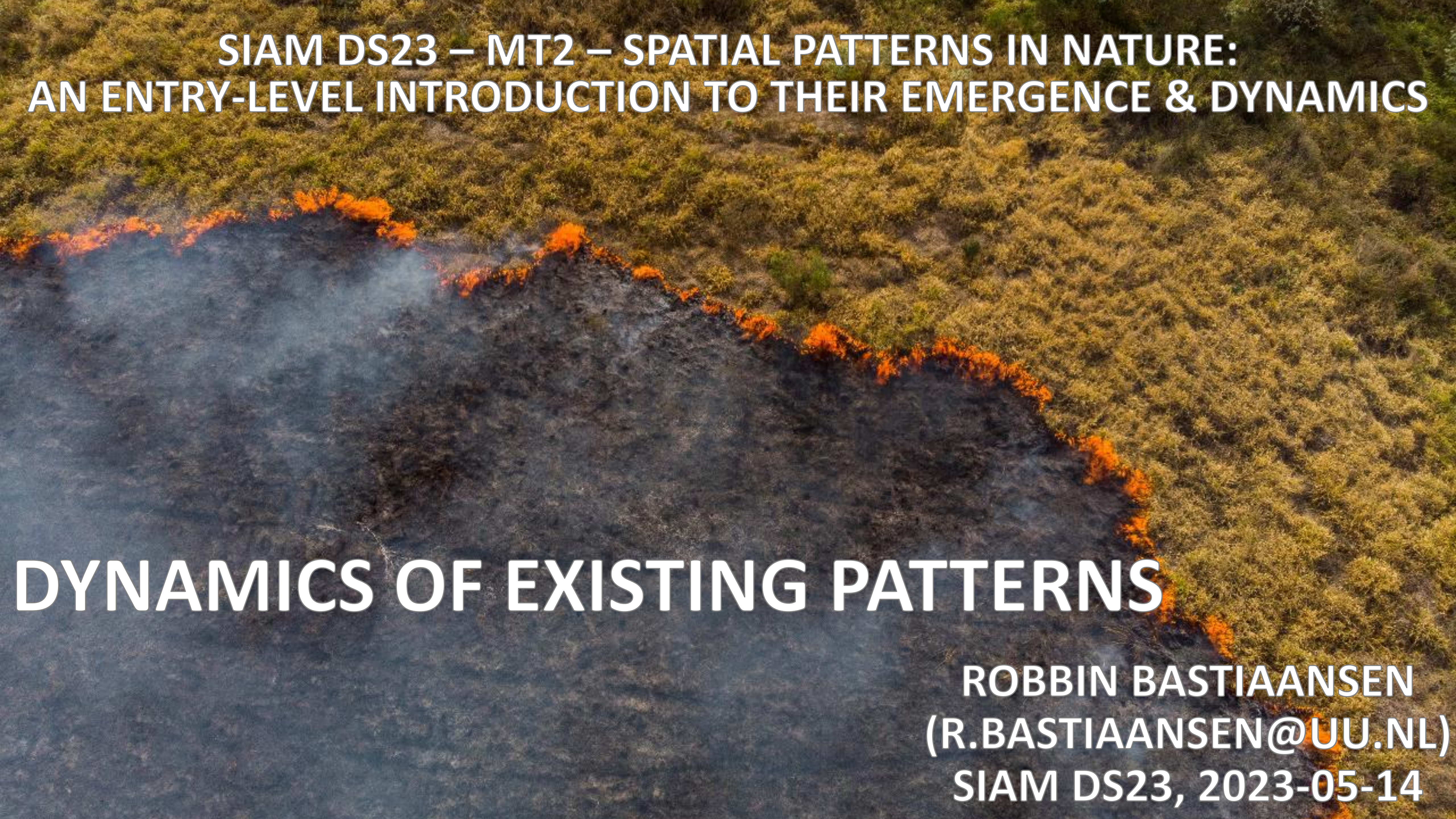
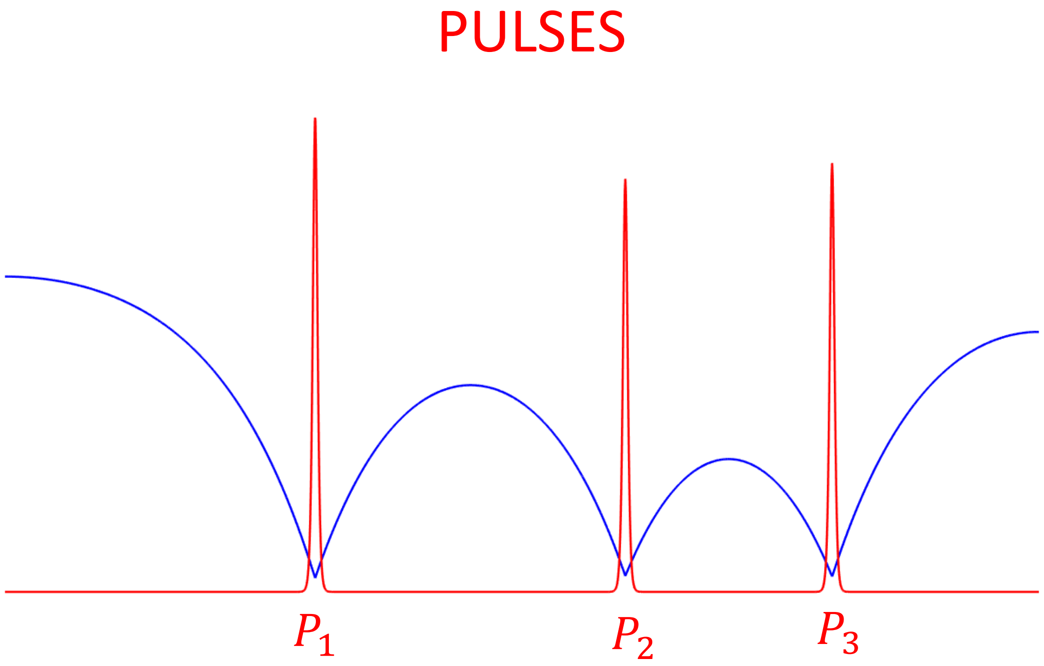


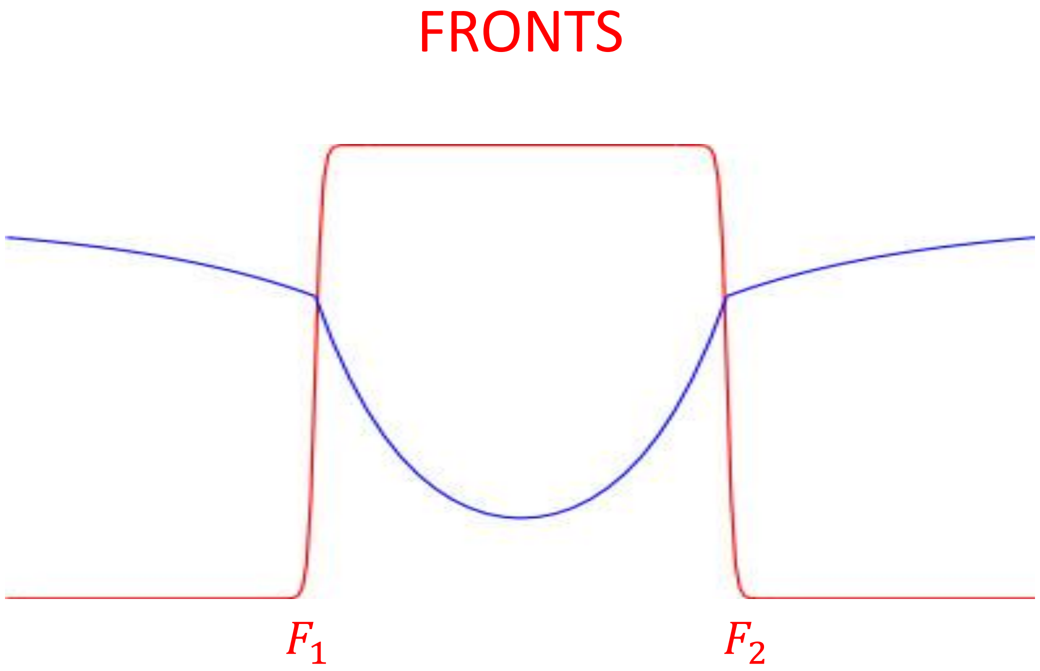
SIAM DS23 – MT2 – SPATIAL PATTERNS IN NATURE: AN ENTRY-LEVEL INTRODUCTION TO THEIR EMERGENCE & DYNAMICS

DYNAMICS OF EXISTING PATTERNS

ROBBIN BASTIAANSEN
(R.BASTIAANSEN@UU.NL)
SIAM DS23, 2023-05-14

SETTING – pre-existing structures





These patterns are fully described by the location of the local structures

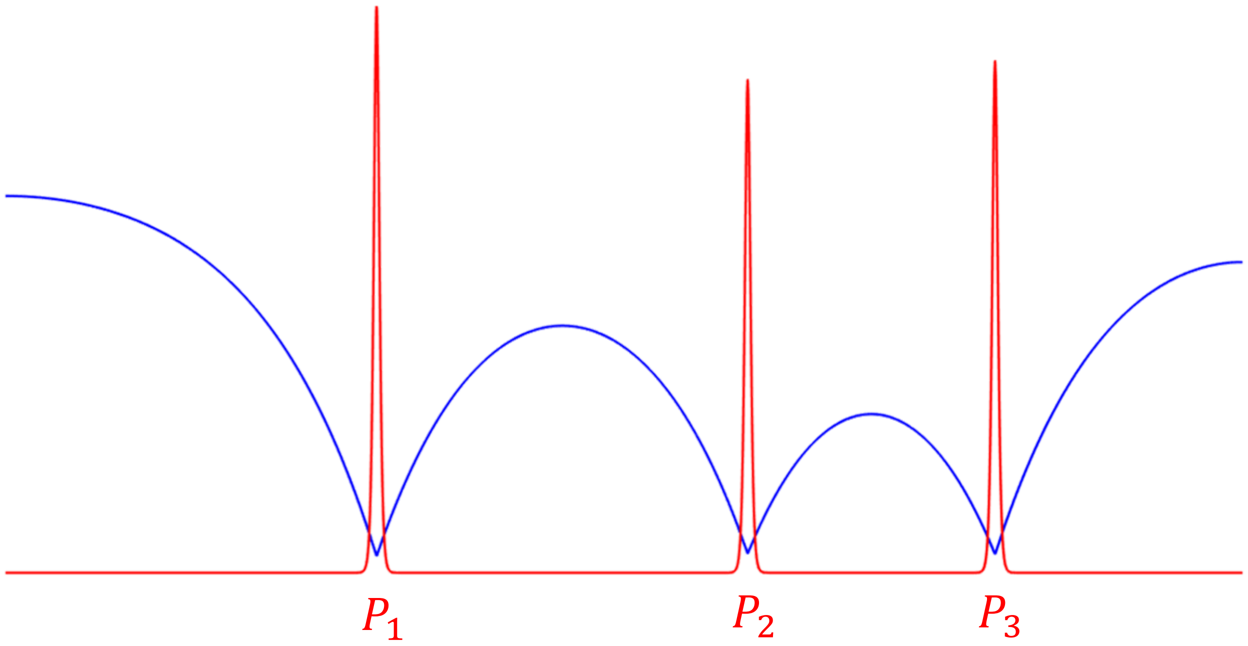
This section: How do they change over time?

