Pulse Solutions in an extended-Klausmeier model with spatially varying coefficients

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The extended-Klausmeier model

$$U_t = U_{xx} + (H_x U)_x + a - U - UV^2$$
$$V_t = D^2 V_{xx} - mV + UV^2$$

Variables:

U Water

V Vegetation

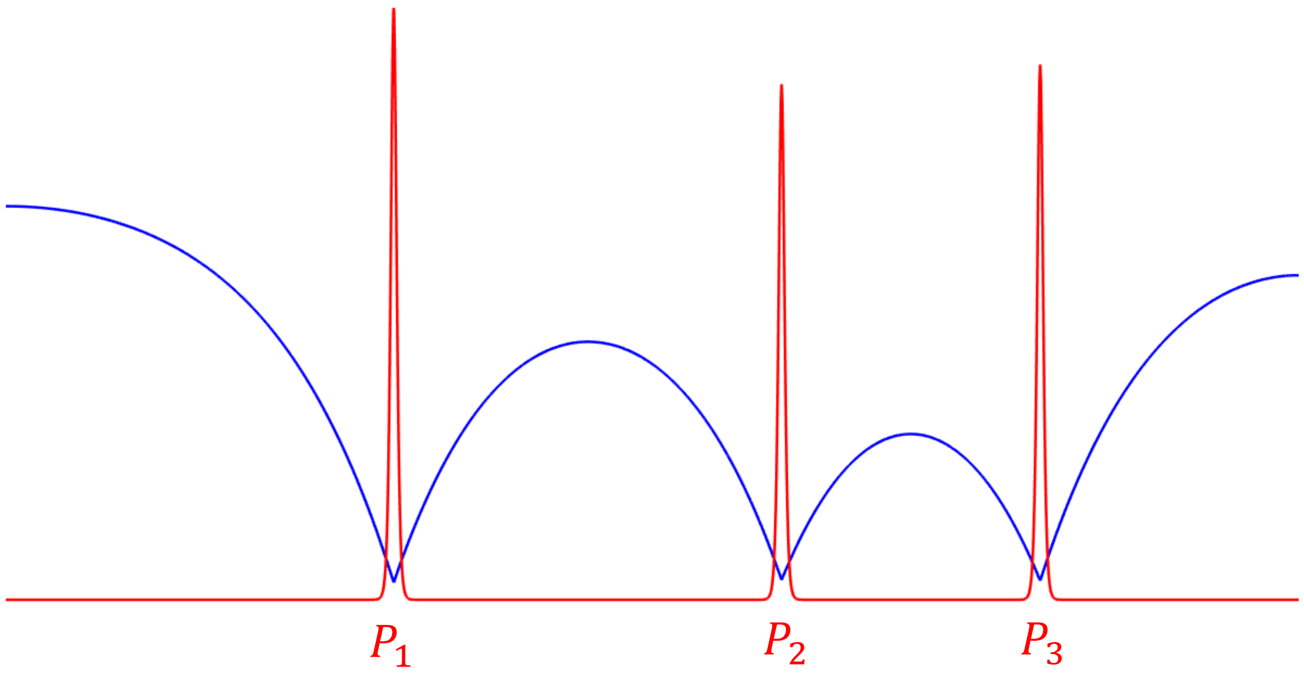
Parameters:

a Rainfall

m Mortality of plants

D Small parameter

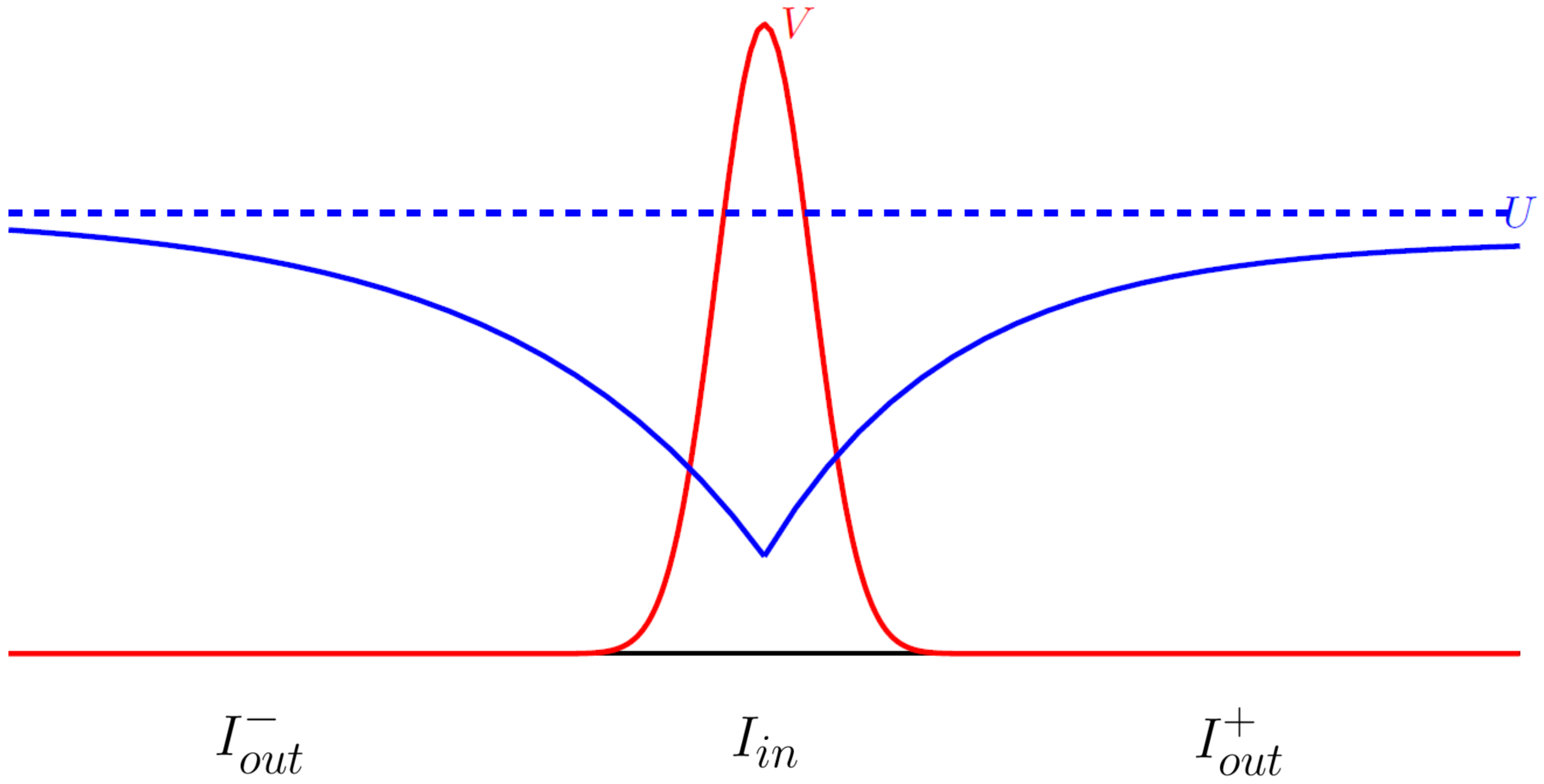
H Height of terrain



Flat terrain ($H(x) = 0$)

