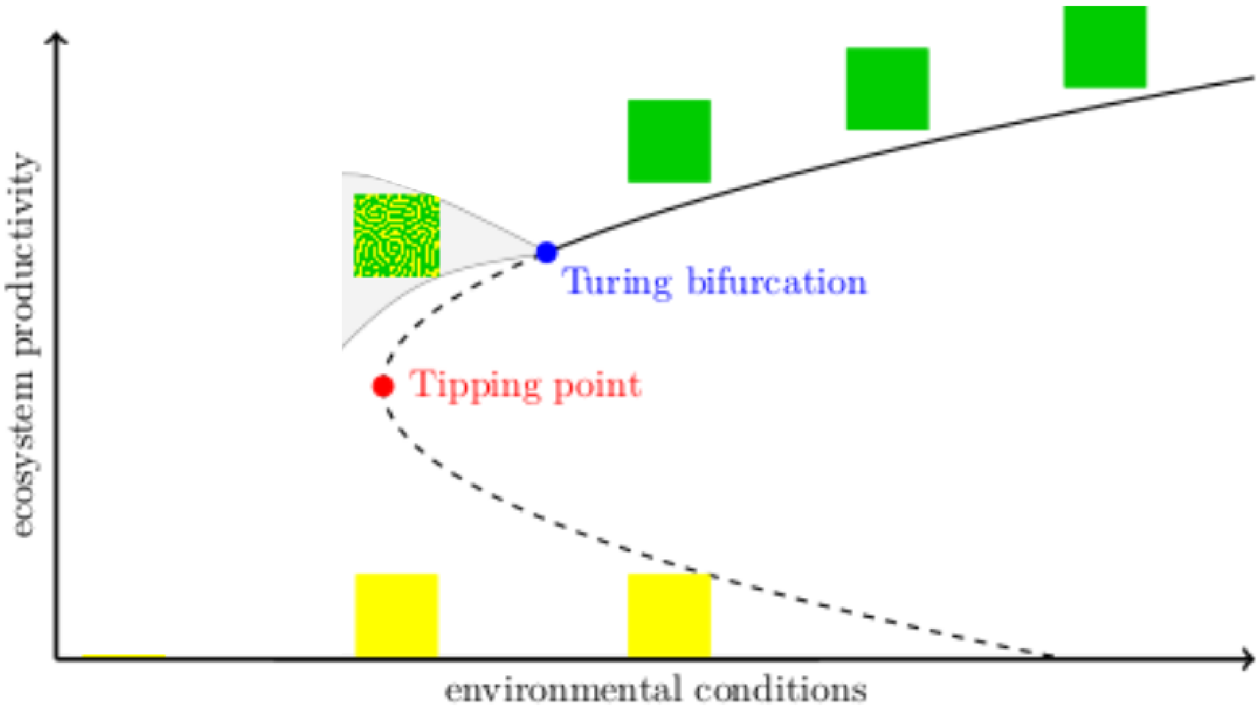
Instability to non-uniform perturbations

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u_* \\ v_* \end{pmatrix} + e^{\lambda t} e^{ikx} \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix}$$

→ Dispersion relation

$$\lambda(k) = \cdots$$





Weakly non-linear analysis

Ginzburg-Landau equation / Amplitude Equation

& Eckhaus/Benjamin-Feir-Newel criterion

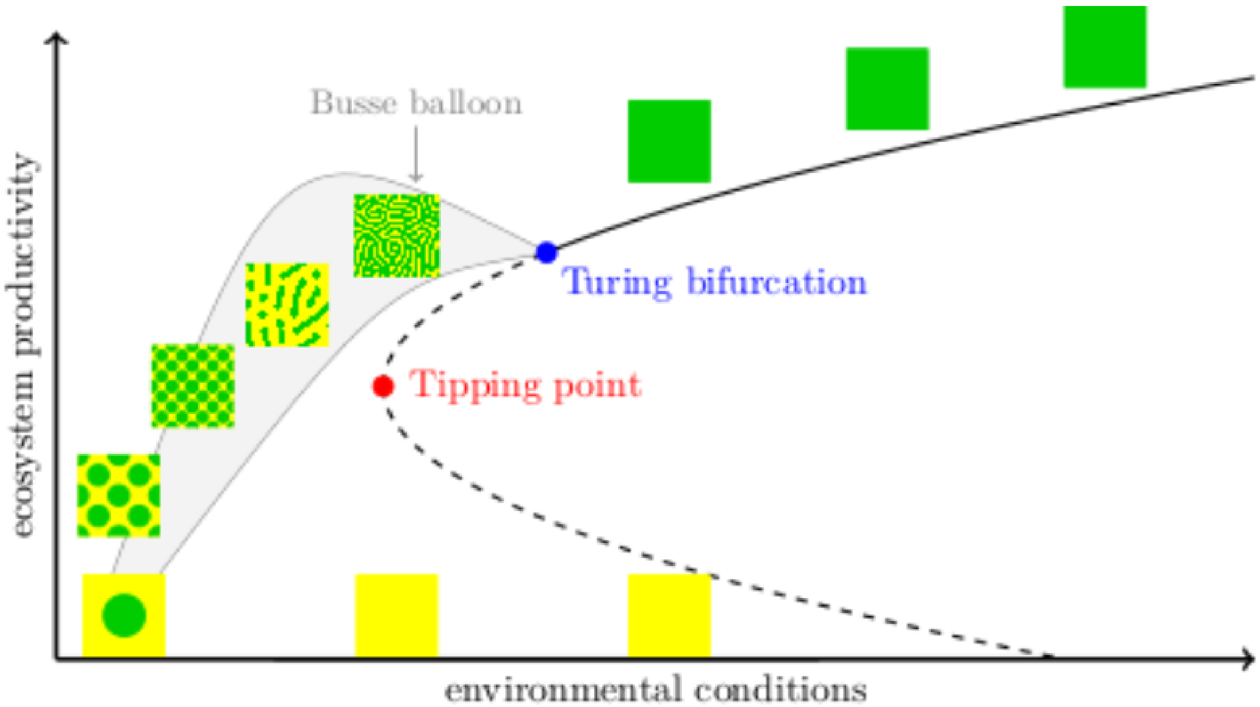
[Eckhaus, 1965; Benjamin & Feir, 1967; Newell, 1974]

Busse balloon

$$\begin{cases} \frac{du}{dt} = f(u,v) + D_u \, \Delta u \\ \frac{dv}{dt} = g(u,v) + D_v \, \Delta v \end{cases}$$

Busse balloon

A model-dependent shape in *(parameter, observable)* space that indicates all stable patterned solutions to the PDE.



Construction Busse balloon

Via numerical continuation

few general results on the shape of Busse balloon

Busse balloon

Idea originates from thermal convection

[Busse, 1978]

Minitutorial overview

- ~~Introduction~~
- Multistability and patterns
- Explicit construction of front solutions
  - Existence
  - Stability
- Dynamics of existing structures
- Summary & Outlook