Spatial Patterns in Nature

An Entry-Level Introduction to Their Emergence and Dynamics

SIAM DS23,
Minitutorial MT1-2

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Minitutorial overview

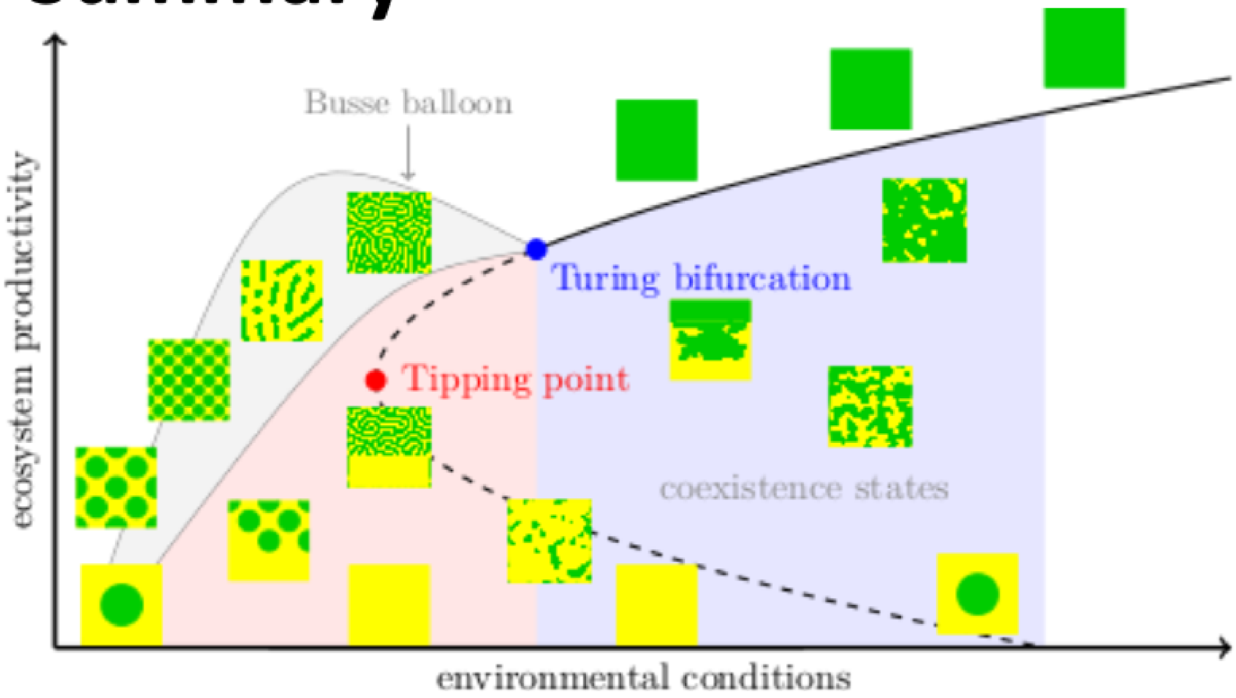
- ~~Introduction~~
- ~~Multistability and patterns~~
- ~~Explicit construction of front solutions~~
  - ~~Existence~~
  - ~~Stability~~
- ~~Dynamics of existing structures~~
- Summary & Outlook

Summary

<u>Spatial Patterns:</u>
- Turing Patterns
- Coexistence States

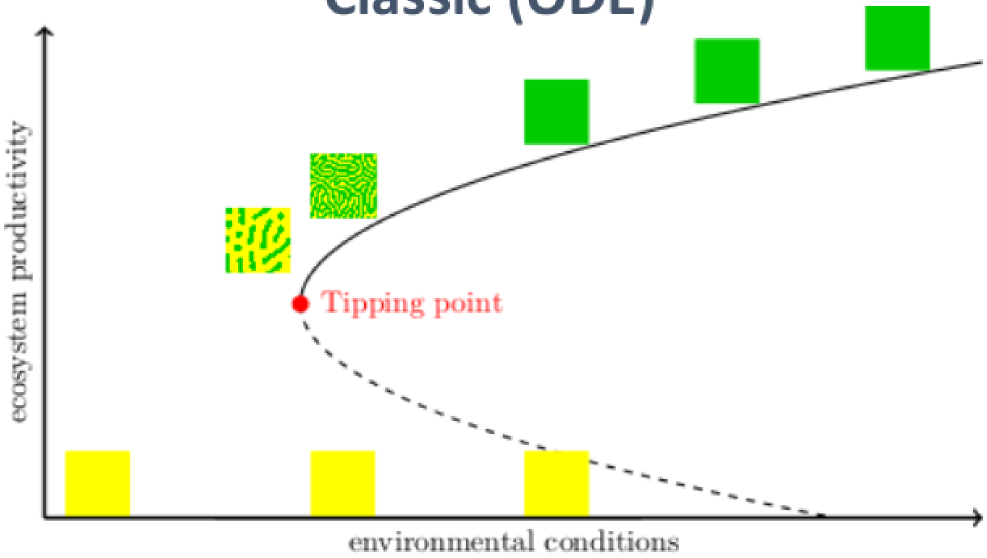
<u>Tipping can be more subtle:</u>
- Spatial reorganization
- Fragmented Tipping

<u>Dynamics of Patterns is:</u>
- Slow Pattern Adaptation
- Fast Pattern Degradation



What if the system tips?

Classic (ODE)



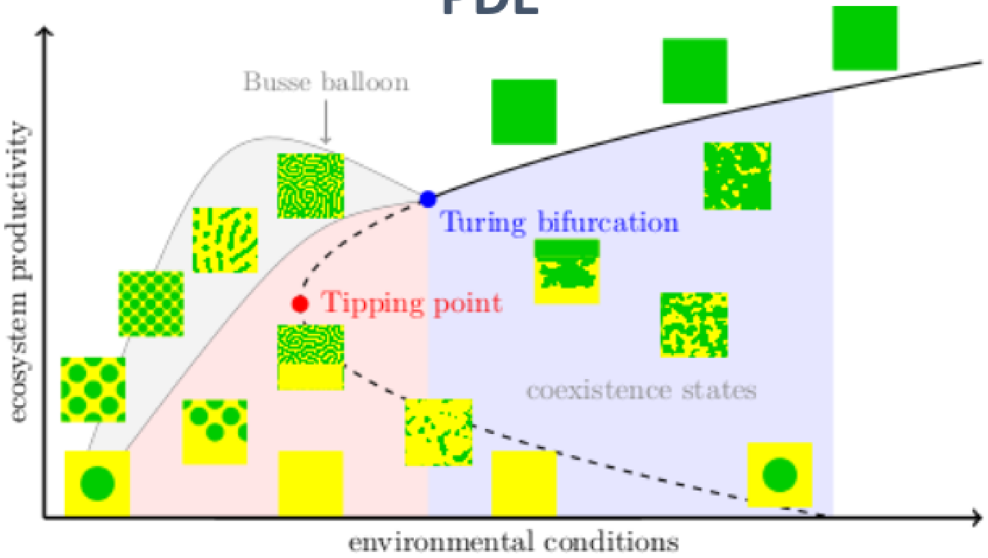
Crossing a Tipping Point:

→ Always full reorganization

Early Warning Signals

signal for <u>WHEN</u>

PDE



Crossing a bifurcation:

Now also possible:

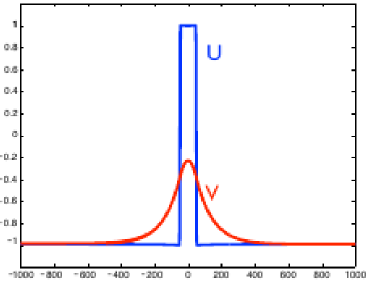
→ Spatial reorganization (Turing patterns)

→ Fragmented tipping (coexistence states)

Early Warning Signals

need to signal <u>WHEN</u> & <u>WHAT</u>

Existence of stationary pulse



$$U_t = \varepsilon^2 U_{xx} + U - U^3 - \varepsilon(\alpha V + \gamma)$$
$$\tau V_t = V_{xx} + U - V$$

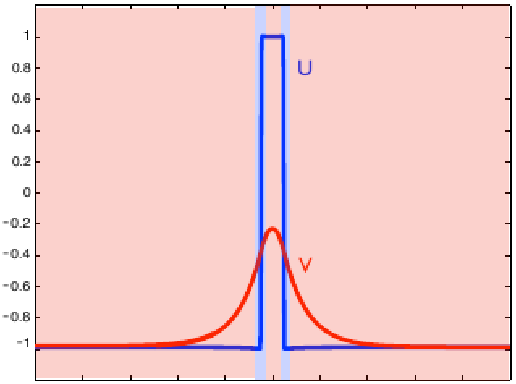
where $(x,t) \in \mathbb{R} \times \mathbb{R}_+; 0 < \varepsilon \ll 1; 0 < \tau, \alpha, \gamma \in \mathbb{R}$

Observations:

- stationary: $\partial_t = 0$

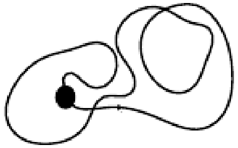
$$0 = \varepsilon^2 u_{xx} + u - u^3 - \varepsilon(\alpha v + \gamma)$$
$$0 = v_{xx} + u - v$$

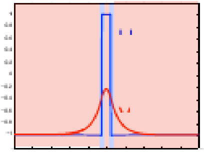
- five distinct spatial regions (due to smallness of ε)



large, U constant (±1), V changing (*outer*/*slow region*)

small, U changing, V constant (inner/fast region)





4D phase portrait

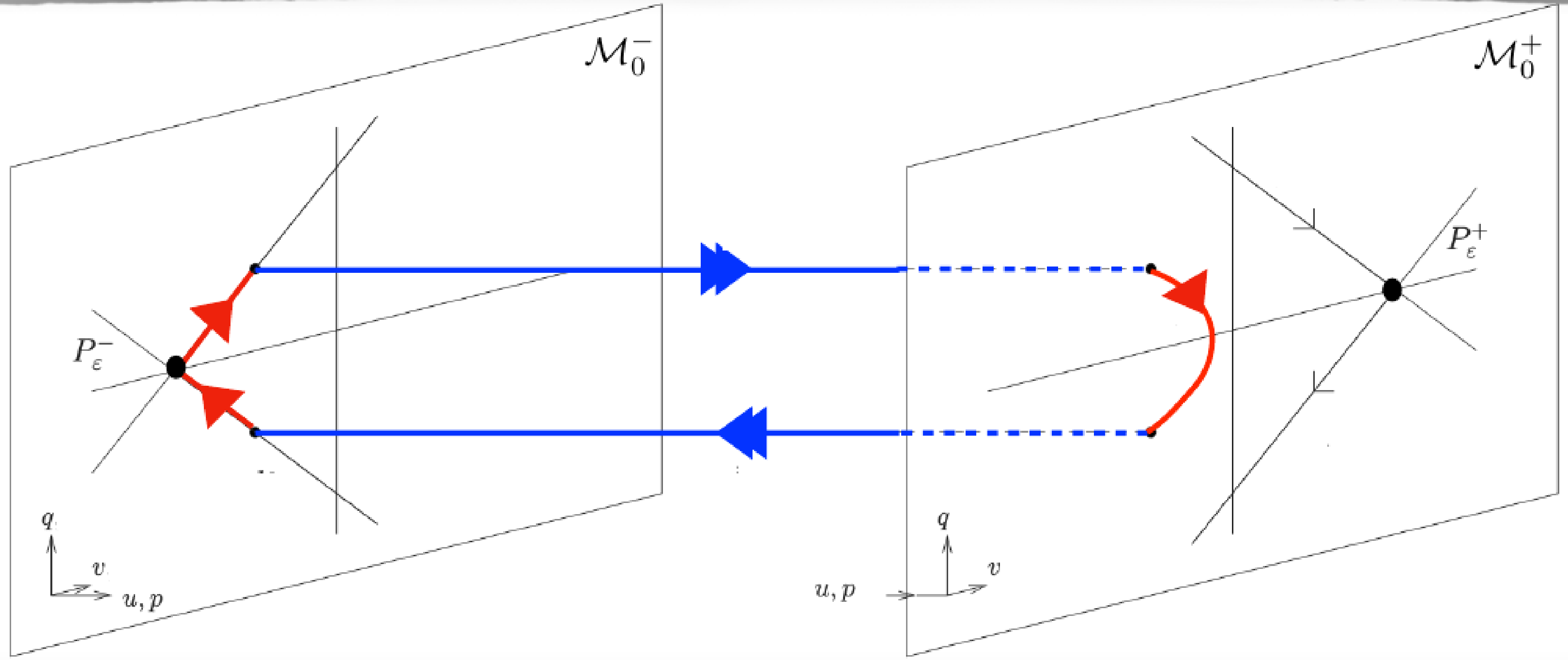
The dynamics of slow reduced system is constraint to fixed points of the fast reduced system.