$$U_t = U_{xx} \qquad + a - U - UV^2$$

$$V_t = D^2 V_{xx} \qquad - mV + UV^2$$



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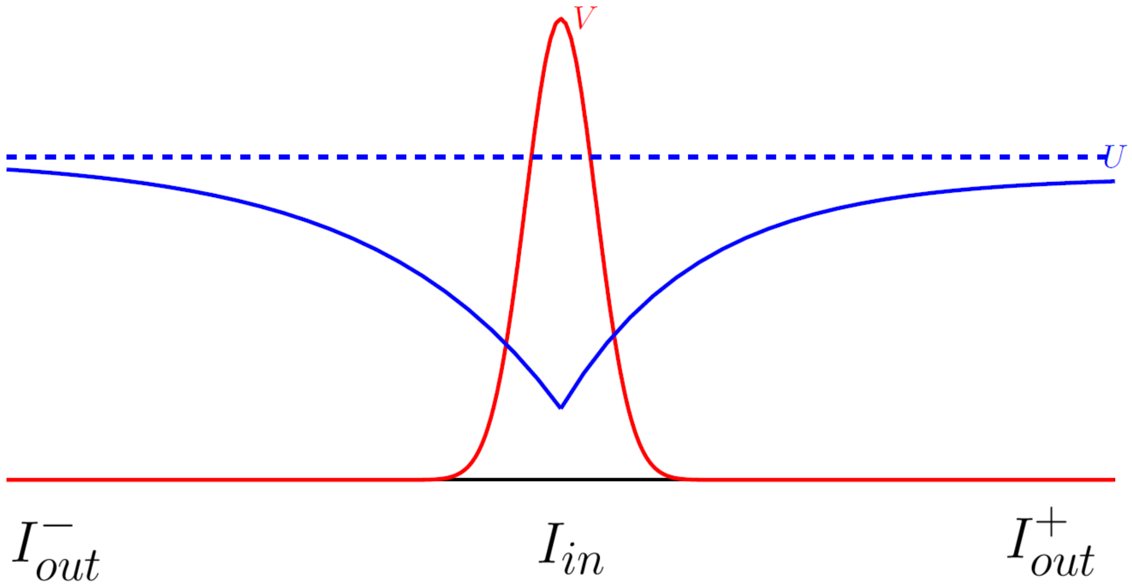
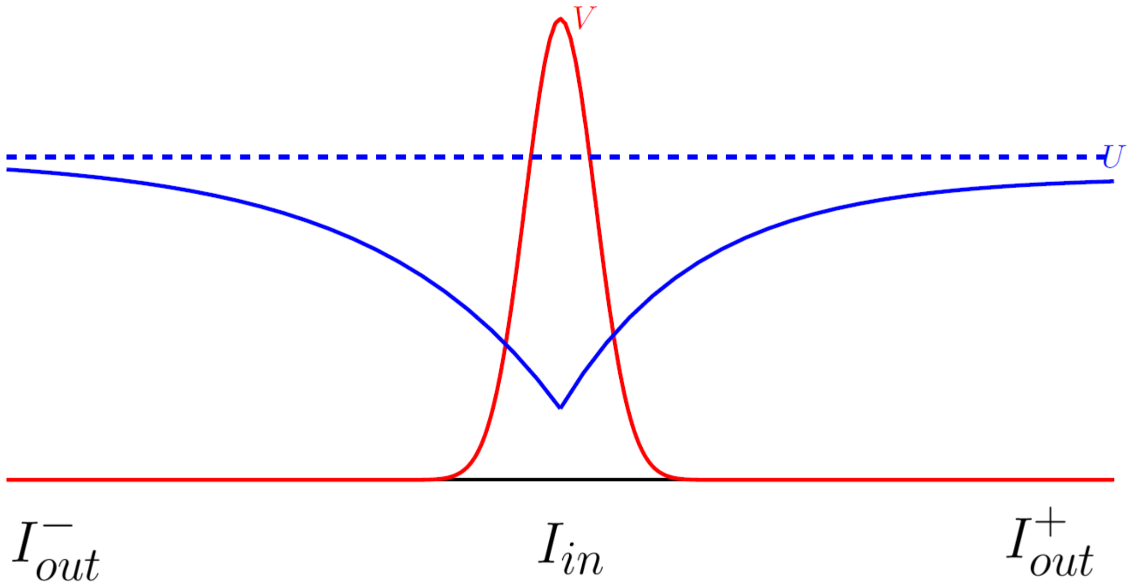


I_{in}

$$V = \frac{a}{\sqrt{m}D}\frac{3}{2}\,\mathrm{sech}\left(\frac{\sqrt{m}x}{D}\frac{x}{2}\right)^2$$

$$U = \frac{m\sqrt{m}D}{a}u_0$$

Flat terrain ($H(x) = 0$)



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$$U = \frac{m\sqrt{m}D}{a}u_0$$

I_{out}^{-} I_{out}^{+}

$$V = 0$$

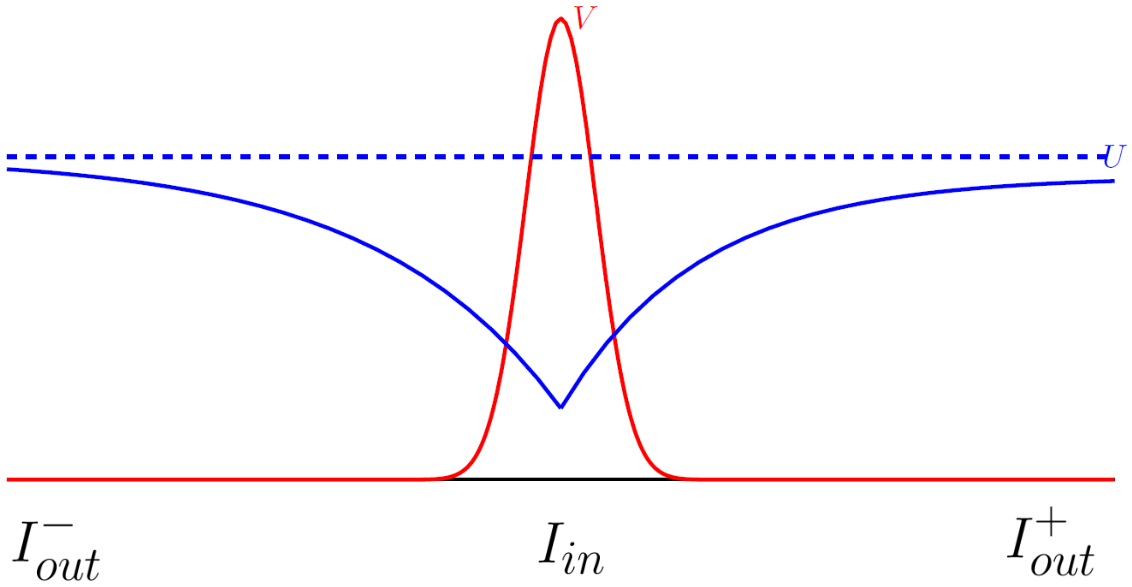
$$U = a\tilde{U}$$

$$\tilde{U}_{xx} - \tilde{U} + 1 = 0$$

$$\tilde{U}(0) = \frac{m\sqrt{m}D}{a^2}u_0$$

$$\Delta\tilde{U}_x(0) = \frac{6}{u_0}$$

Flat terrain ($H(x) = 0$)



I_{in}

$$V = \frac{a}{\sqrt{m}D}\frac{3}{2}\operatorname{sech}\left(\frac{\sqrt{m}x}{D}\frac{x}{2}\right)^2$$