Dynamics of patterns

- PDE: infinite-dimensional state space
- Reduction possible because of localised structures

DYNAMICS in two manners:

1. (SLOW) migratory movement of localised structures [Pattern Adaptation]
2. (FAST) structural changes [Pattern Degradation]

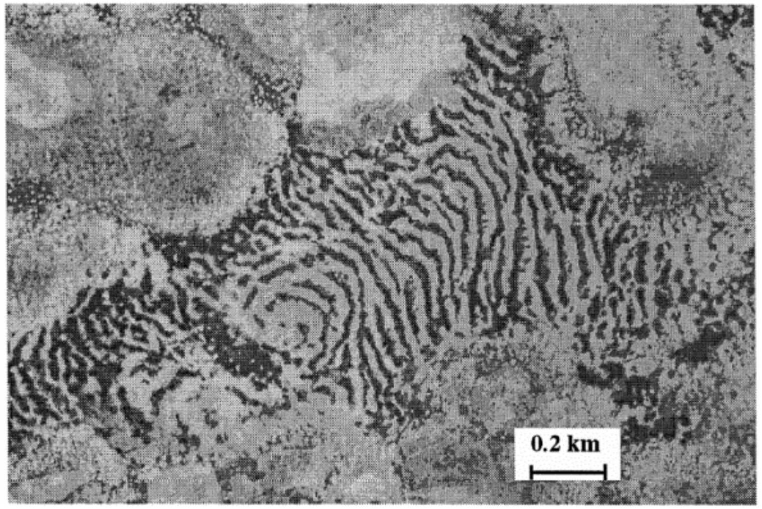


1. SLOW pattern adaptation

Somaliland, 1948 [Macfadyen, 1950]

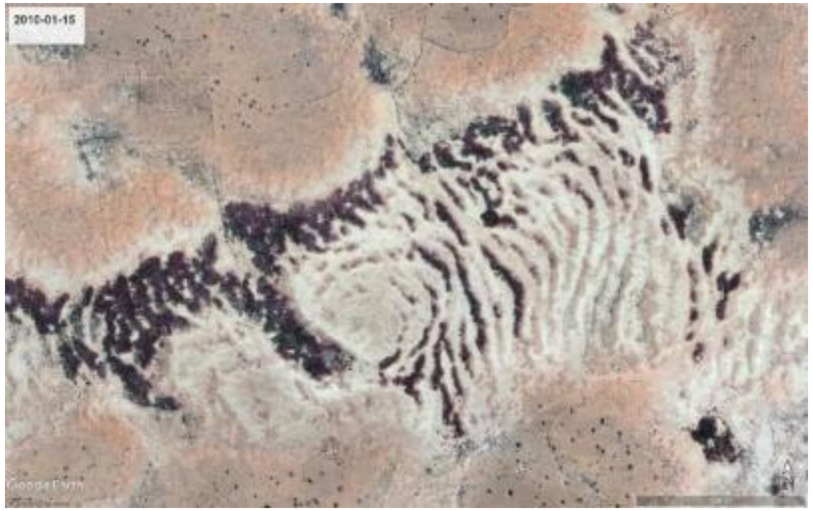
Somaliland, 2008

2. FAST Pattern Degradation

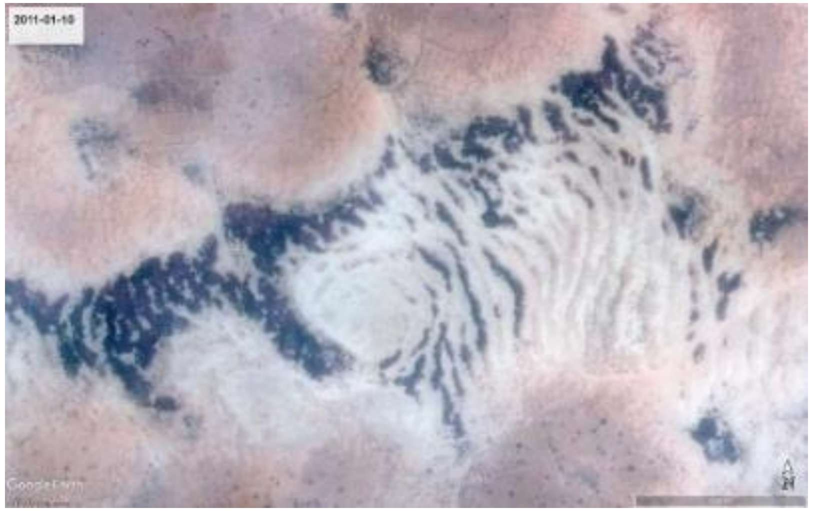


Niger, 1950 [Valentin, 1999]

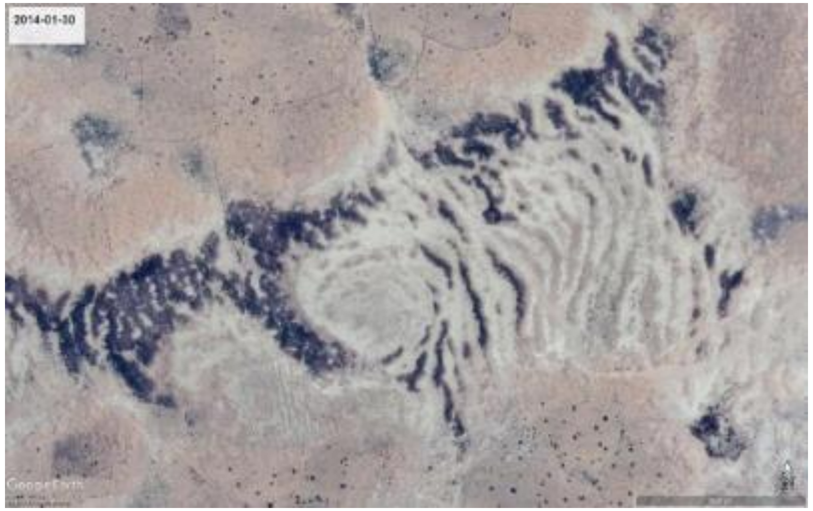
Niger, 2008



Niger, 2010



Niger, 2011



Niger, 2014

Niger, 2016

Example system: dryland ecosystems [reminder]