Critical Manifold

$$\mathcal{M}_0 := \{(u,p,v,q) \in \mathbb{R}^4 : p = 0,\ 0 = u - u^3\}$$

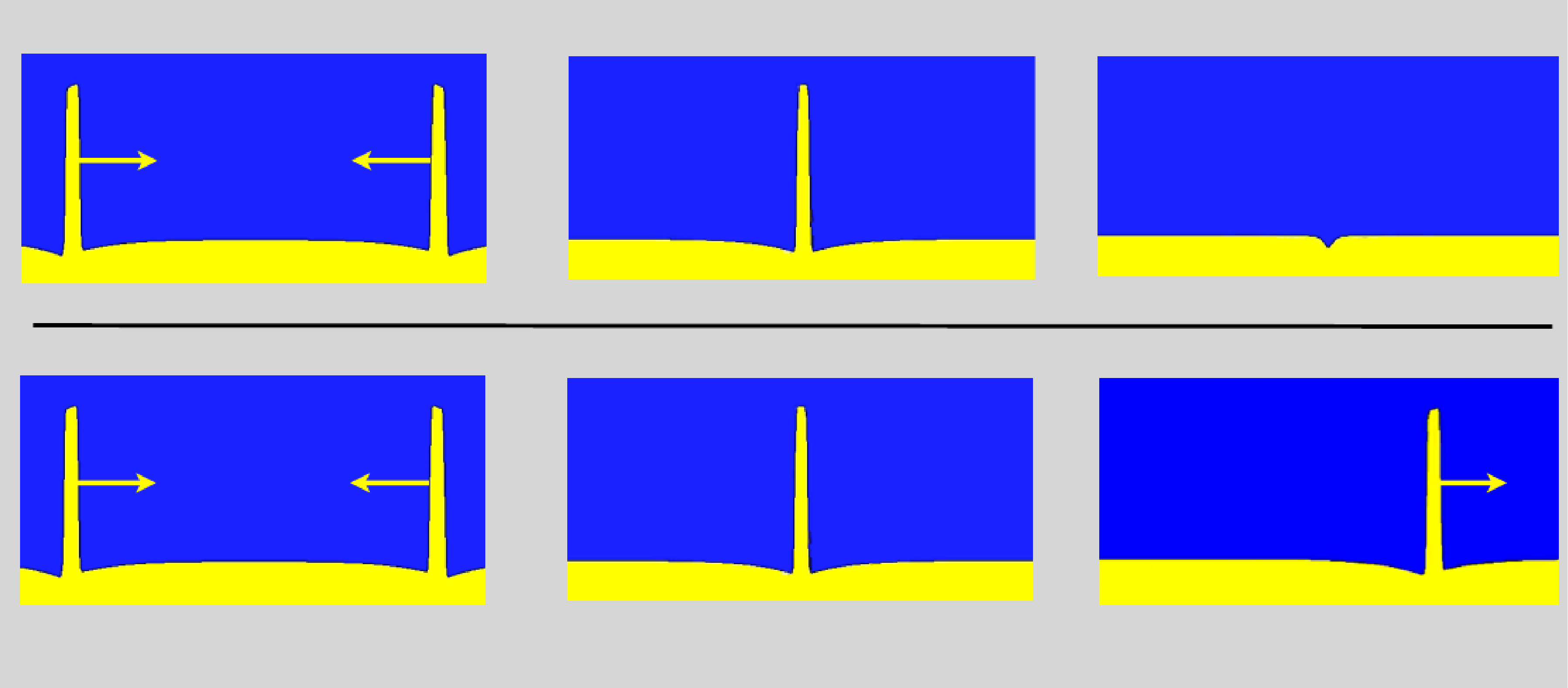
$u = 1$ $u = -1$ $u = 0$

$$\mathcal{M}_0 = \mathcal{M}_0^+ \cup \mathcal{M}_0^- \cup \mathcal{M}_0^0$$

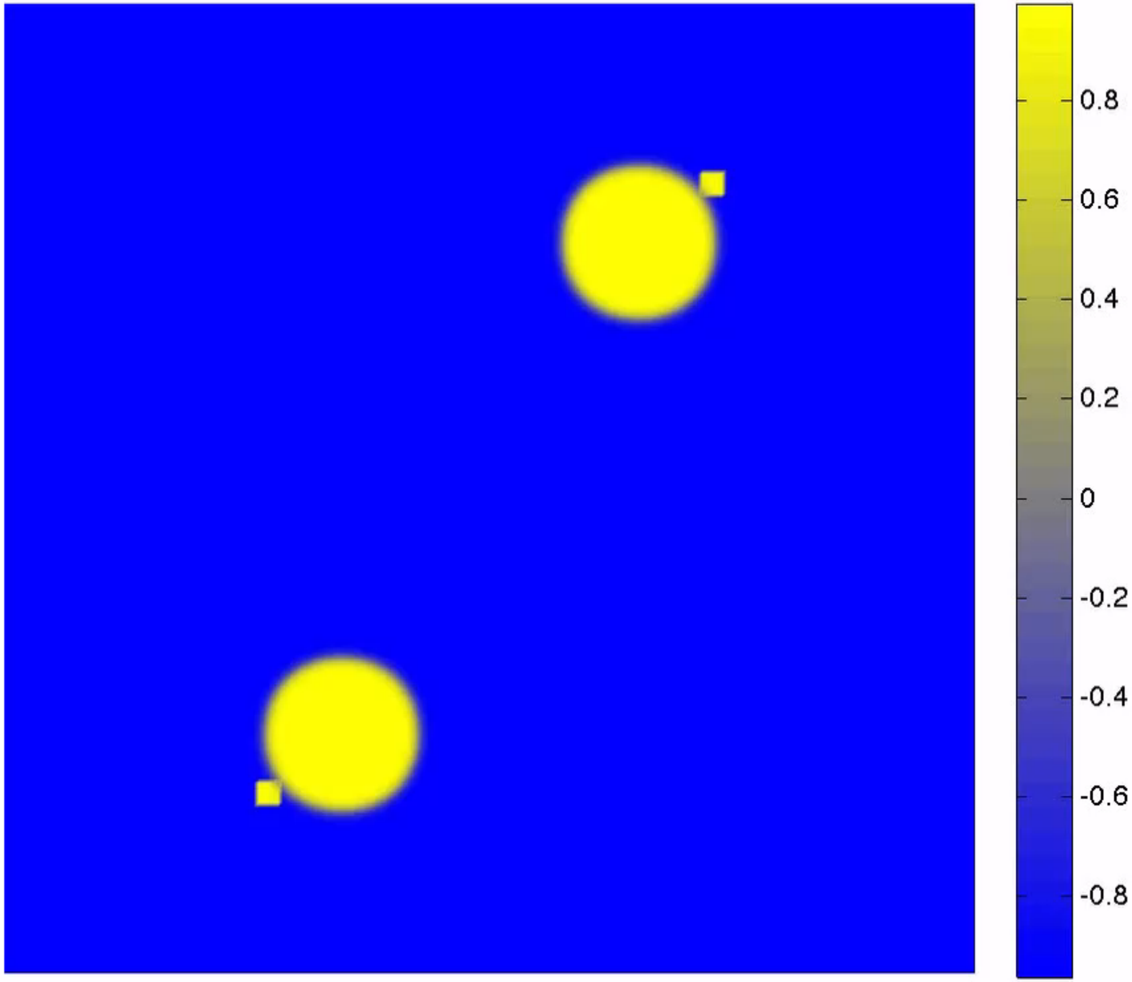


Outlook

Strong interaction



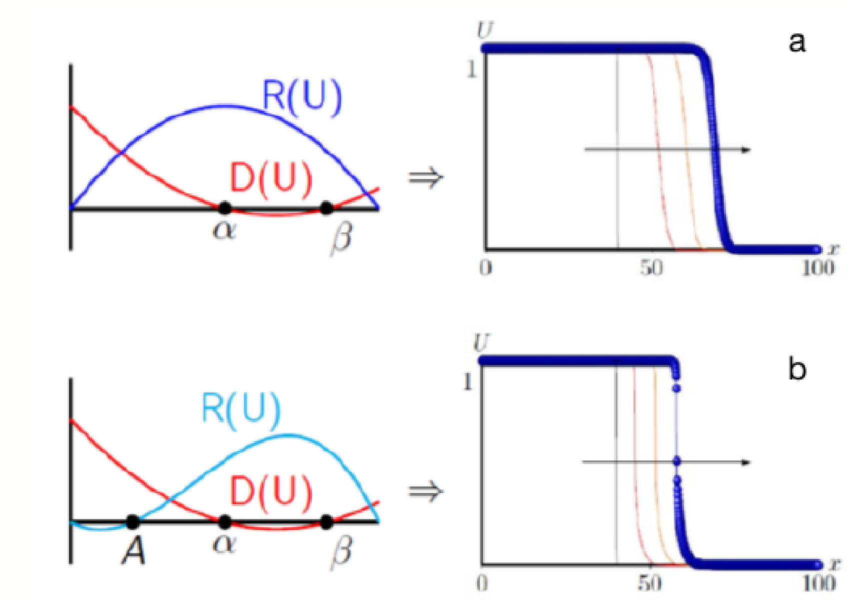
Slightly different parameters (*change in the 6th digit of the parameter*)… gives completely different dynamics

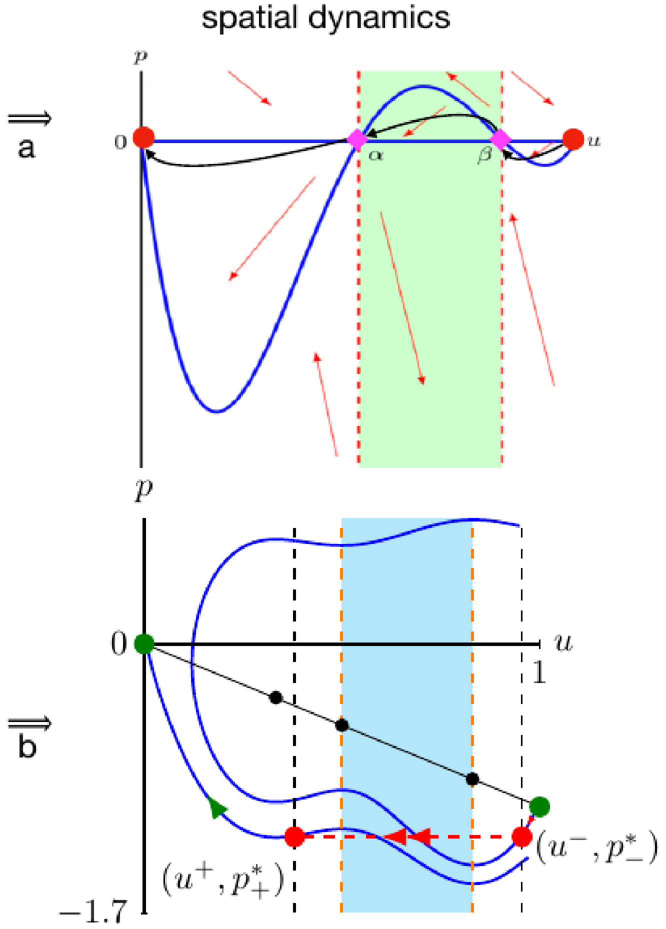

0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8

Nonlinear & backward diffusion