$$U = \frac{m\sqrt{mD}}{a}u_0$$

I_{out}^- I_{out}^+

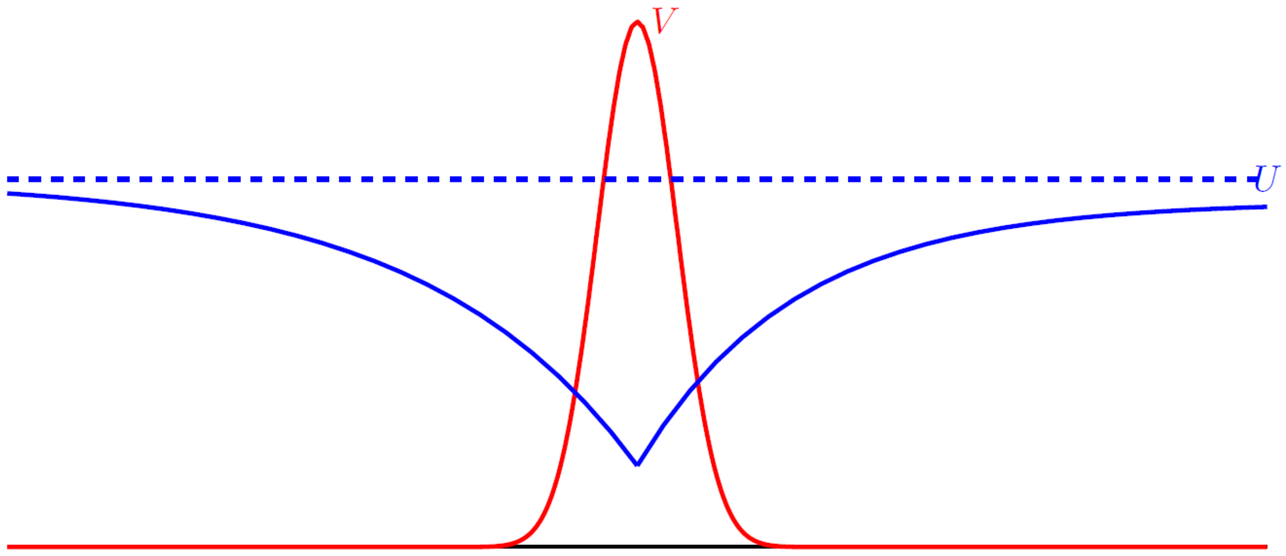
$$V = 0$$

$$U = a\tilde{U} \qquad \tilde{U}(x) = 1 + C^- e^x \quad I_{out}^-$$

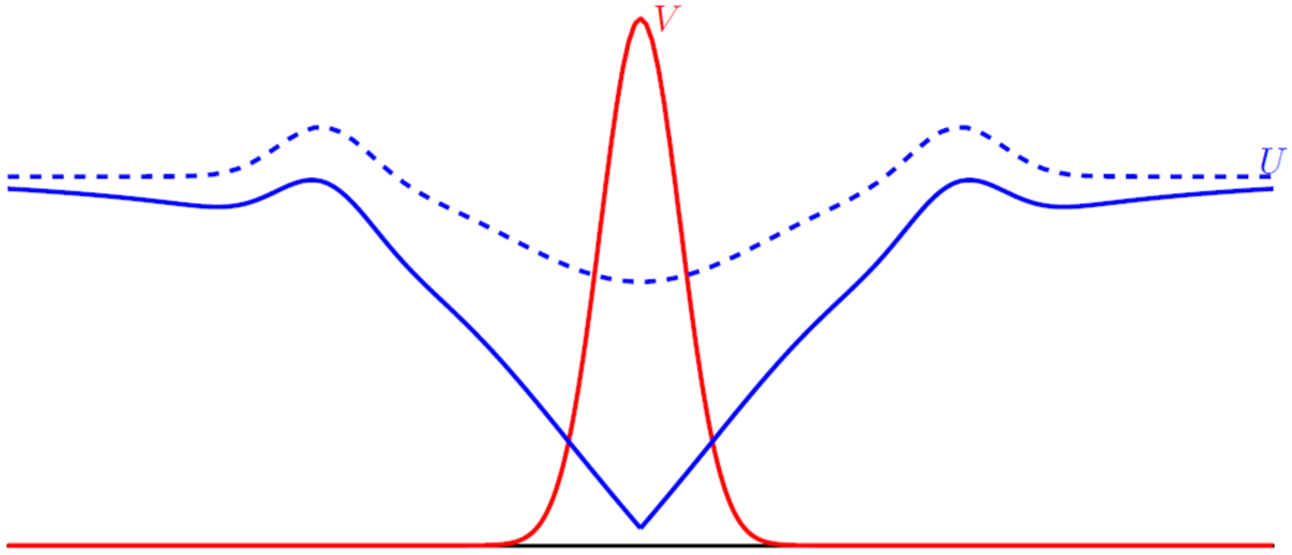
$$\tilde{U}(x) = 1 + C^+ e^{-x} \quad I_{out}^+$$

The effect of adding terrain

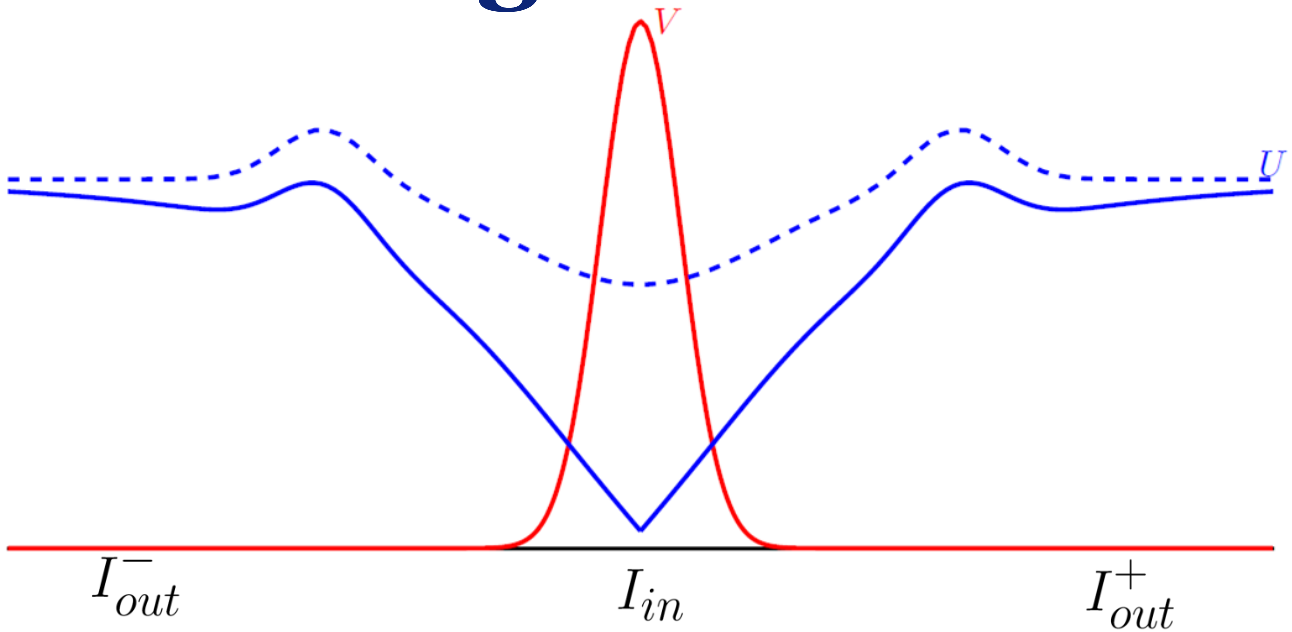
$H(x) = 0$



Generic $H(x)$



The effect of adding terrain



I_{in} I_{out}^- I_{out}^+

$$V = \frac{a}{\sqrt{m}D}\frac{3}{2}\operatorname{sech}\left(\frac{\sqrt{m}x}{D}\frac{x}{2}\right)^2$$