Extended-Klausmeier model

$$w_t = w_{xx} + (h_x w)_x - w + a - wv^2$$
$$v_t = D^2 v_{xx} - mv + wv^2$$

w : water

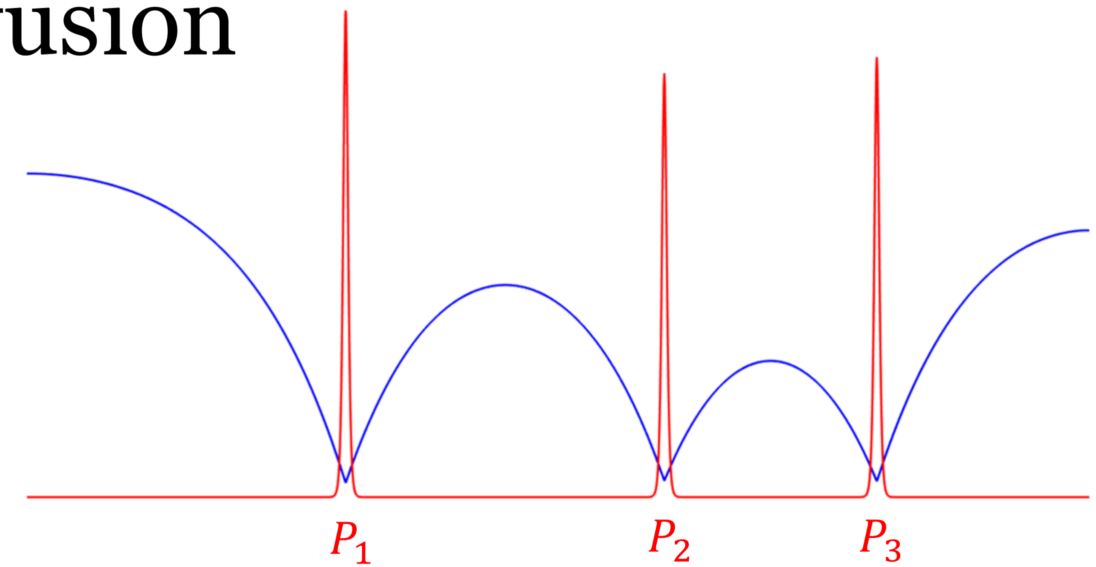
v : vegetation

h : height

D : ratio of diffusion

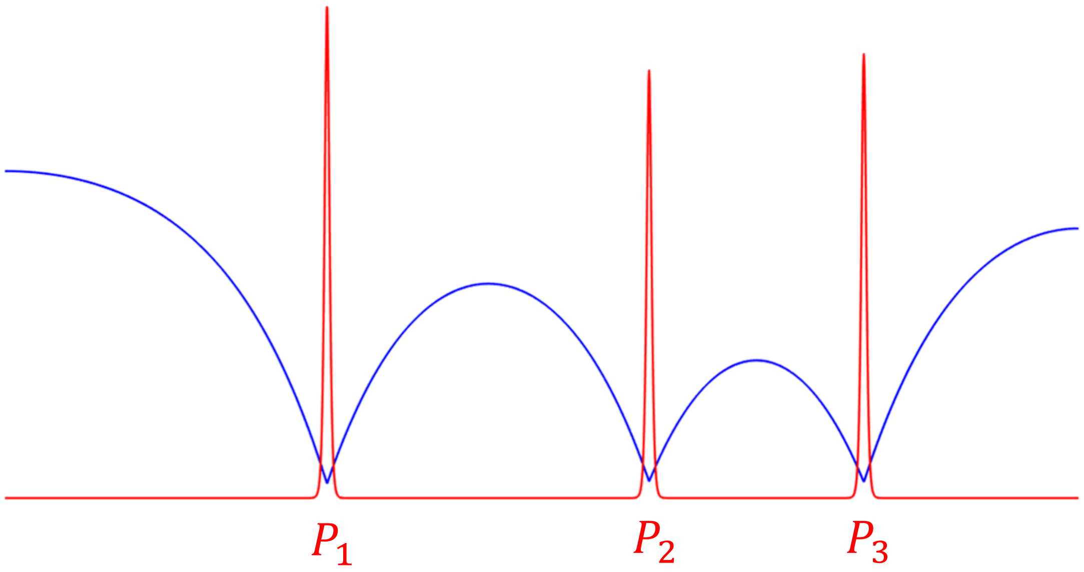
a : rainfall

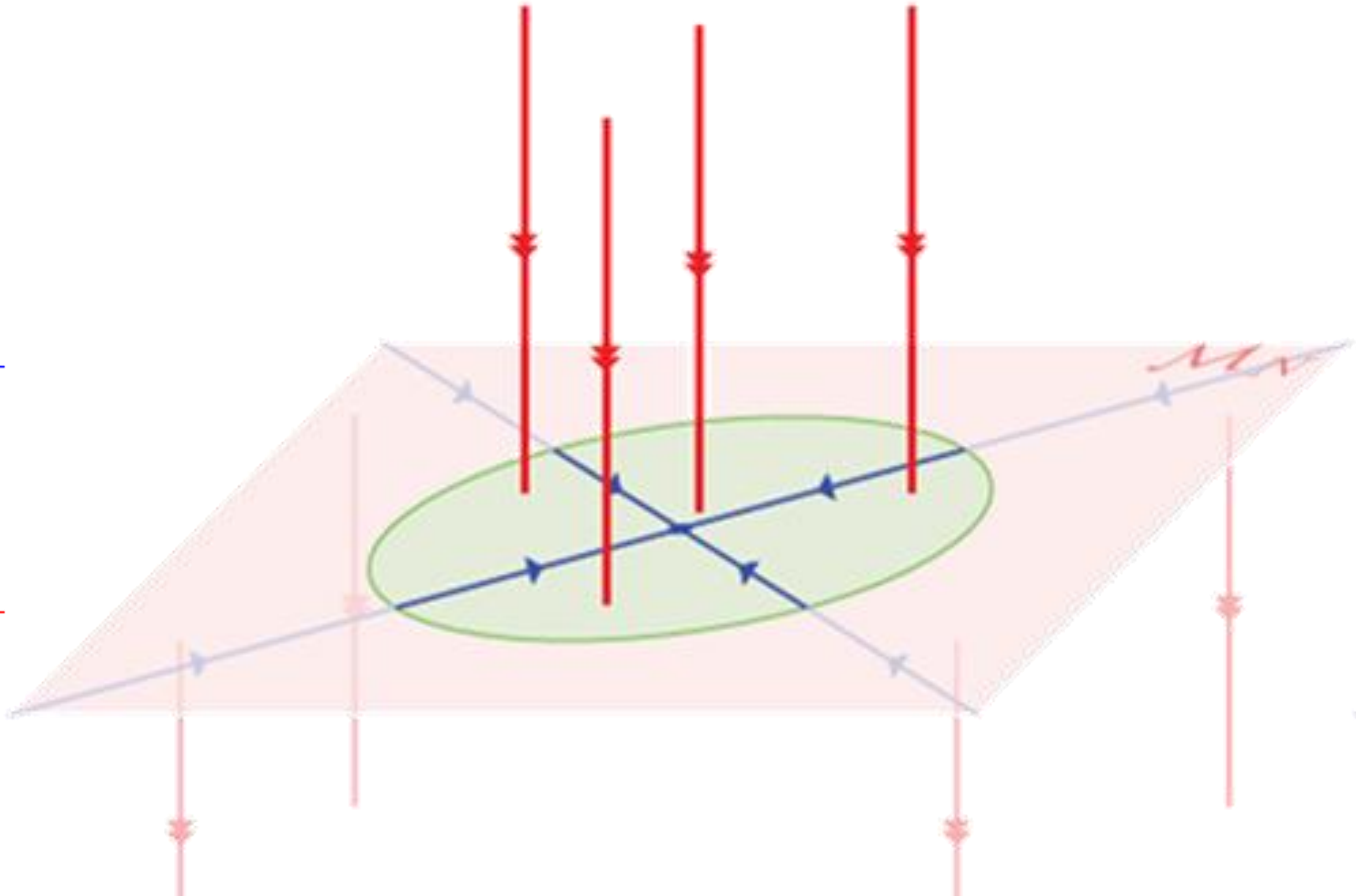
m : mortality



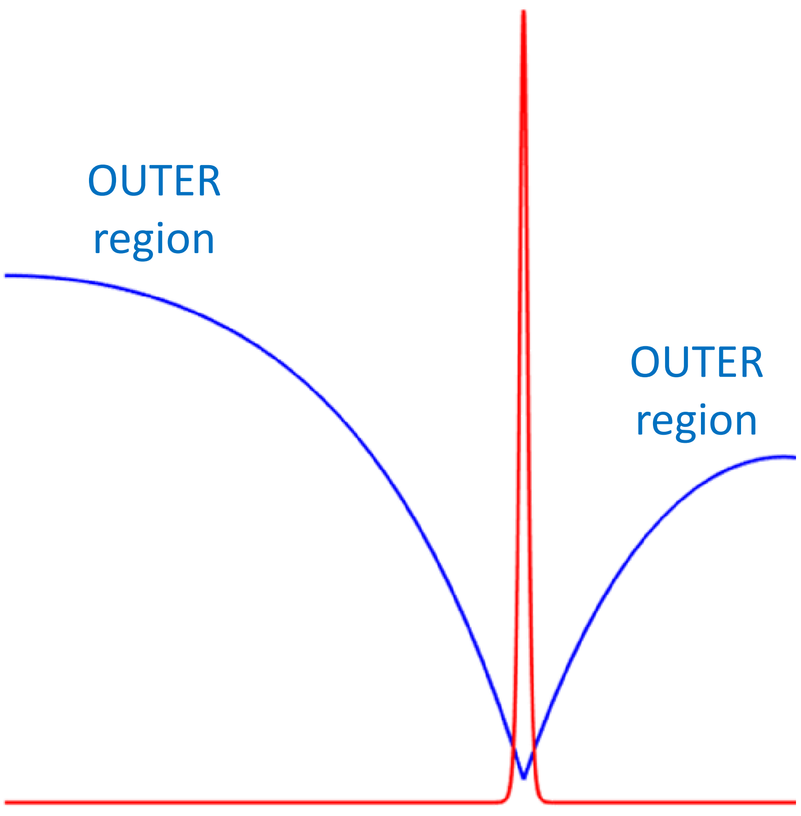
Dynamics of pulses in extended-Klausmeier

1. <u>Pulse-location ODE</u>: describe movement of pulses
2. <u>Stability criterium</u>: test if configuration is feasible





Derivation of pulse-location ODE



$$w_t = w_{xx} + (h(\mathbf{x})_x w)_x - w + a(\mathbf{t}) - wv^2$$
$$v_t = D^2 v_{xx} - mv + wv^2$$

INNER regions:

$$0 = D^2 v_{xx} - mv + wv^2$$

$$\Rightarrow v_p(x - P_j(\mathbf{t}))$$

OUTER regions:

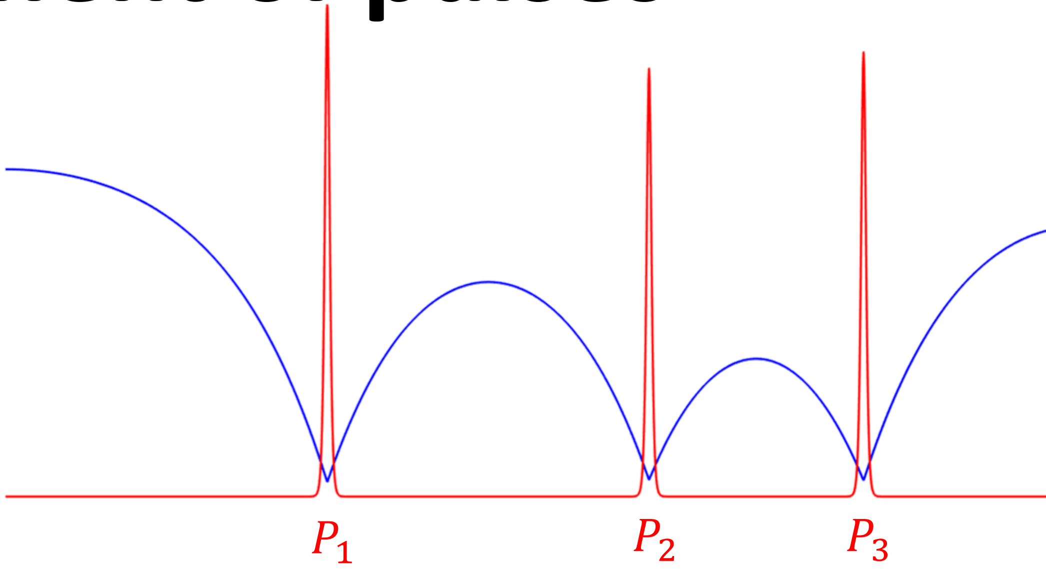
$$0 = w_{xx} + (h(\mathbf{x})_x w)_x - w + a(\mathbf{t})$$

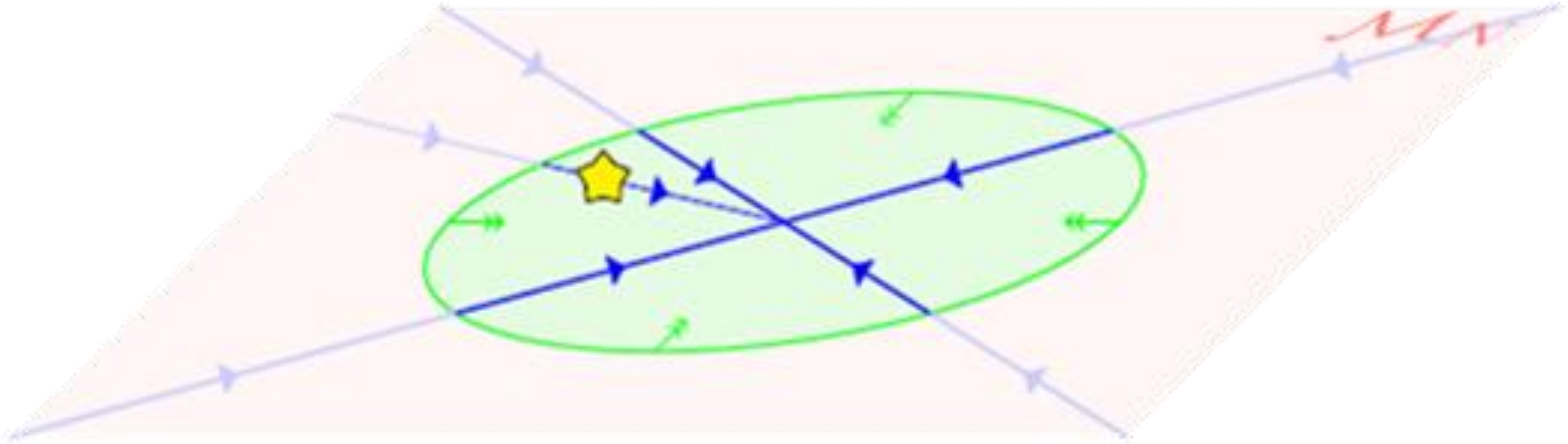
Match solutions at boundaries:

$$\Rightarrow \frac{dP_j}{dt} = \frac{Da^2}{m\sqrt{m}} \left[w_x({P_j}^+)^2 - w_x({P_j}^-)^2 \right]$$

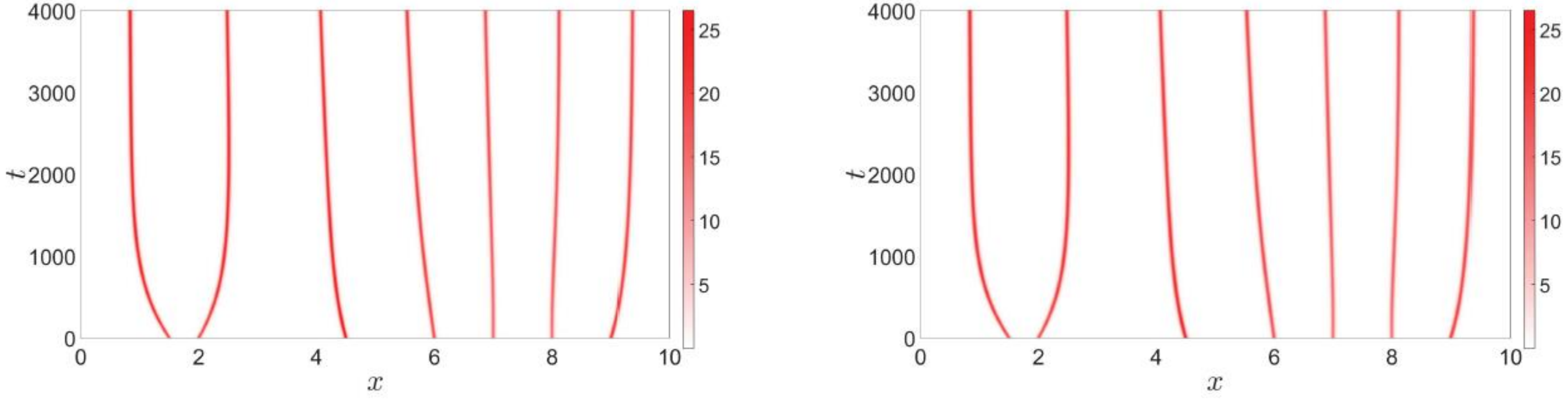
SLOW migratory movement of pulses

$$\frac{dP_j}{dt} = \frac{Da^2}{m\sqrt{m}} \left[w_x(P_j{}^+)^2 - w_x(P_j{}^-)^2 \right]$$