Macroscopic limit of a discrete model that models the difference in collective vs individual behaviour considering proliferation, death and motility/movement events of agents (cells) on a simple one-dimensional lattice [Johnston et al., 2017, Li et al, 2020, 2021]

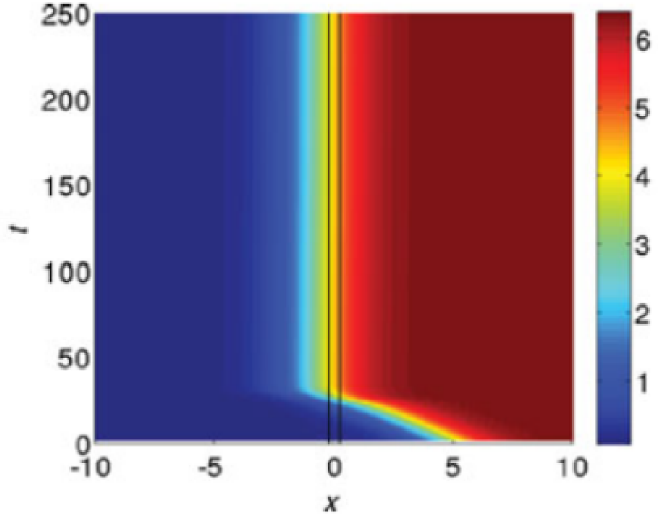
$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\left(D(U)\frac{\partial U}{\partial x}\right) + R(U)$$

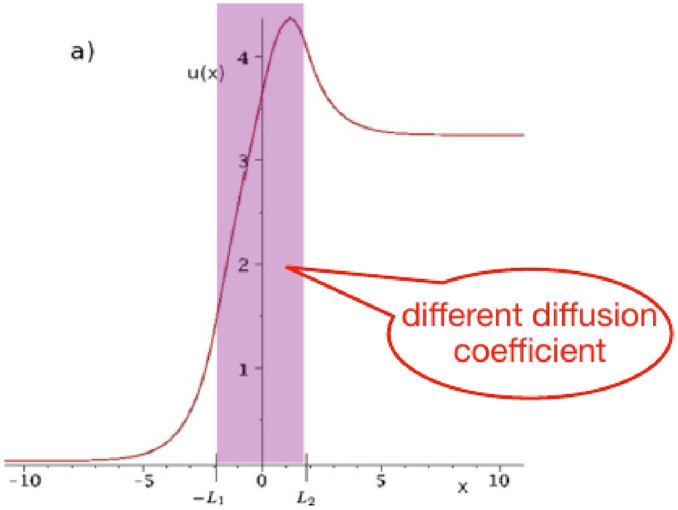




Heterogeneous media

Defects, even small, can for instance, pin travelling waves in nonlinear wave equations