$$U = \frac{m\sqrt{m}D}{a}u_0$$

$$V = 0$$

$$U = a\tilde{U}$$

$$\tilde{U}_{xx} + H_x\tilde{U}_x + H_{xx}\tilde{U} - \tilde{U} + 1 = 0$$

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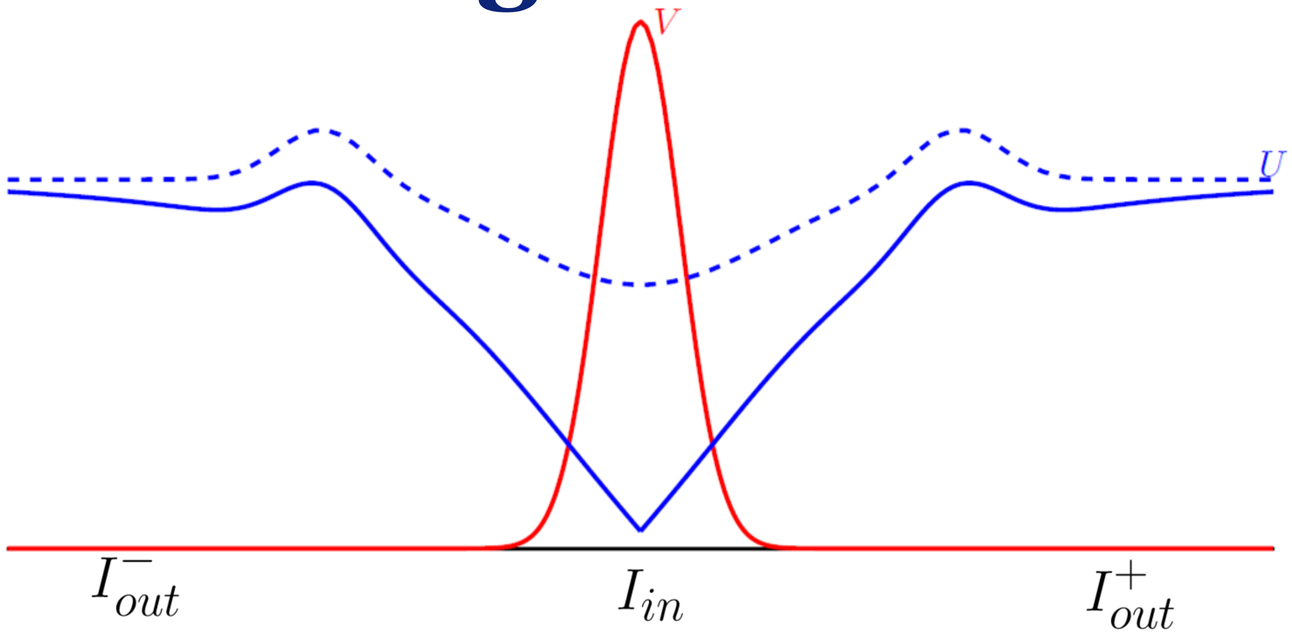
The effect of adding terrain



I_{in}

I^-_{out} I^+_{out}

$$V = \frac{a}{\sqrt{m}D}\frac{3}{2}\operatorname{sech}\left(\frac{\sqrt{m}x}{D}\frac{x}{2}\right)^2$$

$$U = \frac{m\sqrt{m}D}{a}u_0$$

$$V = 0$$

$$U = a\tilde{U}$$

$$\tilde{U}(x) = \tilde{U}_b(x) + C^-\tilde{U}^-(x) \quad I^-_{out}$$

$$\tilde{U}(x) = \tilde{U}_b(x) + C^+\tilde{U}^+(x) \quad I^+_{out}$$

Movement of pulses

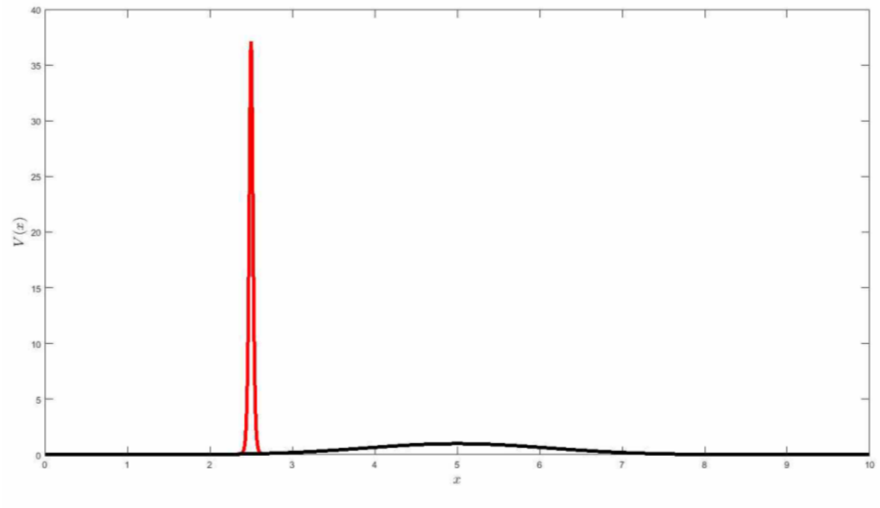
$$\frac{dP}{dt} = \frac{Da^2}{m\sqrt{m}}\frac{1}{6}\left[\tilde{U}_x(P^+)^2 - \tilde{U}_x(P^-)^2\right]$$

conform [W.Chen & M. Ward, 2009]

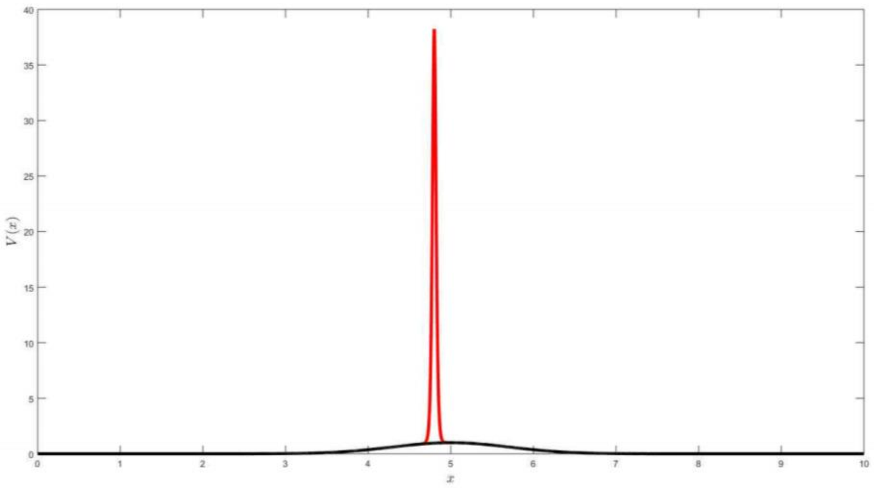
$H(x) = 0 \longrightarrow$ no movement

$H(x) = Sx \longrightarrow$ uphill movement [K. Siteur et al, 2014], [L. Sewalt & A. Doelman, 2017]

Generic $H(x) \longrightarrow$



uphill movement



downhill movement

Rigorous existence proofs

Recall:

$$\tilde{U}_{xx} + H_x\tilde{U}_x + H_{xx}\tilde{U} - \tilde{U} + 1 = 0$$

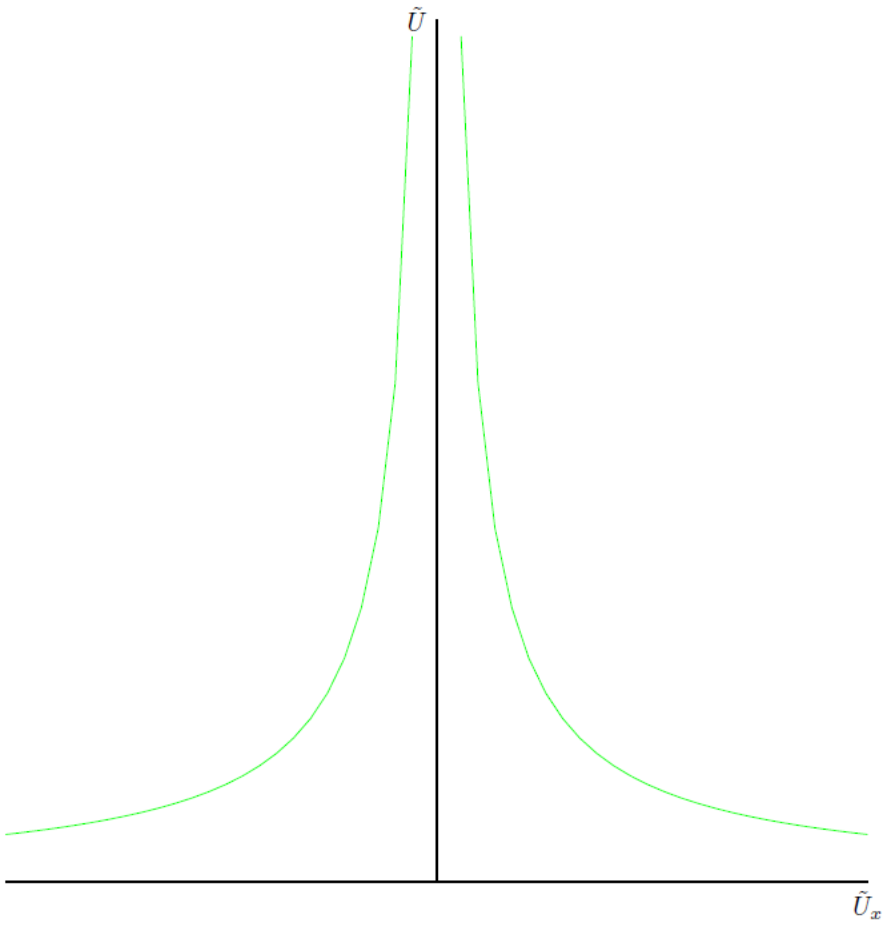
$$\tilde{U}(0) = \frac{m\sqrt{m}D}{a^2}u_0$$

$$\Delta\tilde{U}_x(0) = \frac{6}{u_0}$$

AND

A pulse has a movement speed

$$\frac{dP}{dt} = \frac{Da^2}{m\sqrt{m}}\frac{1}{6}\left[\tilde{U}_x(P^+)^2 - \tilde{U}_x(P^-)^2\right]$$