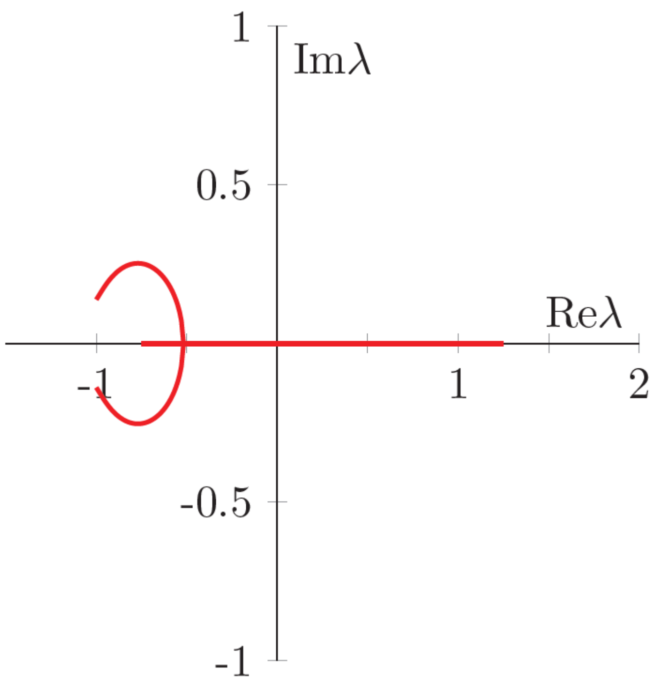
Comparison between full PDE and reduced ODE



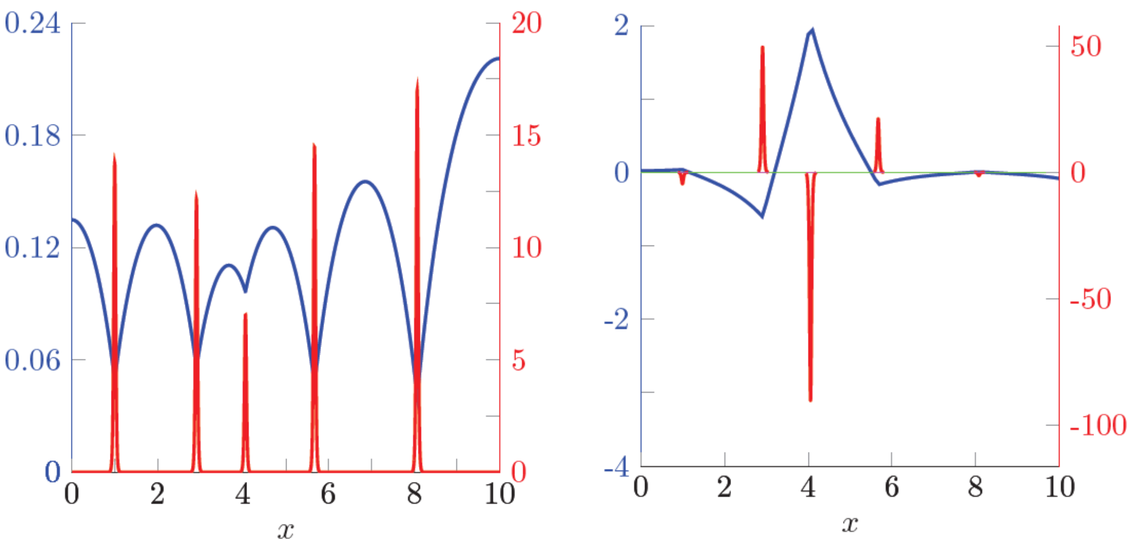
(a) ODE (b) PDE

2. Stability Criterium

Procedure:
Freeze solution in time
Study (quasi-steady) eigenvalues & eigenfunctions



spectrum



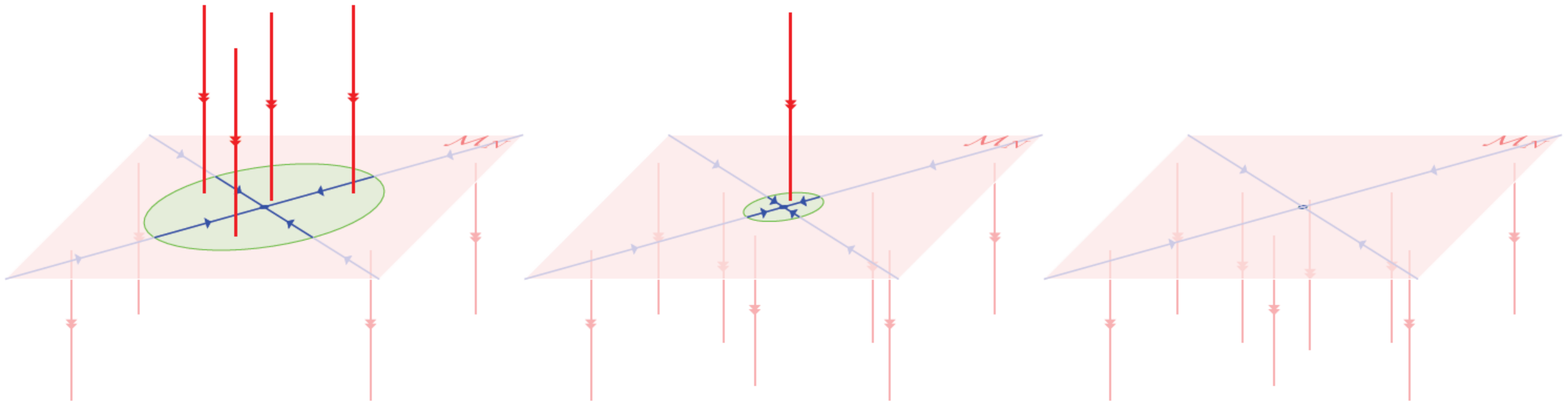
solution

eigenfunction

Nonlinear prediction based on linear analysis

2. Stability Criterium

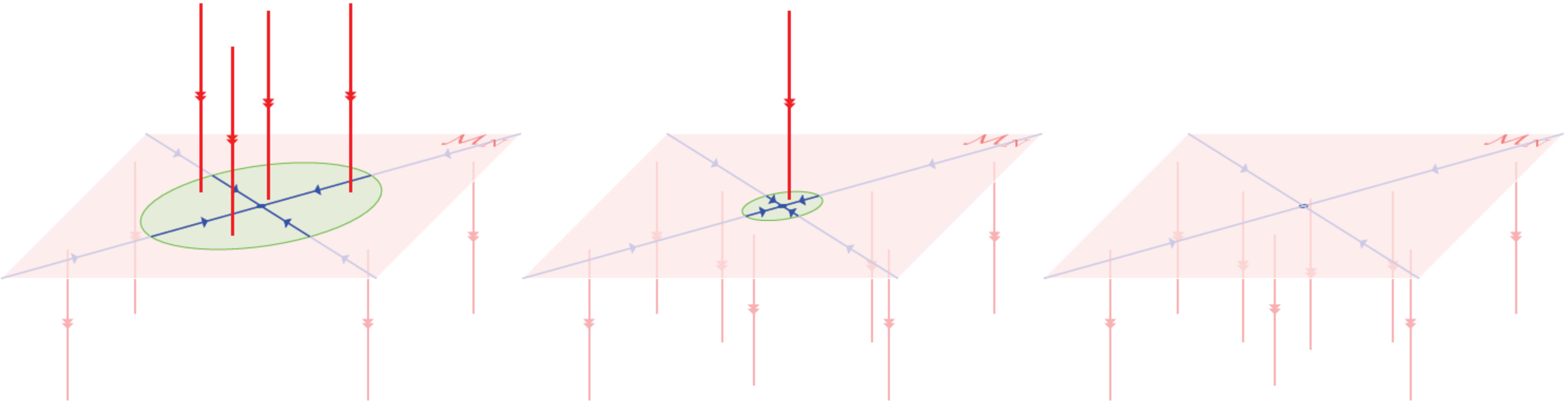
Enough resources to sustain all vegetation patches?
Depends on **amount of rainfall** and **distance between pulses**



high rainfall | medium rainfall | low rainfall

2. Stability Criterium

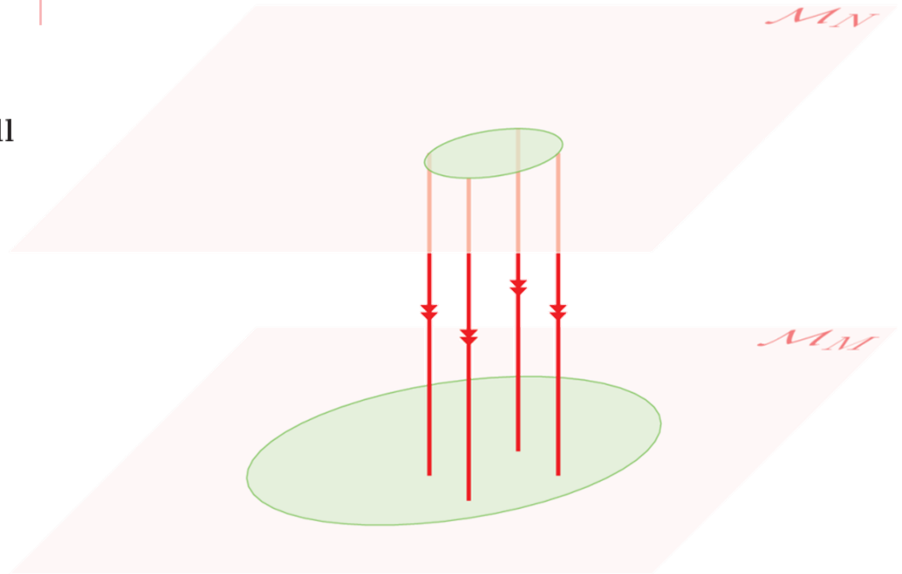
Enough resources to sustain all vegetation patches?
Depends on **amount of rainfall** and **distance between pulses**



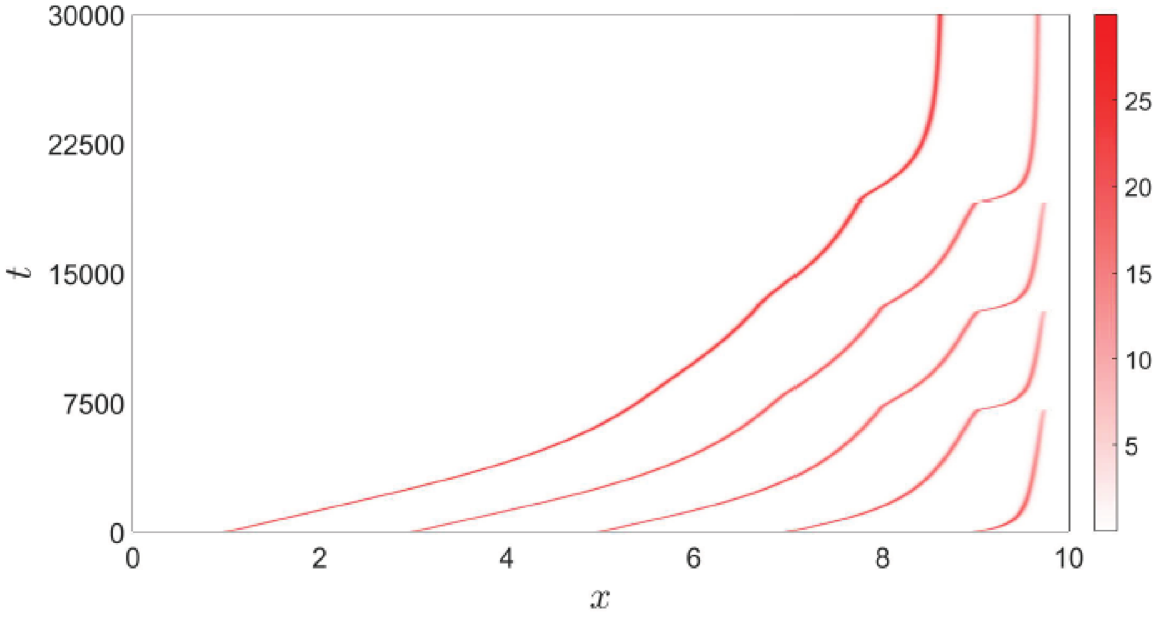
high rainfall   medium rainfall   low rainfall

What happens when outside feasible region?