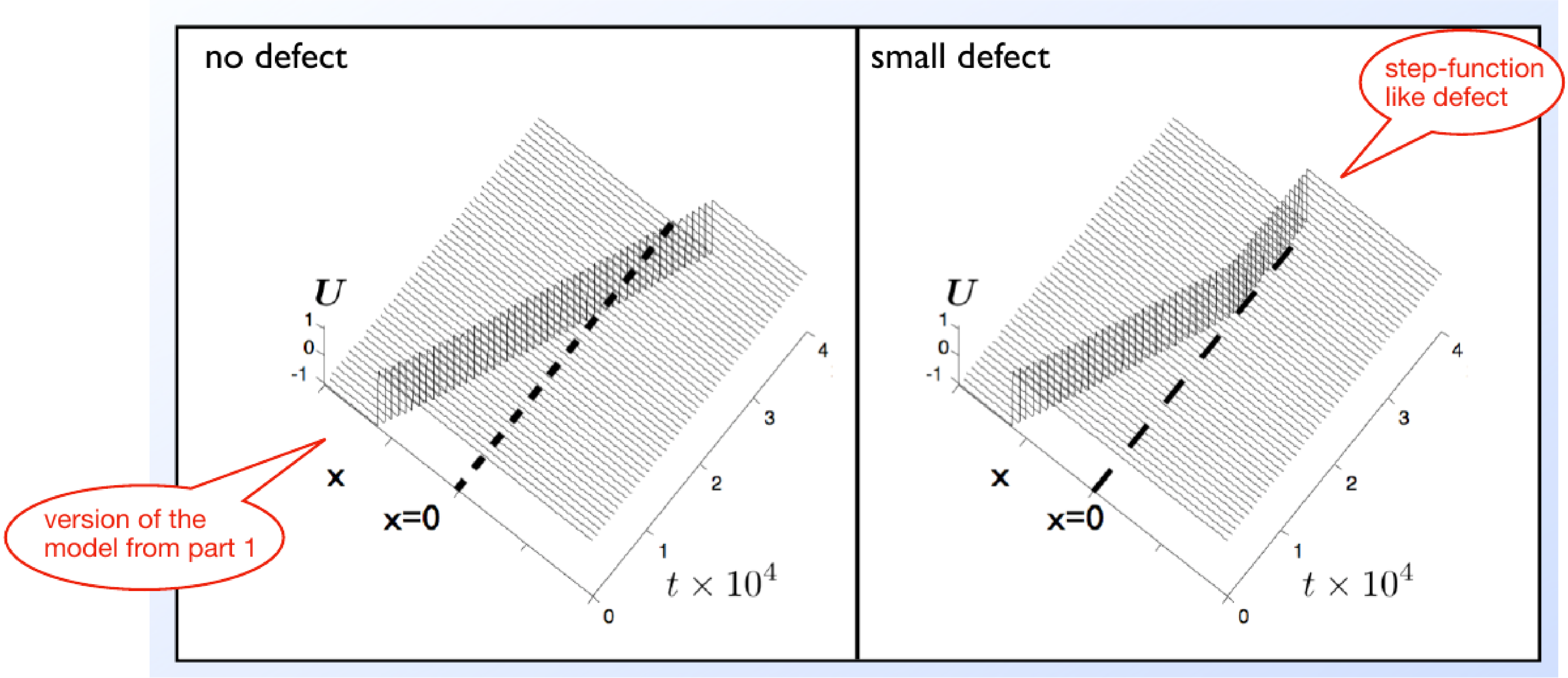




[Derks, Doelman, Knight, Susanto, 2012, 2013]

Heterogeneous media

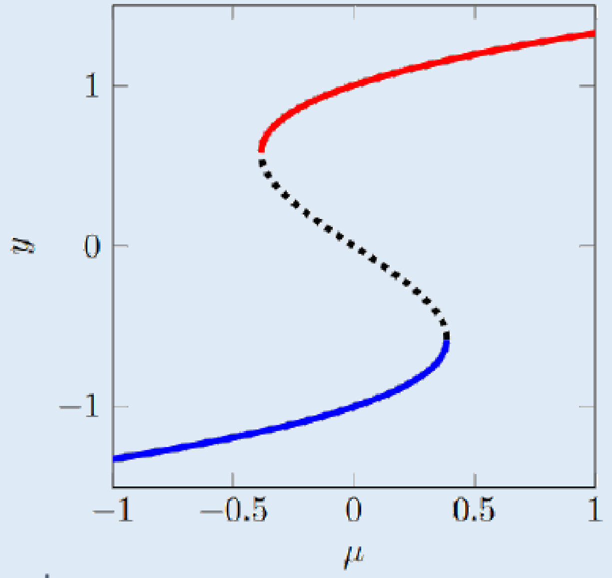
- … and in reaction-diffusion equations



[PvH, Doelman, Kaper, Nishiura,Ueda, 2011]

A spatially heterogeneous world

Classic Tipping



Example: $$\frac{dy}{dt} = y(1-y^2) + \mu$$

Tipping in Spatially Heterogeneous Systems

Spatial Transport

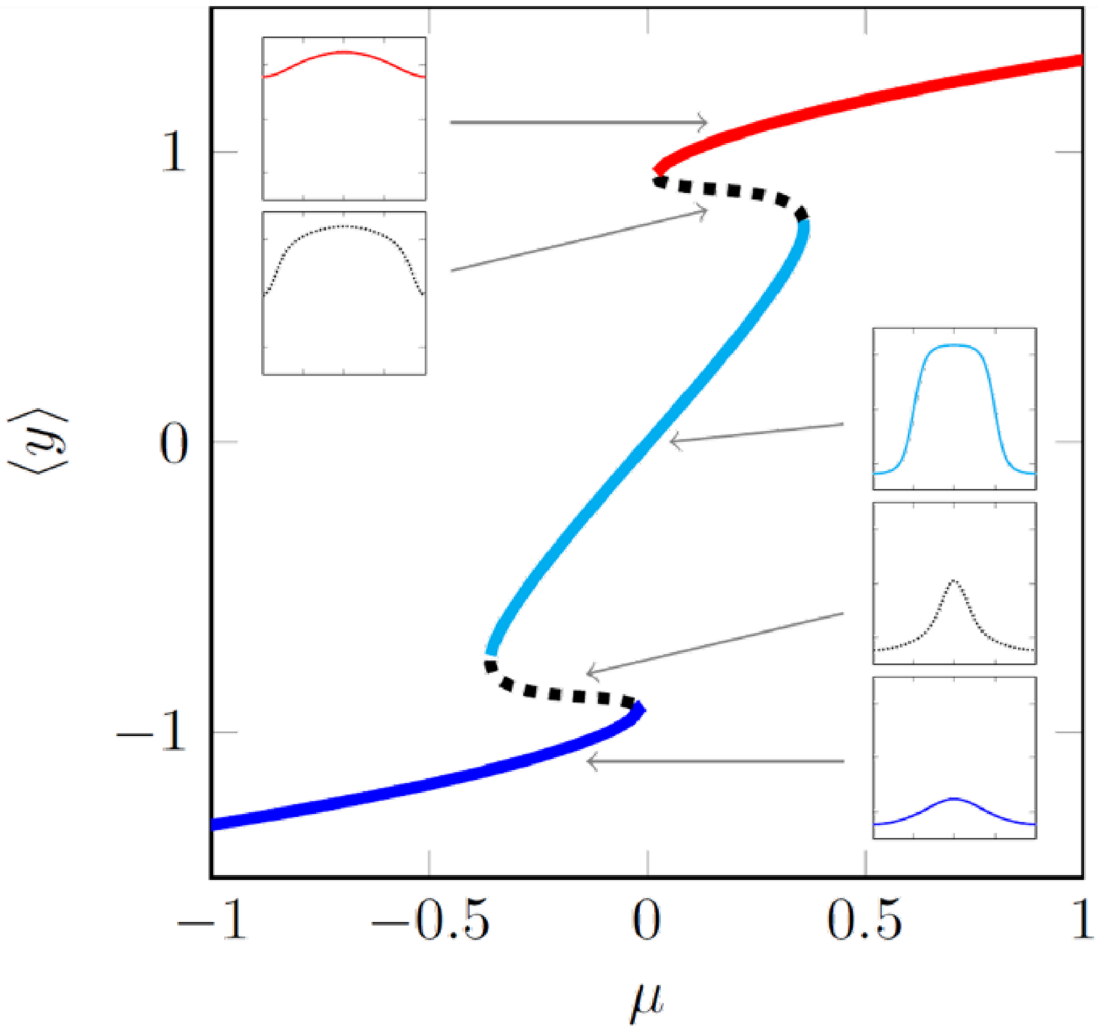
Spatial Variation in Environmental Conditions