Stationary situation
(no movement)

Rigorous existence proof - ($H(x) = 0$)

Recall:

$$\tilde{U}_{xx} - \tilde{U} + 1 = 0$$

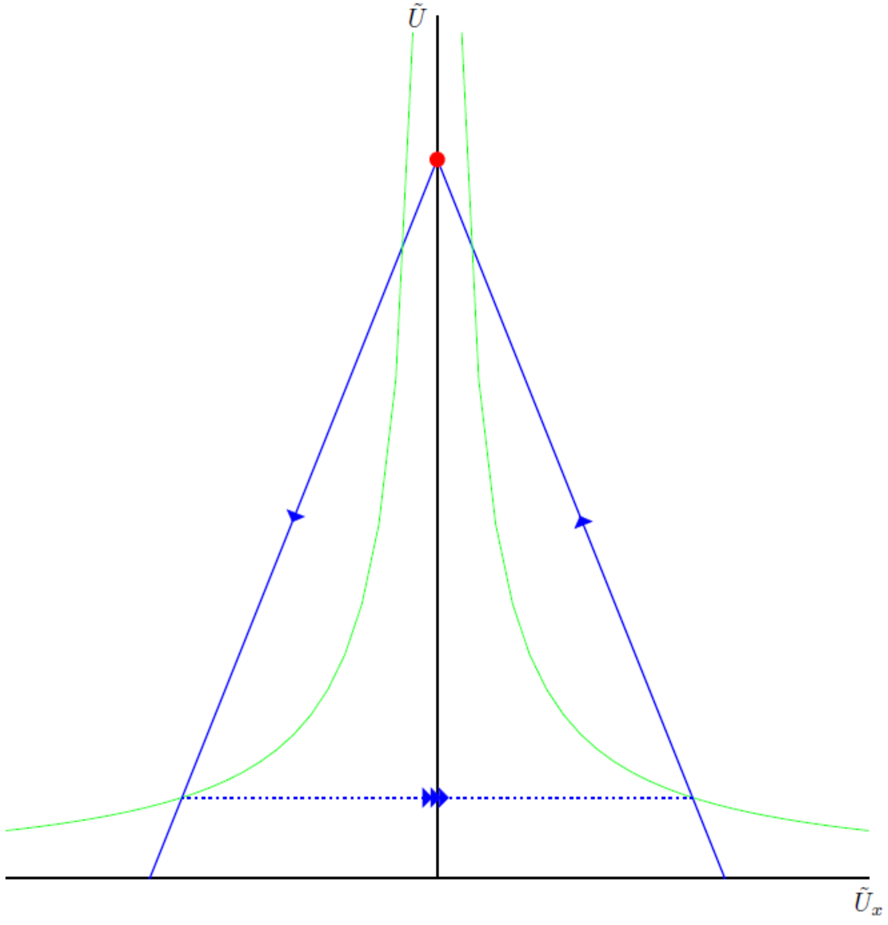
$$\tilde{U}(0) = \frac{m\sqrt{m}D}{a^2} u_0$$

$$\Delta \tilde{U}_x(0) = \frac{6}{u_0}$$

AND

A pulse has a movement speed

$$\frac{dP}{dt} = \frac{Da^2}{m\sqrt{m}} \frac{1}{6} \left[\tilde{U}_x(P^+)^2 - \tilde{U}_x(P^-)^2 \right]$$



Stationary situation
(no movement)

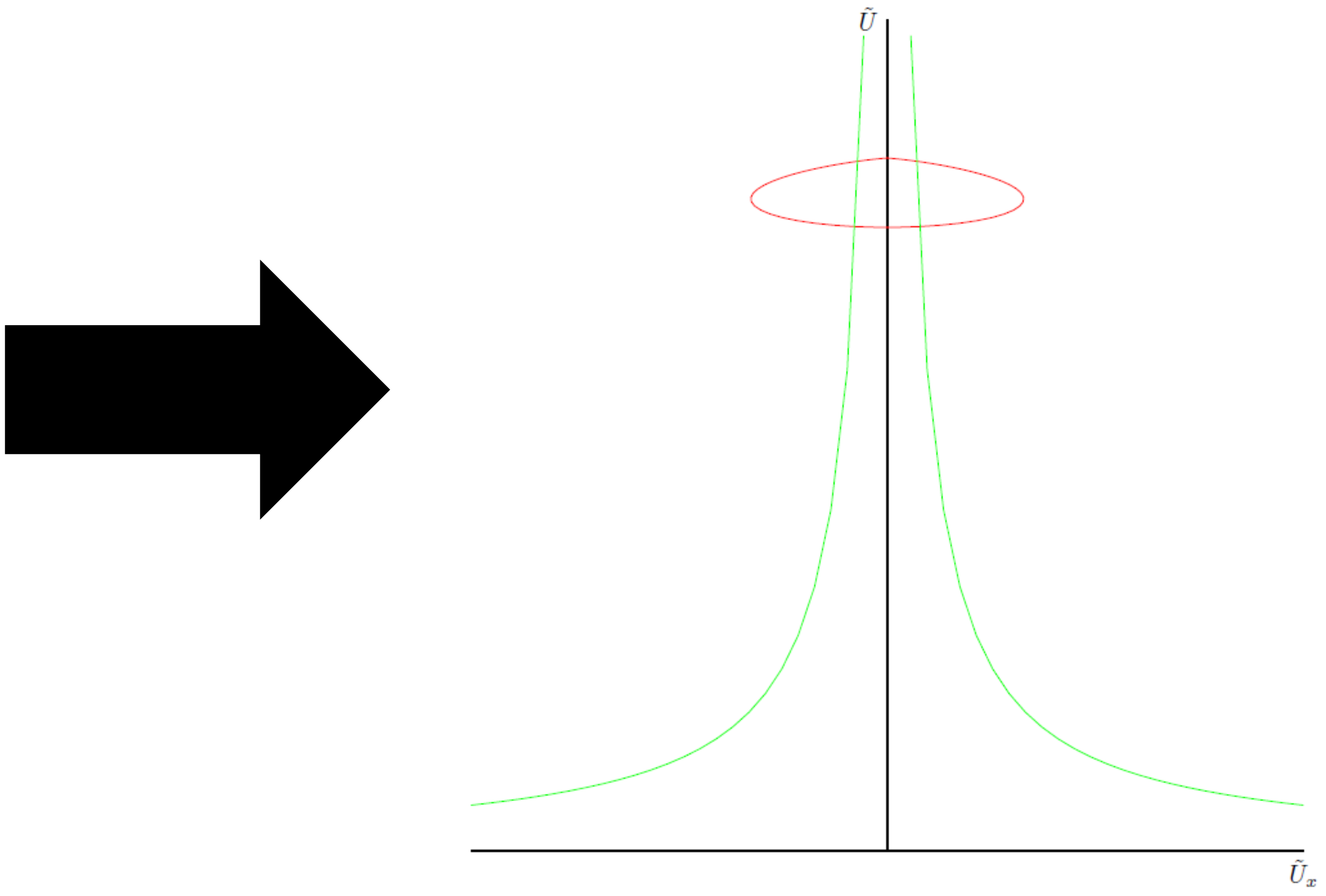
Rigorous existence proof – Specific $H(x)$