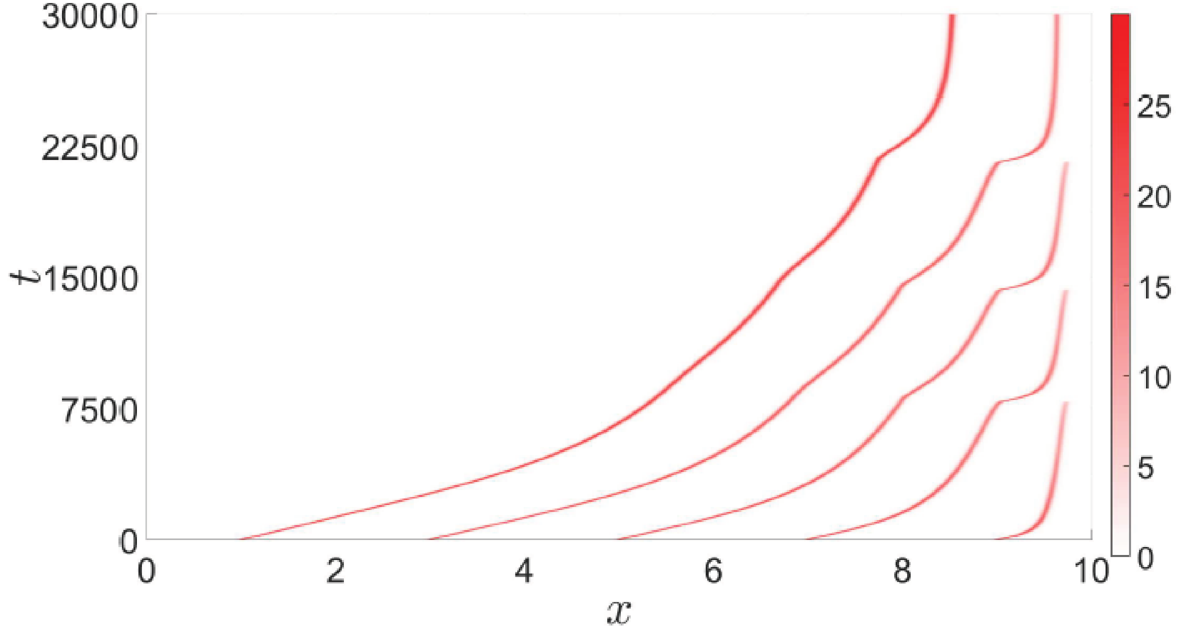
irregular configuration:	**One** patch disappears (least amount of biomass)
regular configuration:	**Half** of the patches disappears (wavelength doubling)



Comparison between full PDE and reduced ODE



(a) ODE



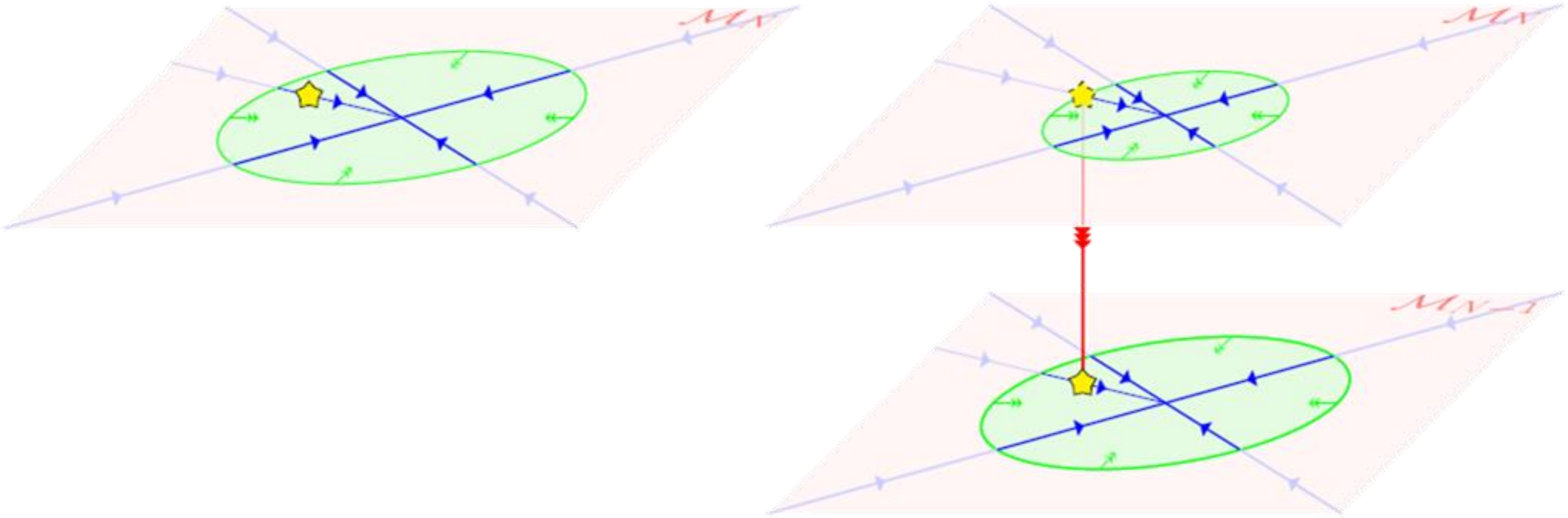
(b) PDE

Pulses during climate change (1)

Competition of two effects:

1. Pulse rearrangement
2. Shrinking of feasible region

fast climate change



(a) initial configuration

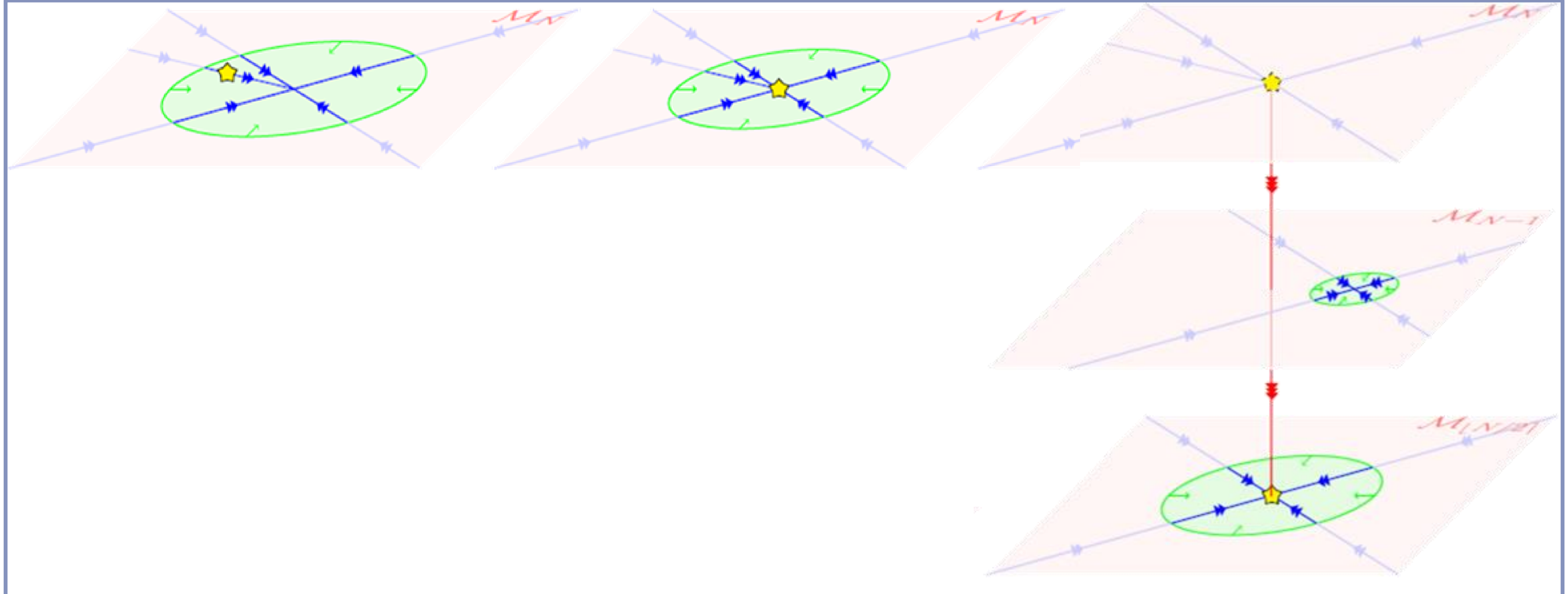
(b) at/after destabilization

Pulses during climate change (2)

Competition of two effects:

1. Pulse rearrangement
2. Shrinking of feasible region

slow climate change



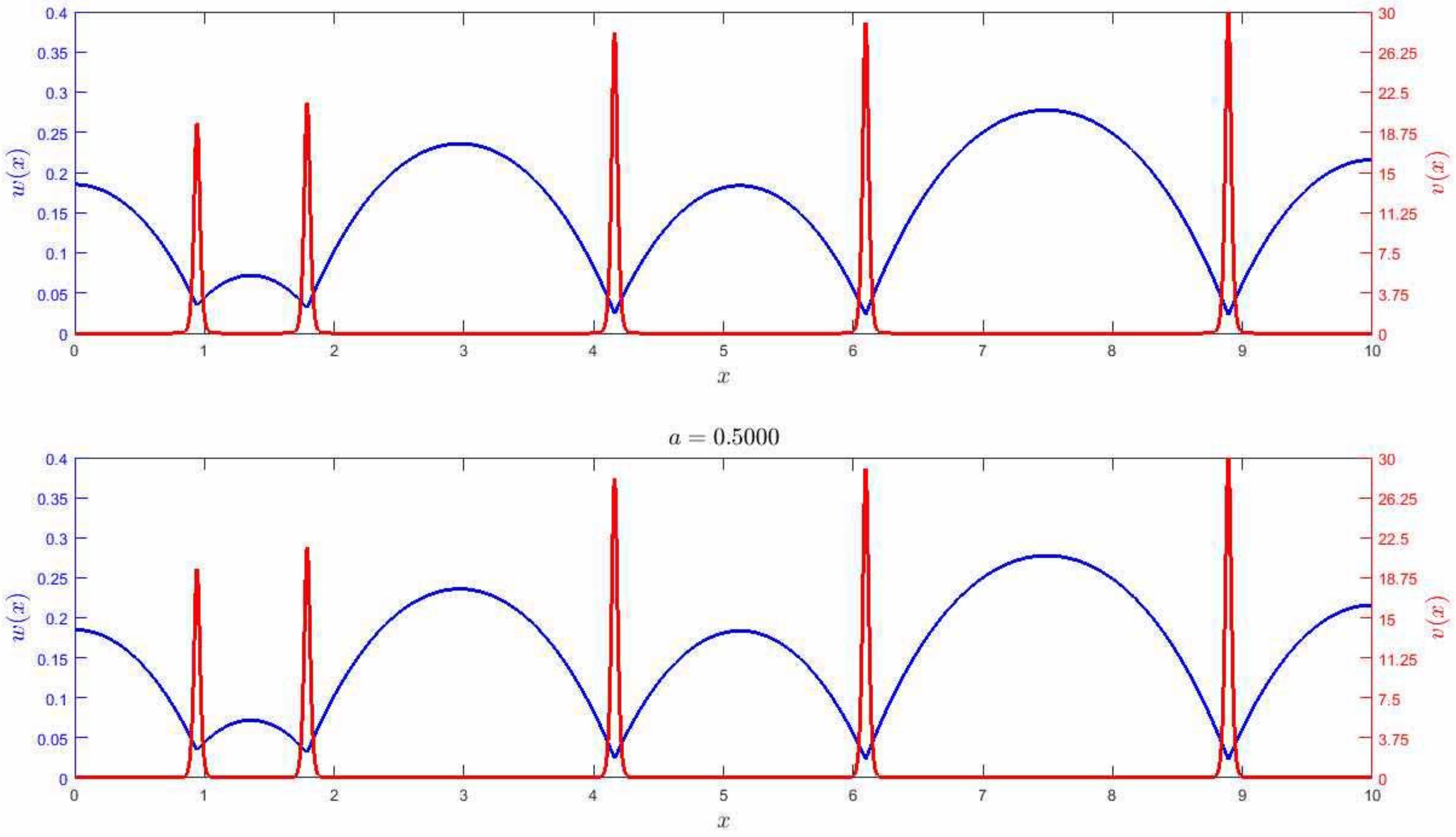
(a) initial configuration (b) regular pattern achieved (c) at/after destabilization