Example: $$\frac{\partial y}{\partial t} = D\frac{\partial^2 y}{\partial x^2} + y(1-y^2) + \mu + \frac{1}{2}\cos(\pi x)$$

Adding Spatial Heterogeneity

$$\frac{\partial y}{\partial t} = D\frac{\partial^2 y}{\partial x^2} + f(y, x; \mu)$$

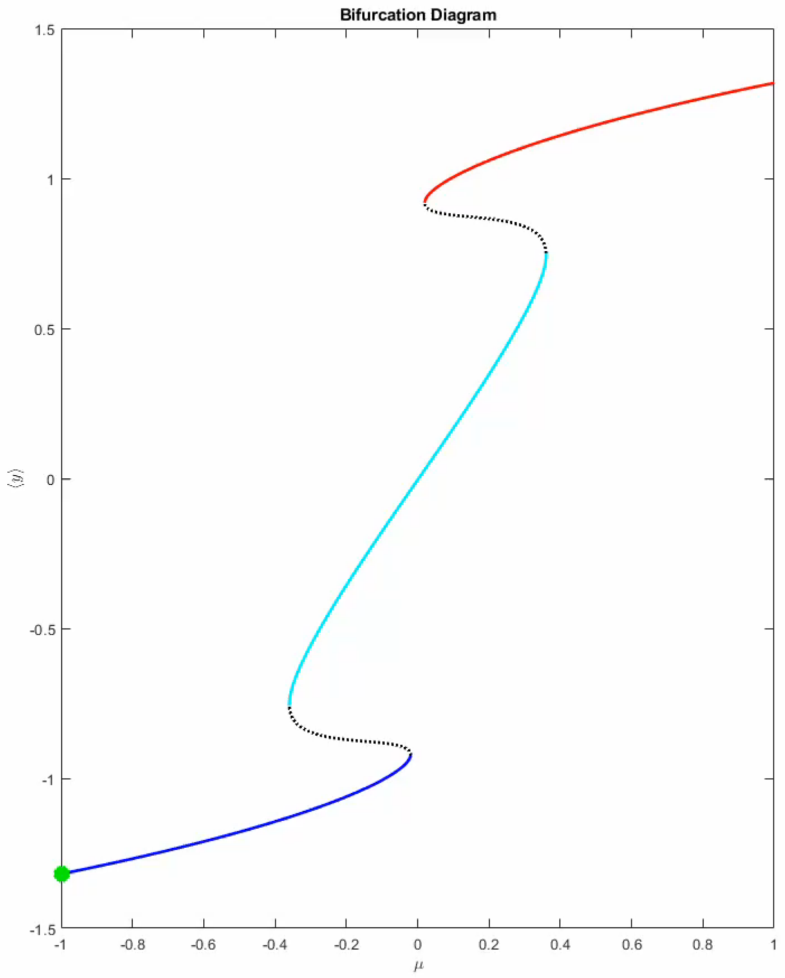
Now, the **local** difference in potentials determines the front movement

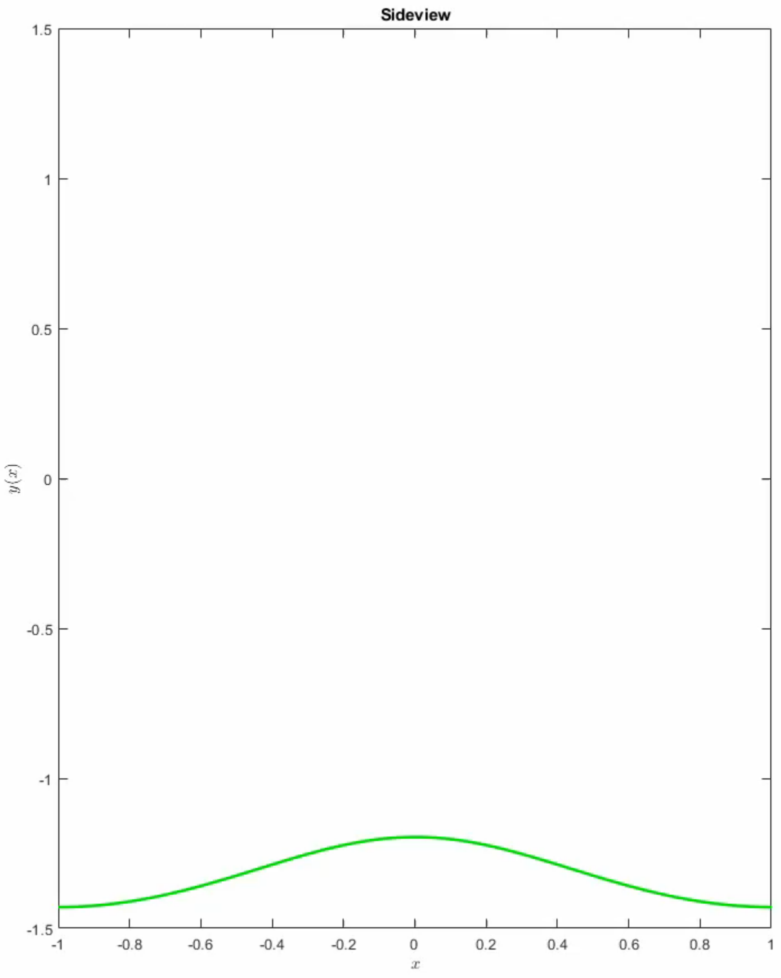


New behaviour:

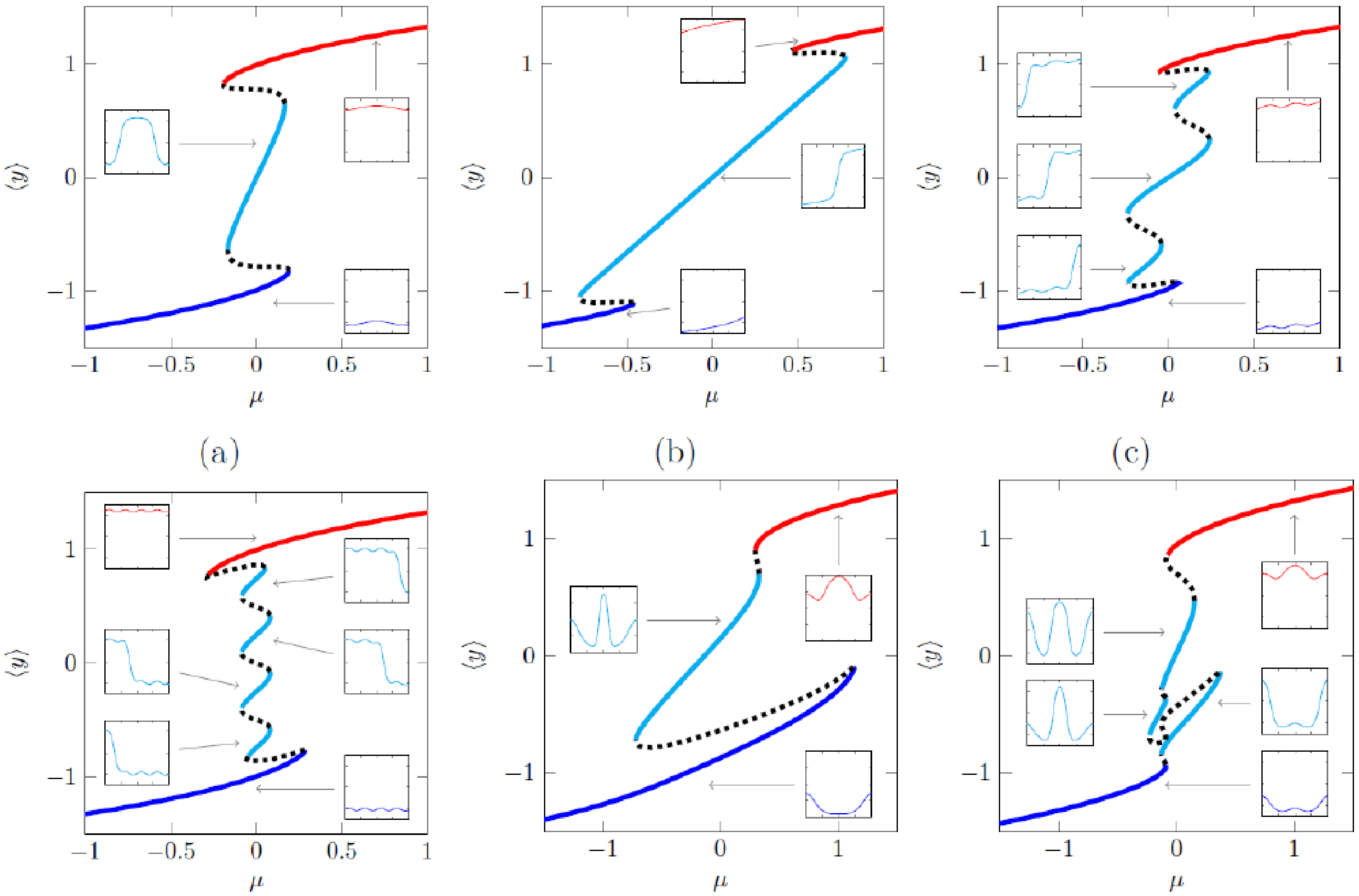
- Multi-fronts can be stationary
- Maxwell point is smeared out