Recall:

$$\tilde{U}_{xx} + H_x\tilde{U}_x + H_{xx}\tilde{U} - \tilde{U} + 1 = 0$$

Bounded solution:

$$\tilde{U}_b(x)$$



Projected $(\tilde{U}_x, \tilde{U})$-plane

Rigorous existence proof – Specific $H(x)$

Recall:

$$\tilde{U}_{xx} + H_x\tilde{U}_x + H_{xx}\tilde{U} - \tilde{U} + 1 = 0$$

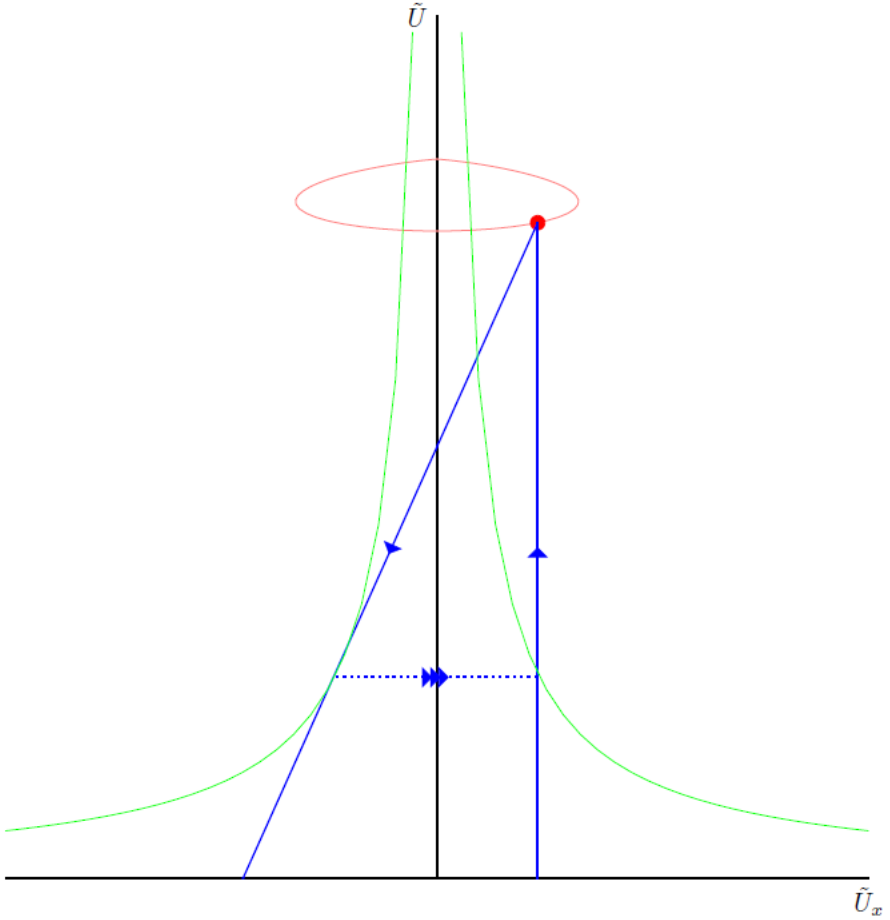
Bounded solution:

$$\tilde{U}_b(x)$$

Stable/Unstable manifolds:

UNSTABLE $\quad \tilde{U}_b(x) + C^-\tilde{U}^-(x)$

STABLE $\quad \tilde{U}_b(x) + C^+\tilde{U}^+(x)$



$(\tilde{U}_x, \tilde{U})$-plane for specific x

Rigorous existence proof – Generic $H(x)$

Existence theorem

If $H(x)$ is symmetric in $x = 0$ and $\delta := \sup_{x\in\mathbb{R}} \sqrt{H_x(x)^2 + H_{xx}(x)^2} < \frac{\sqrt{2}-1}{8}$

then

a stationary symmetric one-pulse solution to the PDE exists

(under the standard Gray-Scott magnitude assumptions on the parameters)

Heart of the proof is the ‘rhoughness of exponential dichotomies’

→ This gives bounds on stable and unstable manifolds and the bounded solution

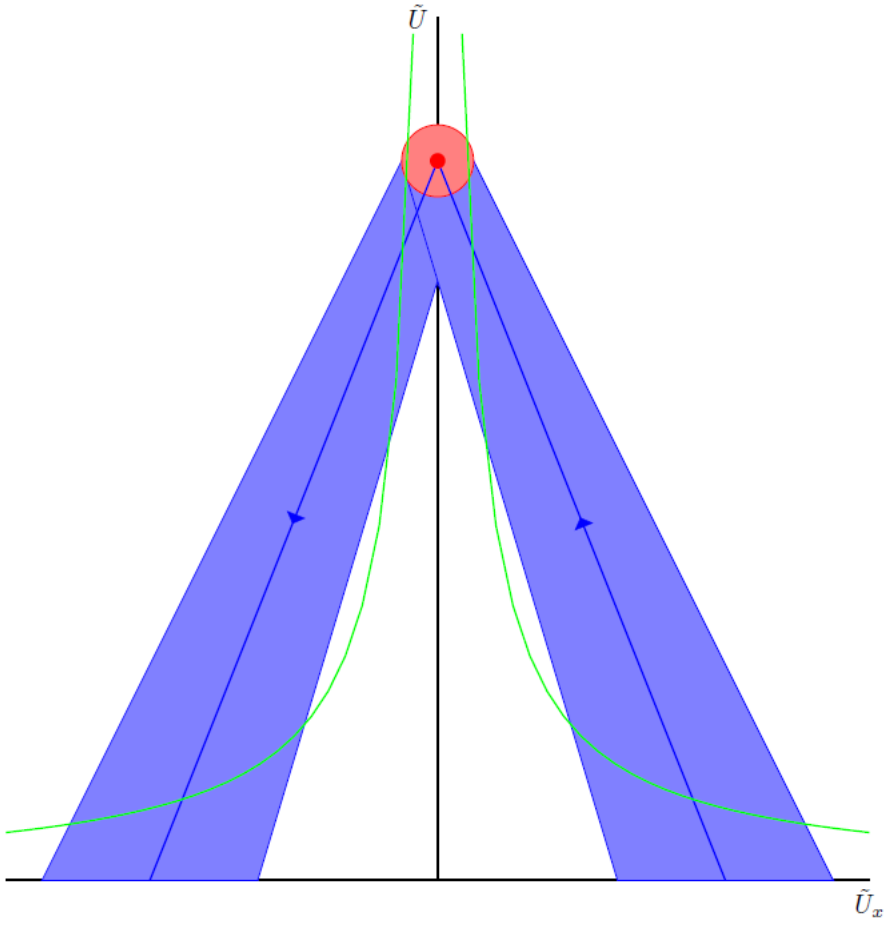
Rigorous existence proof – Generic $H(x)$

Recall:

$$\tilde{U}_{xx} + H_x \tilde{U}_x + H_{xx} \tilde{U} - \tilde{U} + 1 = 0$$

AND

Bounds from exponential dichotomies



Bounds via exponential dichotomies

Rigorous existence proof – Generic $H(x)$

Recall:

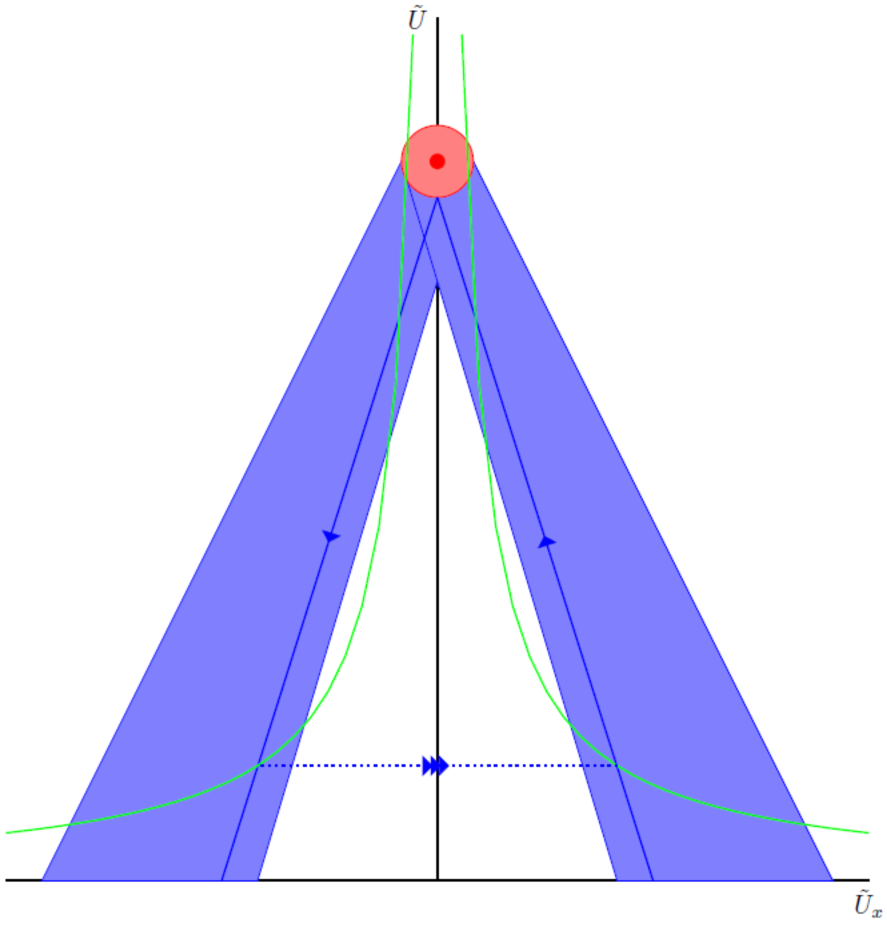
$$\tilde{U}_{xx} + H_x \tilde{U}_x + H_{xx} \tilde{U} - \tilde{U} + 1 = 0$$