Pulses during climate change (3)

Rate of climate change

FAST

SLOW

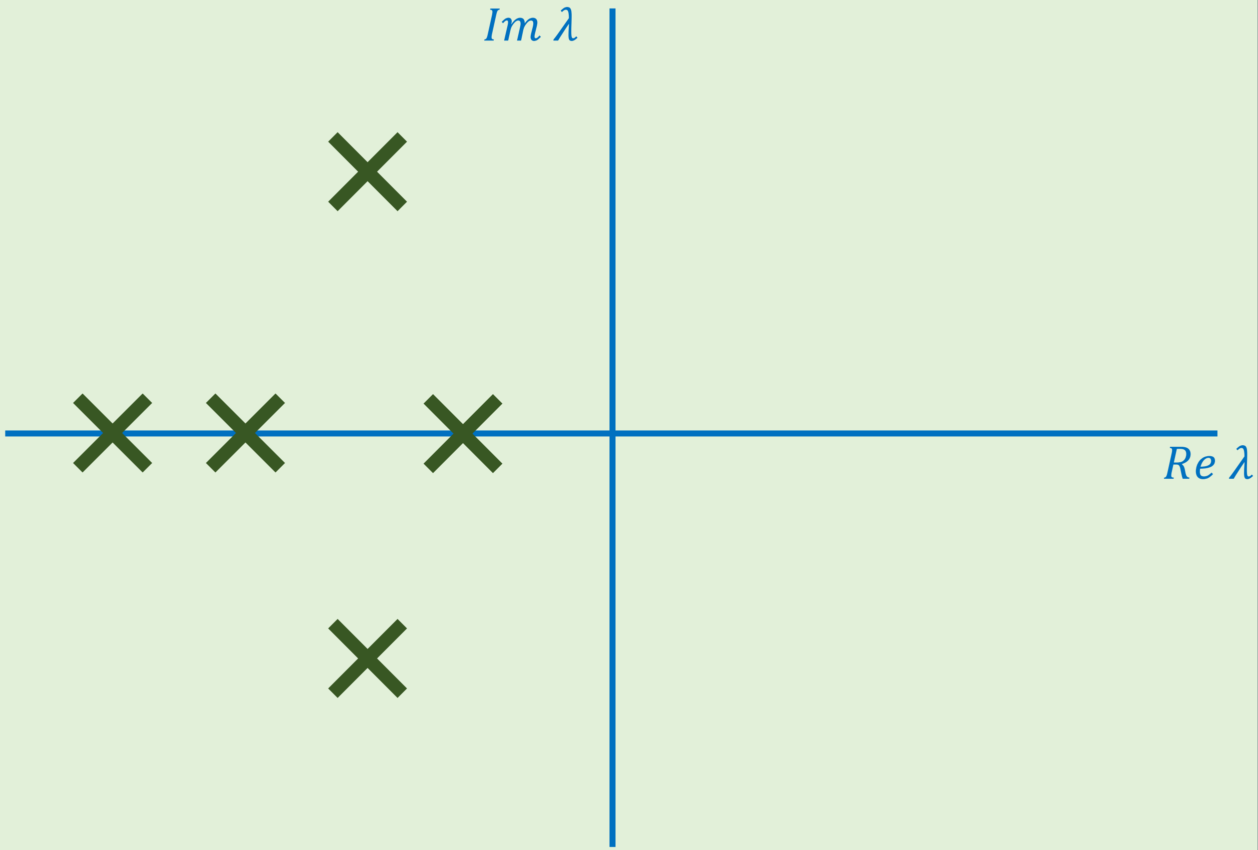



General
Considerations

Stability of Stationary States

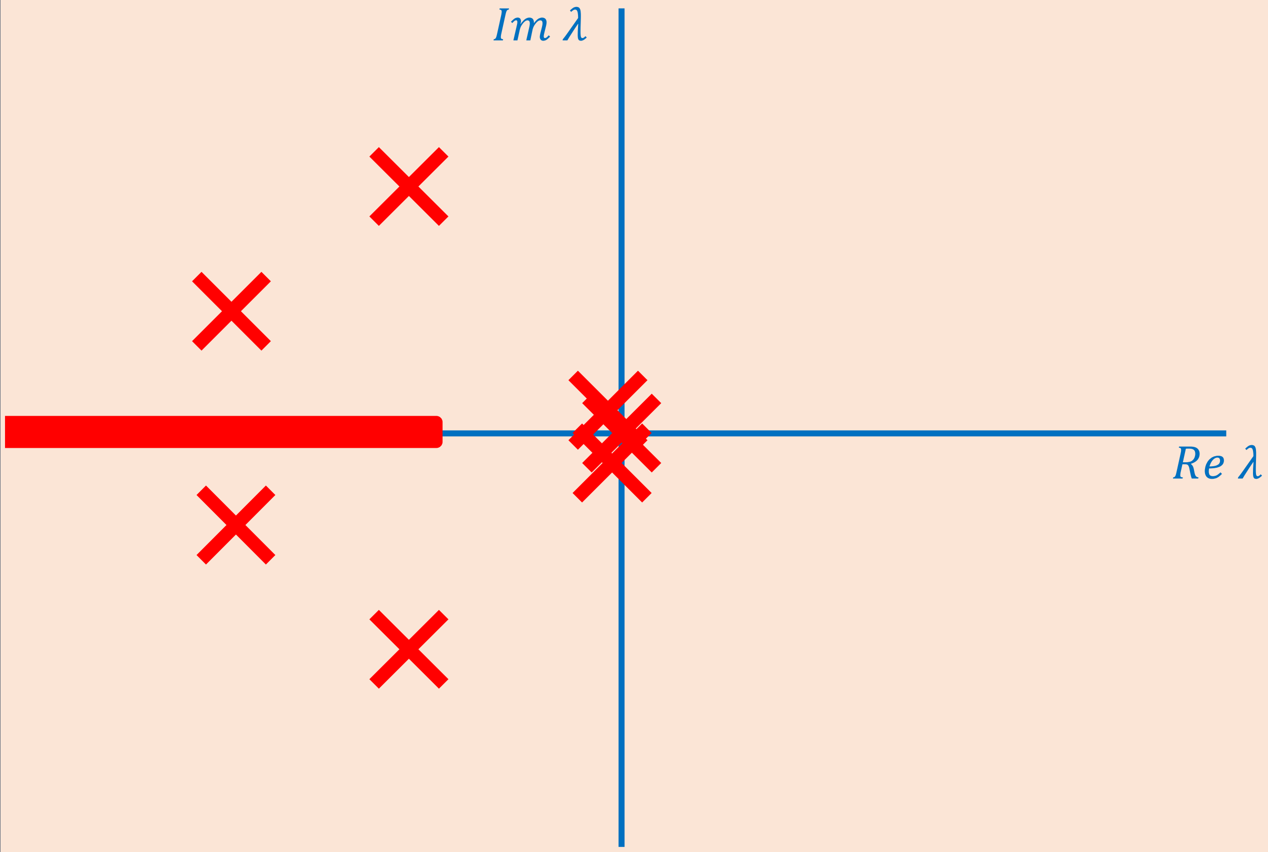
ODE

$$\lambda\,\bar{y} = f_y(y^*;\mu)\,\bar{y}$$

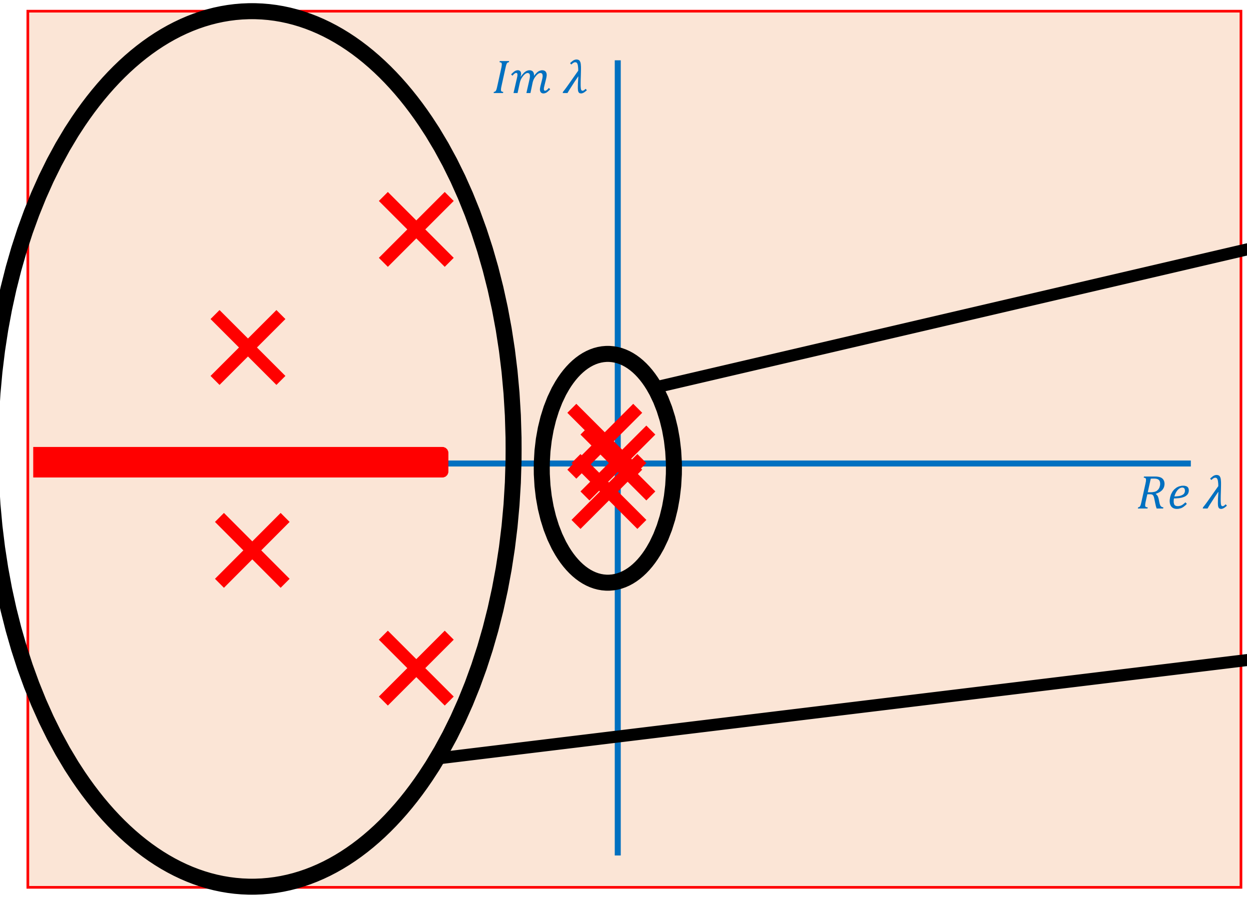


PDE

$$\lambda\,\bar{y} = \bar{y}_{xx} + f_y(y^*(x);\mu)\,\bar{y}$$



Bifurcations



What happens at bifurcation?