Fragmented Tipping



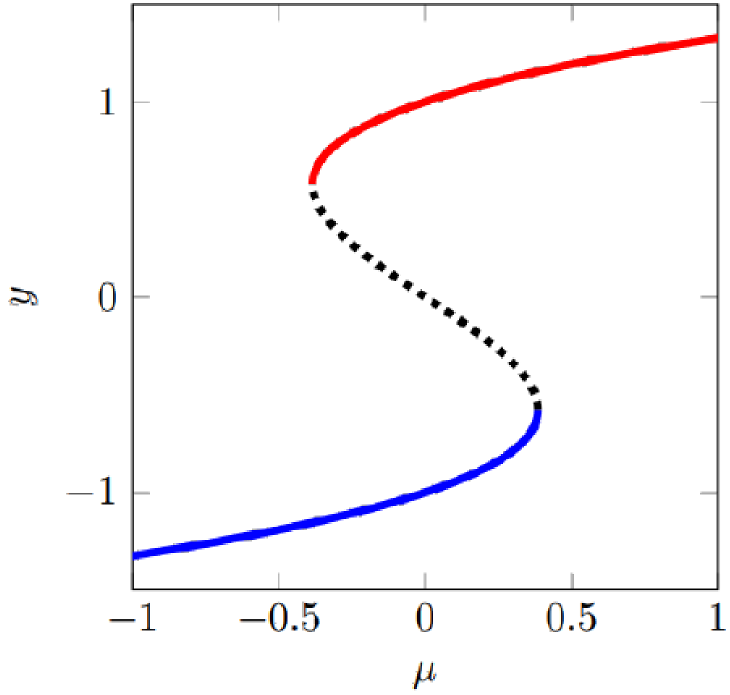


Other Spatial Heterogeneities



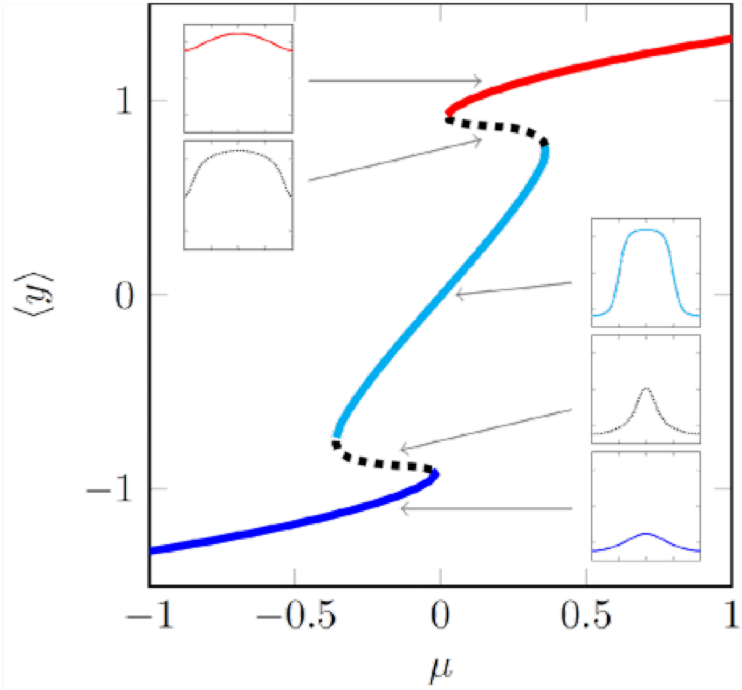
Fragmented Tipping

Classic tipping



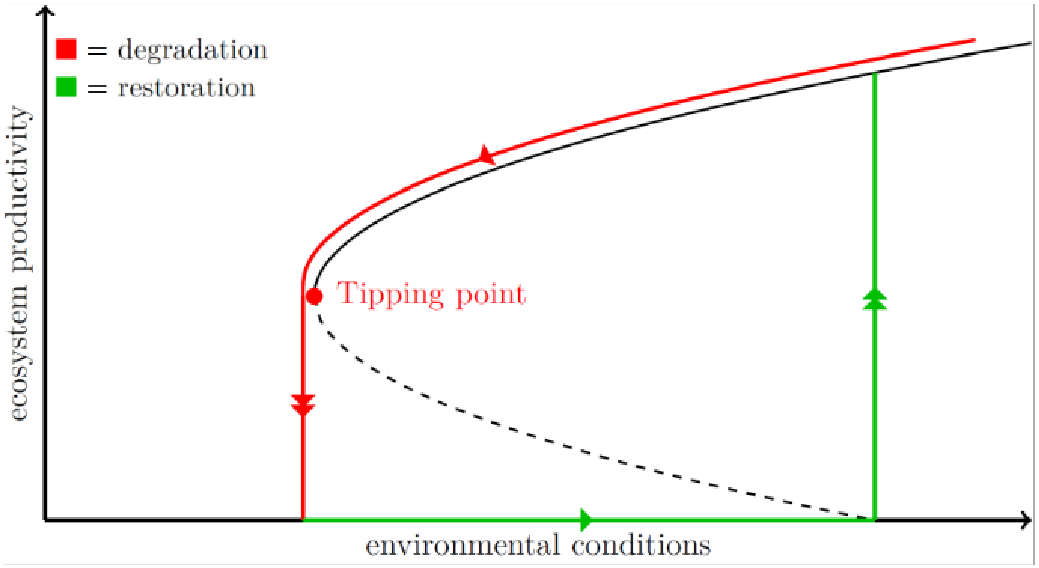
Tipping leads to full reorganisation

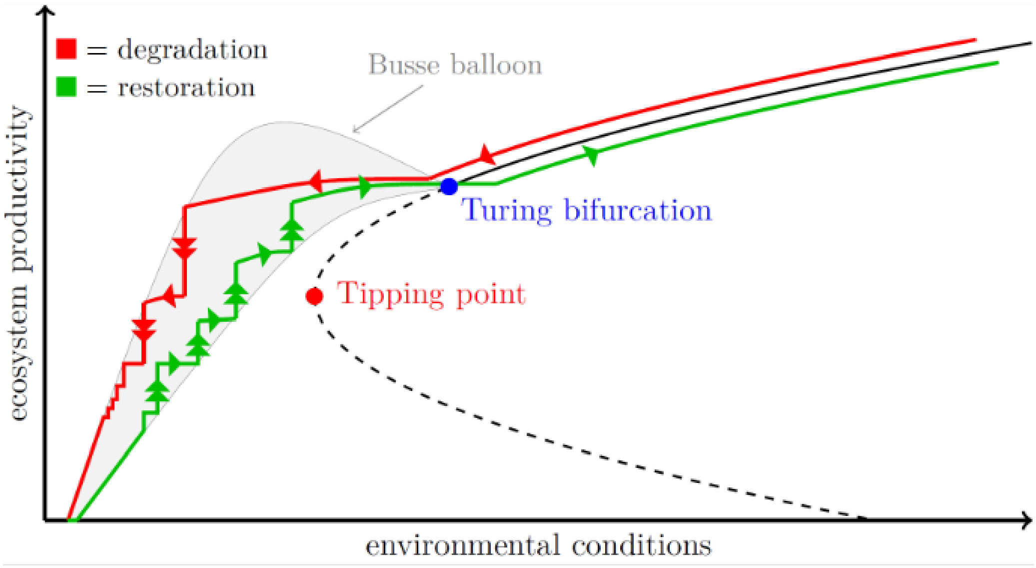
Tipping in a heterogeneous world



Fragmented tipping possible:
Only part of the domain reorganises

Tipping of (Turing) patterns

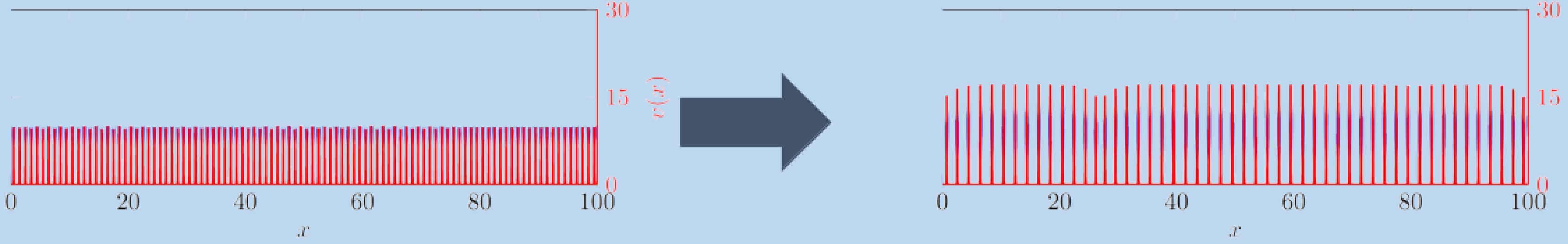




Classic tipping

Tipping of patterns

Degradation of patterns



More questions

- What determines the shape of the Busse balloon?
- How are transient patterns selected? (path through Busse balloon)
- What is the influence of spatial heterogeneities? (terrain features, roads, etc.)
- What can happen in 2D? Is this a fundamentally different context?
- Is a PDE always the best model to describe patterns? When is an `effective' ODE pulse/front interaction model more useful?
- ...

Interesting minisymposia:

- MS 29, *Tipping Points in Natural Systems*
- MS 32 & 47, *Patterns in Nonlinear PDEs*
- MS 36 & 50, *Singular Perturbation Methods for Multi-Scale Infinite-Dimensional Systems*
- MS 62 & 75, *Pattern Formation in Nature: from Busse Balloons to Homoclinic Snaking*
- MS 73 & 86, *Dynamical Systems Methods in Climate Modeling*
- MS 88 & 103, *Branching Out: a New Generation's Perspective on Spatial Localisation in Higher Dimensions*
- MS 99 & 100, *Rate-Induced Tipping*
- MS 114 & 128, *Patterns in Earth's Climate System*
- MS 130 & 144, *Modeling and Data-Driven Methods for Collective Behaviour and Pattern Formation*
- MS 138, 158 & 173, *Front Propagaion and Invasion Phenomena*