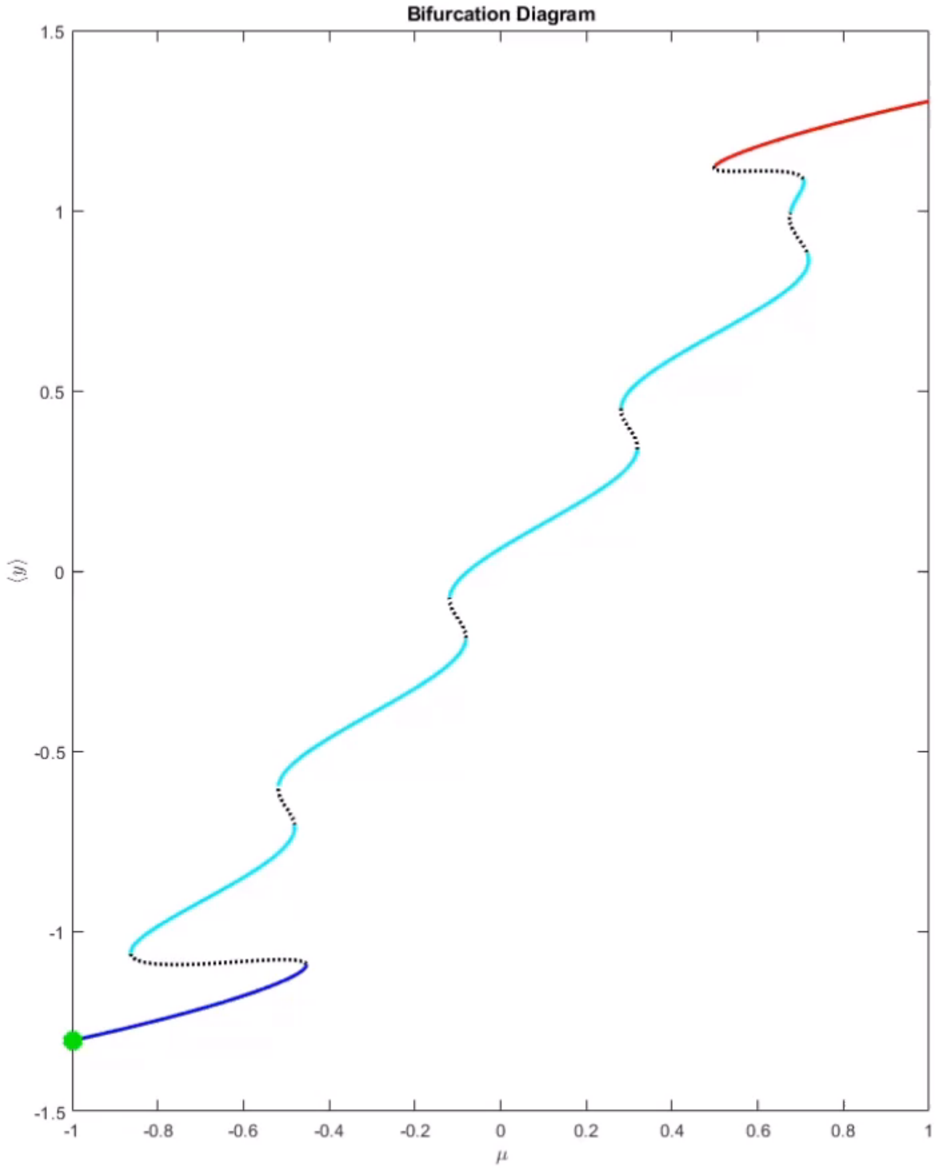
1. SLOW Pattern Adaptation

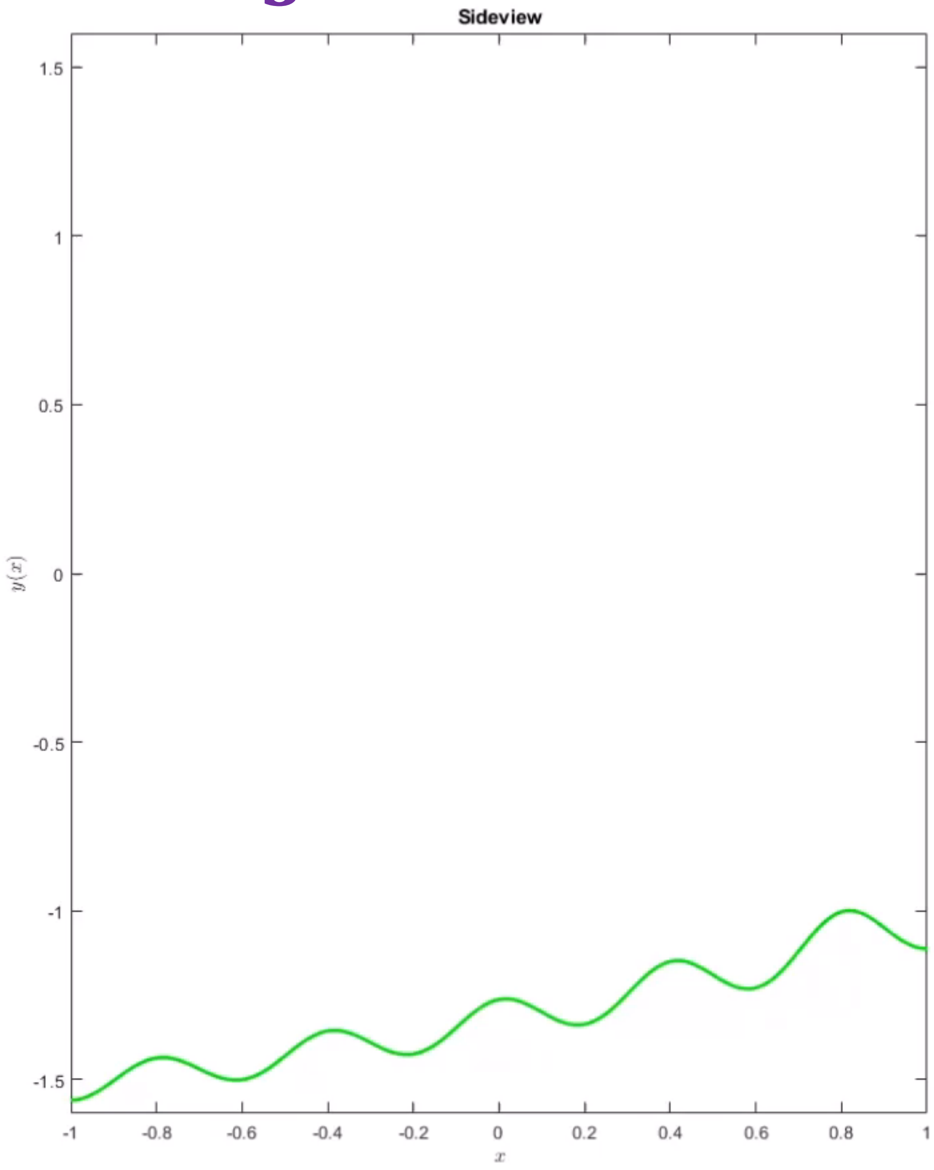
At bifurcation:
→ Location of structure changes

2. FAST Pattern Degradation

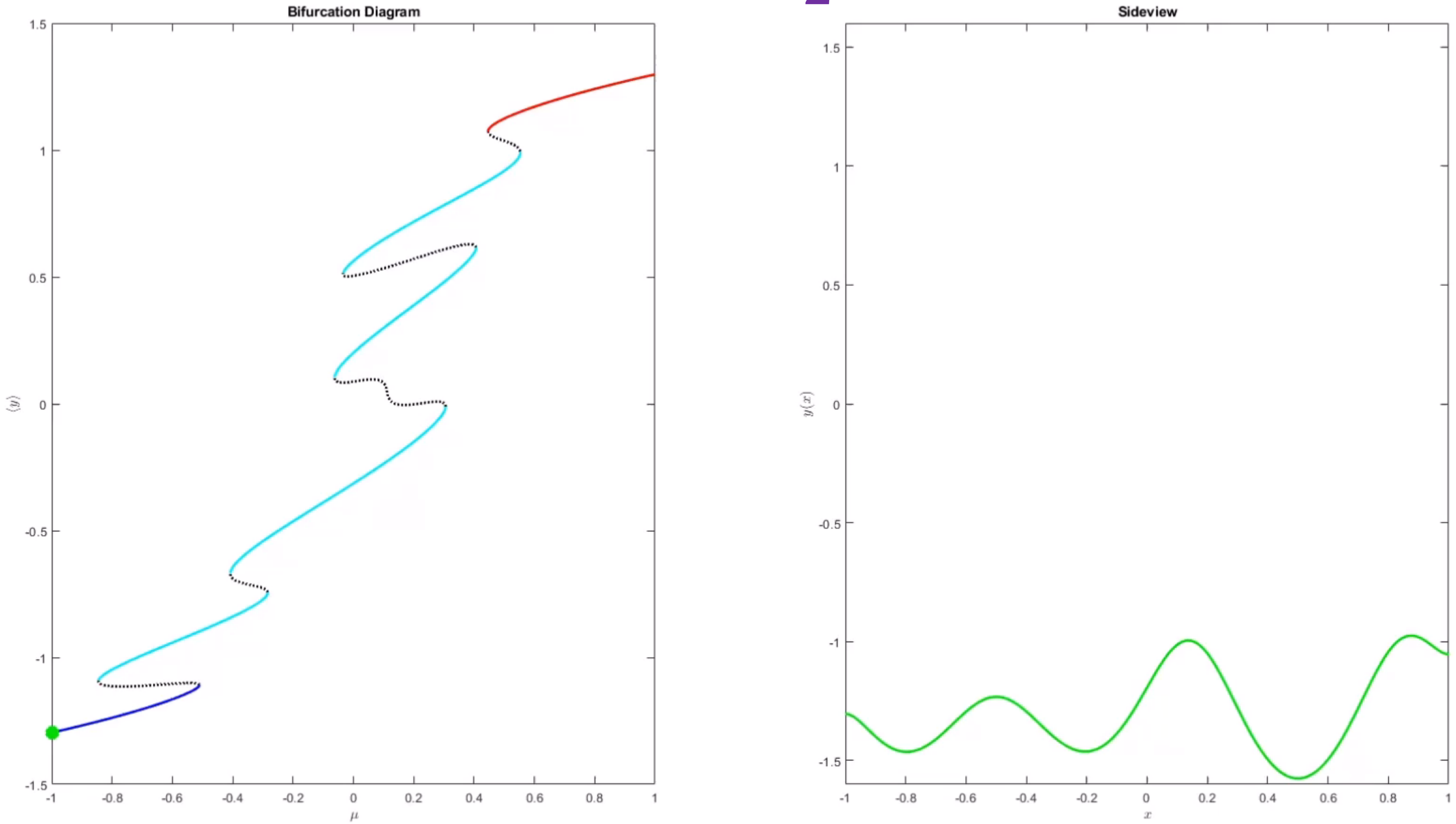
At bifurcation:
→ Structures created or destroyed

$$y_t = D\,y_{xx} + y(1 - y^2) + \mu + x + \frac{2}{5}\cos(5\pi x)$$

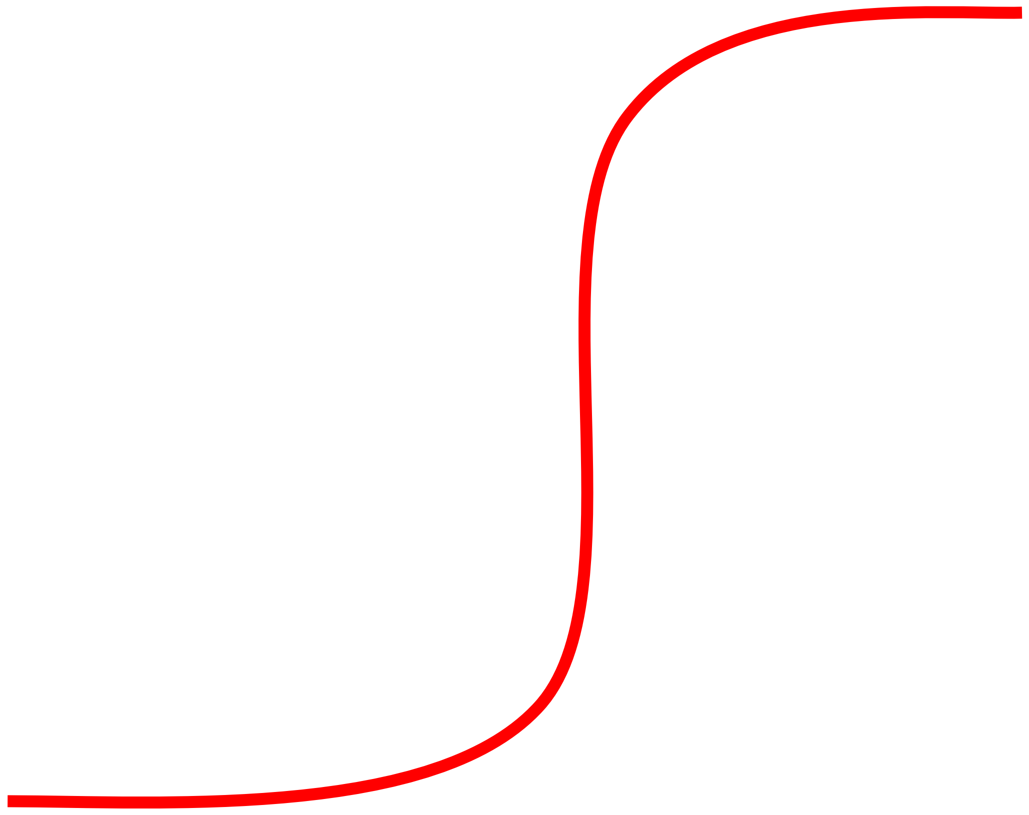




$$y_t = D\, y_{xx} + y(1 - y^2) + \mu + \frac{1}{2}\cos(2\pi x) + \sin(3\pi x)$$



Destabilizations in 2D



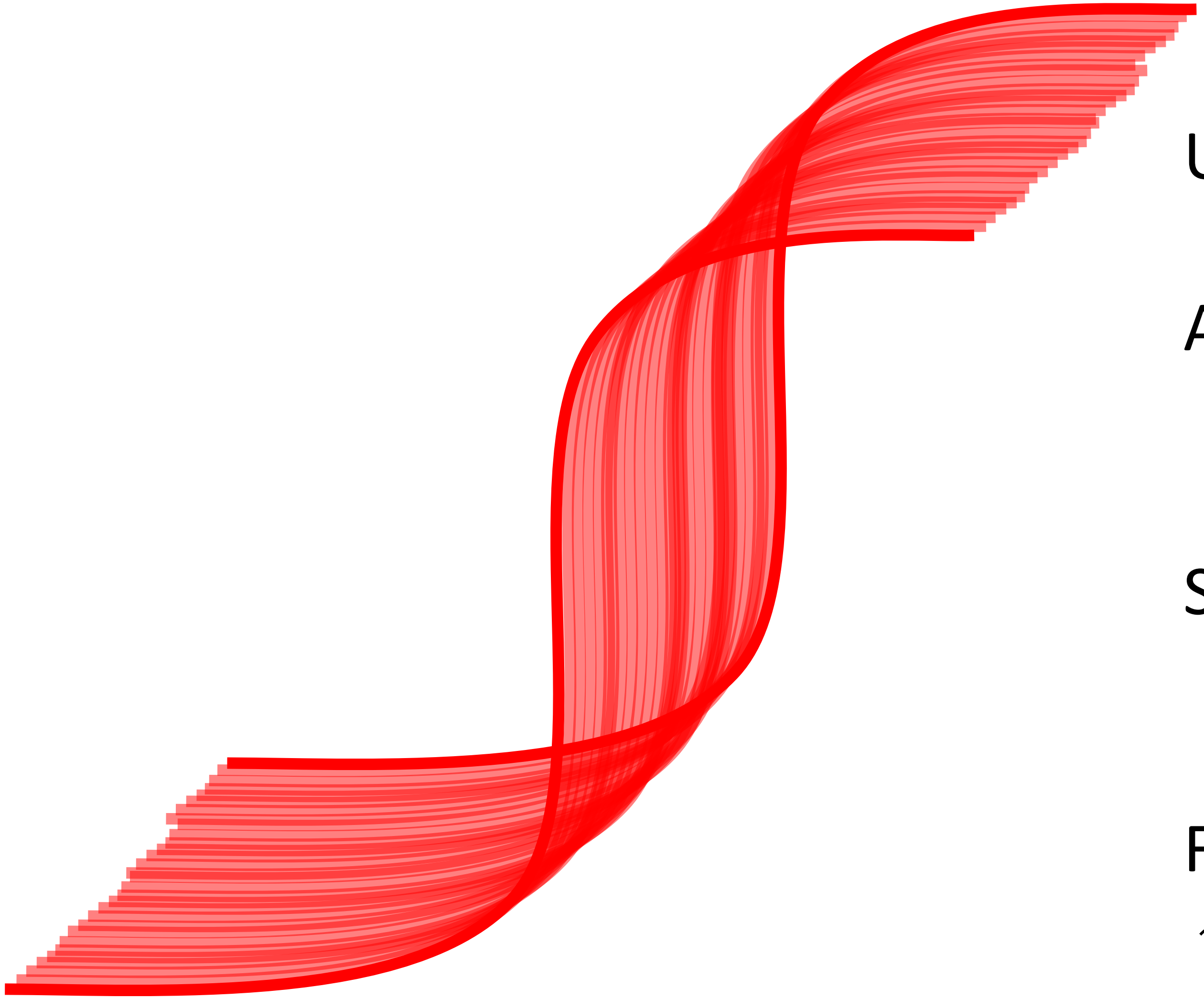
Until now

$$u_t = u_{xx} + F(u)$$

Adding another spatial dimension:

$$u_t = u_{xx} + u_{yy} + F(u)$$

Destabilizations in 2D



Until now

$$u_t = u_{xx} + F(u)$$

Adding another spatial dimension:

$$u_t = u_{xx} + u_{yy} + F(u)$$

Solution in 2D:

$$u^*(x,y) = u^*_{1D}(x)$$

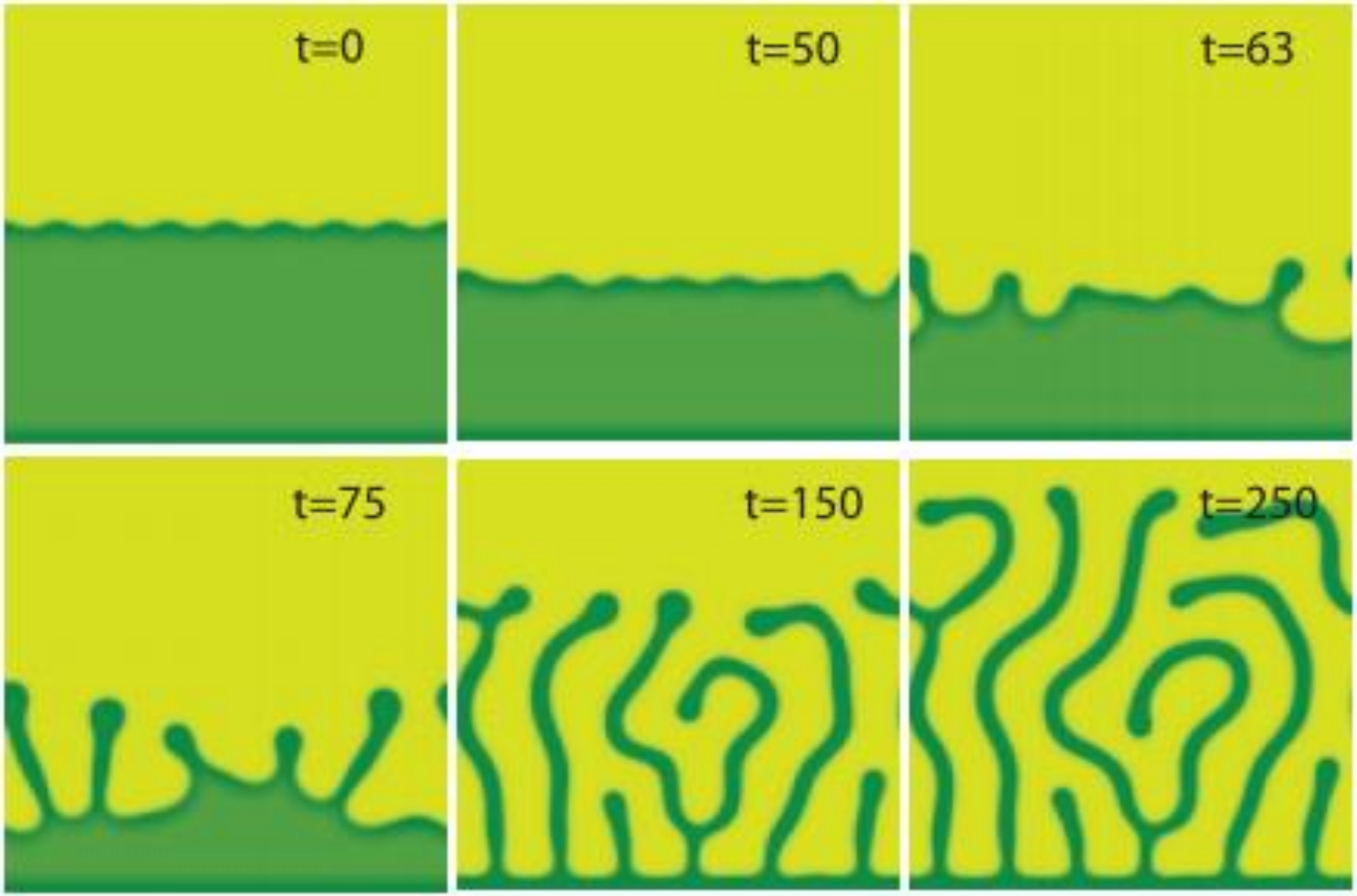
For stability analysis, use

$$u(x,y) = u^*_{1D}(x) + e^{iky} e^{\lambda t} \bar{u}(x)$$

to get:

$$\lambda \bar{u} = (\partial_x^2 - k^2)\bar{u} - F_u(u^*)\bar{u}$$

Fingering instability



Fernandez-Oto et al. *Front instabilities can reverse desertification*. PRL, 2019

Summary – Dynamics of existing patterns

Dynamics of patterned states is a combination of:

1. SLOW Pattern Adaptation

At bifurcation:
→ Location of structure changes

&

2. FAST Pattern Degradation

At bifurcation:
→ Structures created or destroyed

Slides available at:
bastiaansen.github.io/MTpatterns/patternMT.html