AND

Bounds from exponential dichotomies

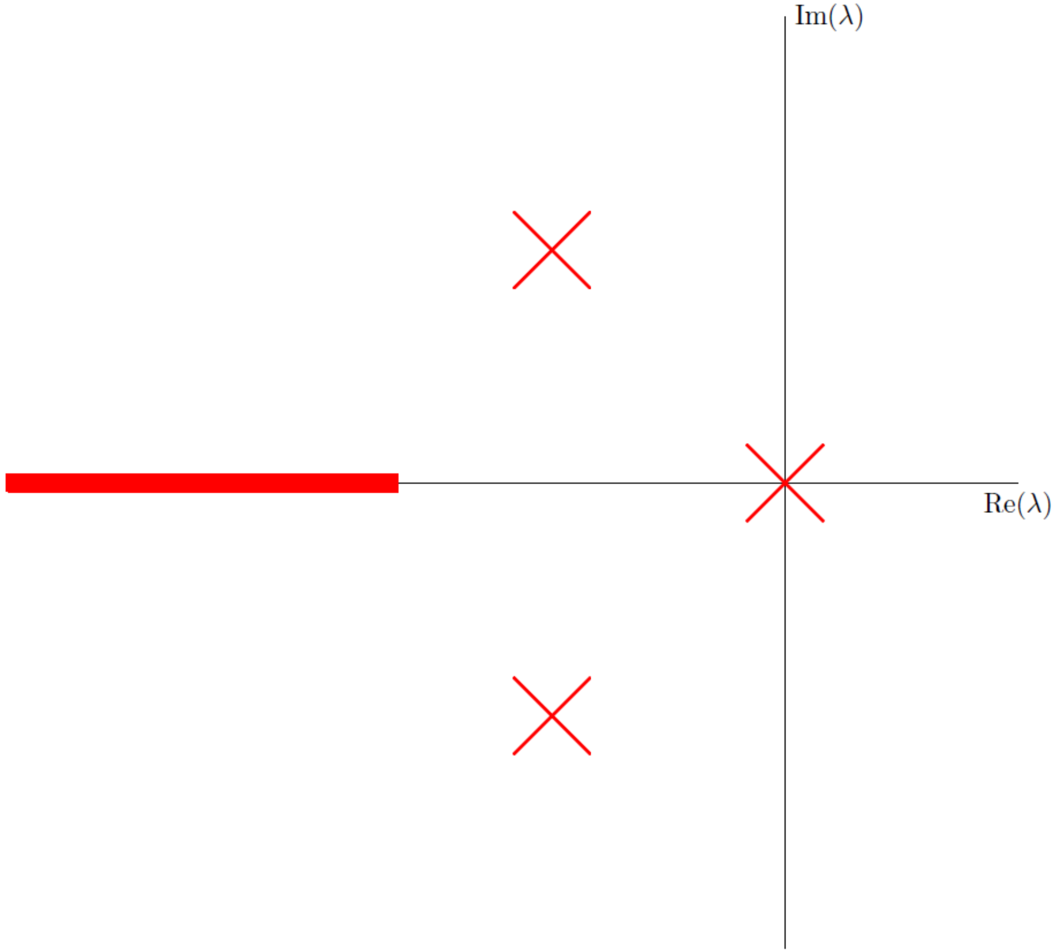
AND

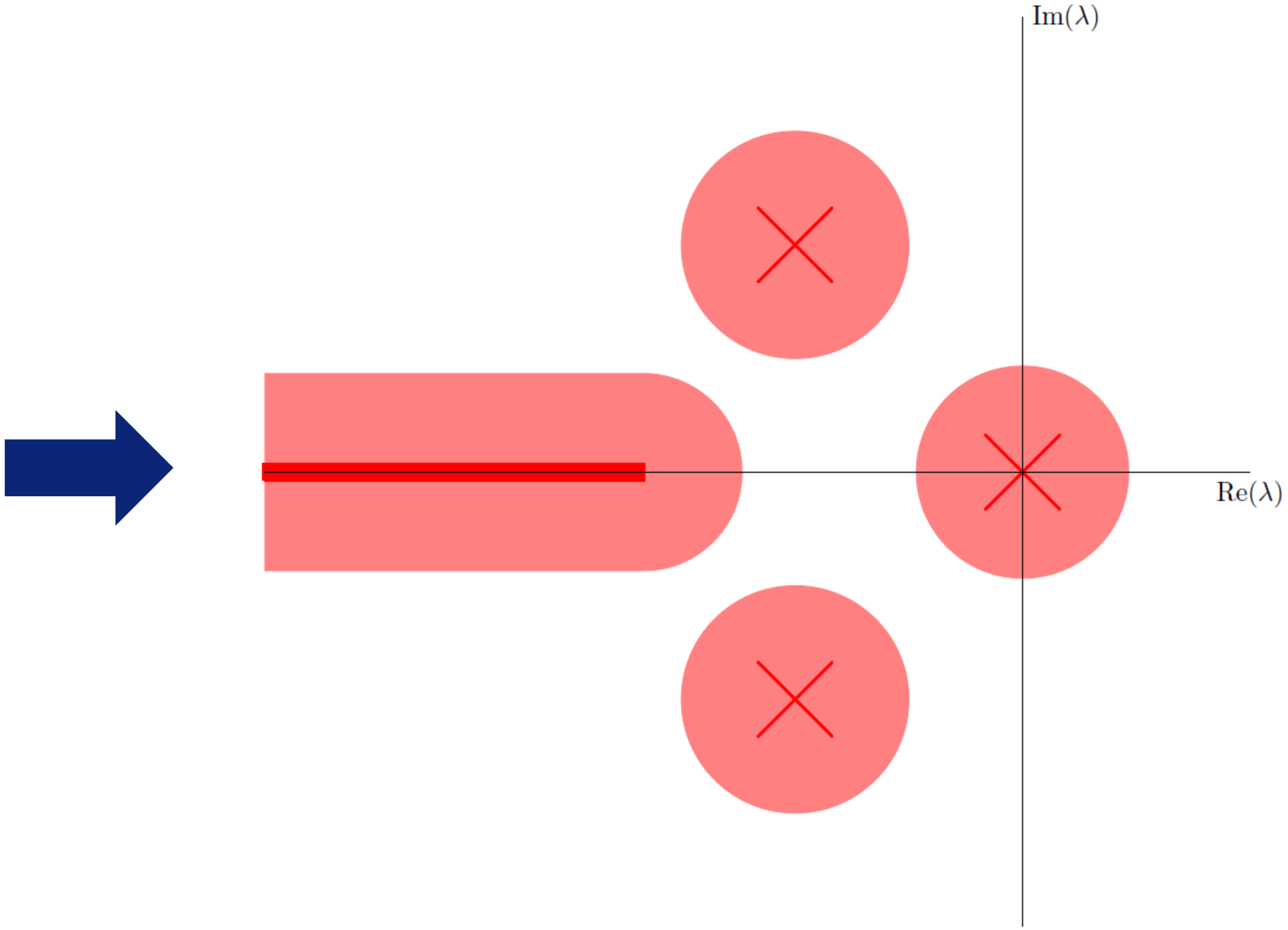
$H(x)$ is symmetric in $x = 0$



Using symmetry arguments

Stability of one-pulse solution





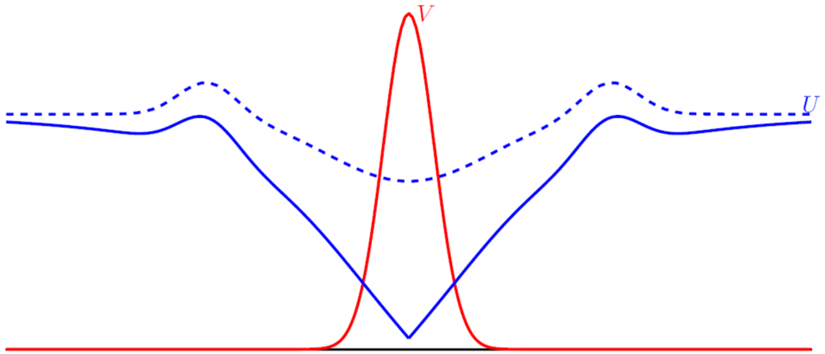
Autonomous ($H_x(x) \equiv 0$)
[A. Doelman, R.A. Gardner, T.J. Kaper, 1998]

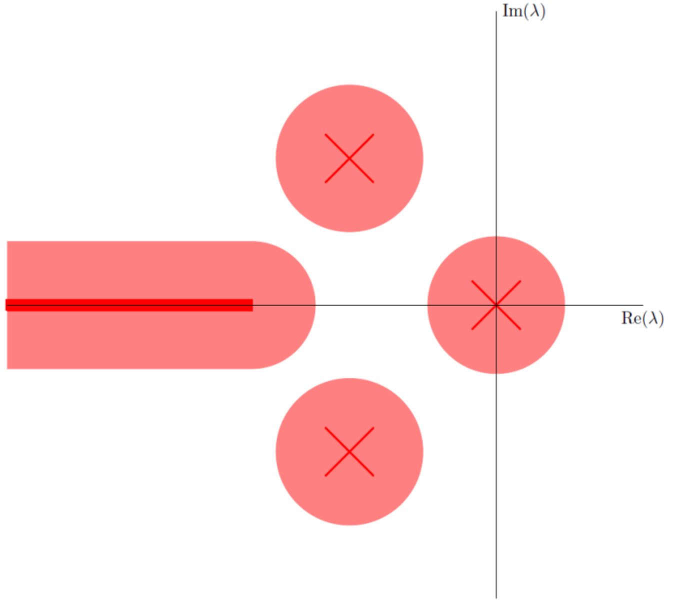
Non-Autonomous
($\delta := \sup_{x\in\mathbb{R}} \sqrt{H_x(x)^2 + H_{xx}(x)^2} < \delta_c$)

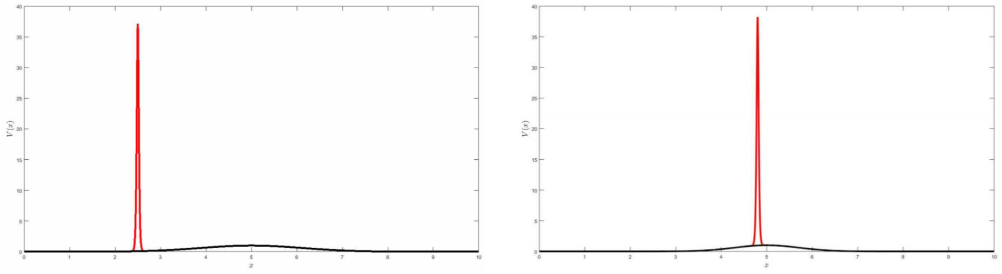
Summary

Existence stationary one-pulse solution
- with explicit expressions
- with rhoughness of exponential dichotomies

Stability
- big eigenvalues have negative real part
- small eigenvalue can become unstable
  - Related to movement of the pulse
  - Both uphill and downhill movement possible







Rigorous existence proof – Generic $H(x)$

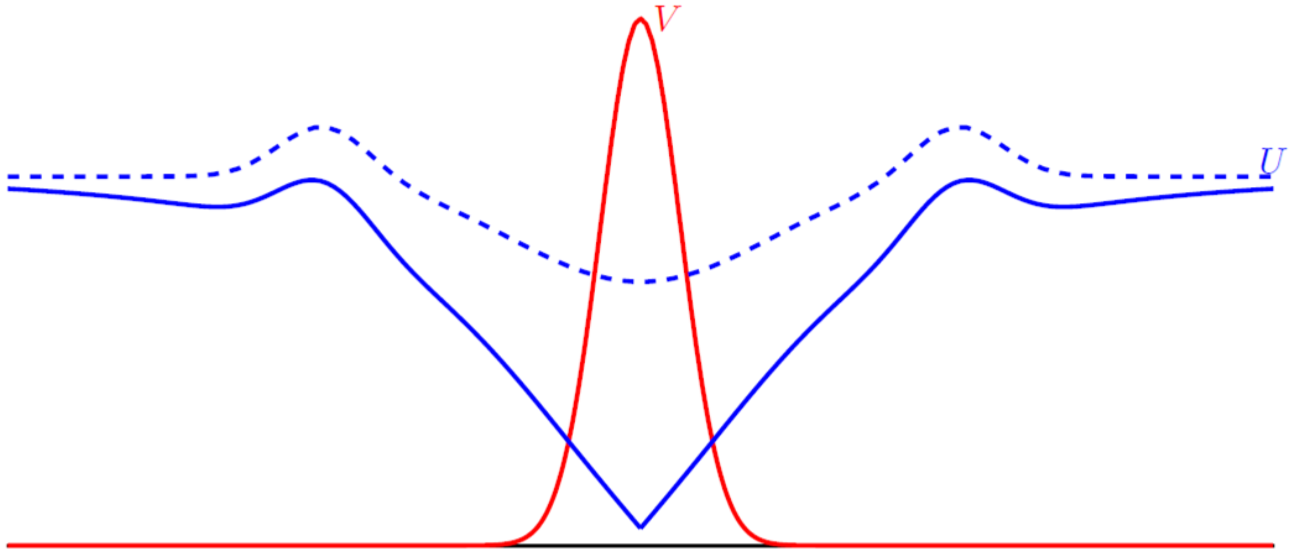
Existence theorem

If $H(x)$ is symmetric in $x = 0$ and $\delta \coloneqq \sup_{x\in\mathbb{R}} \sqrt{H_x(x)^2 + H_{xx}(x)^2} < \frac{\sqrt{2}-1}{8}$

then

a stationary symmetric one-pulse solution to the PDE exists

(under the standard Gray-Scott magnitude assumptions on the parameters)



Rigorous existence proof – Generic $H(x)$

Existence theorem

If $H(x)$ is symmetric in $x=0$ and $\delta := \sup_{x\in\mathbb{R}} \sqrt{H_x(x)^2 + H_{xx}(x)^2} < \frac{\sqrt{2}-1}{8}$

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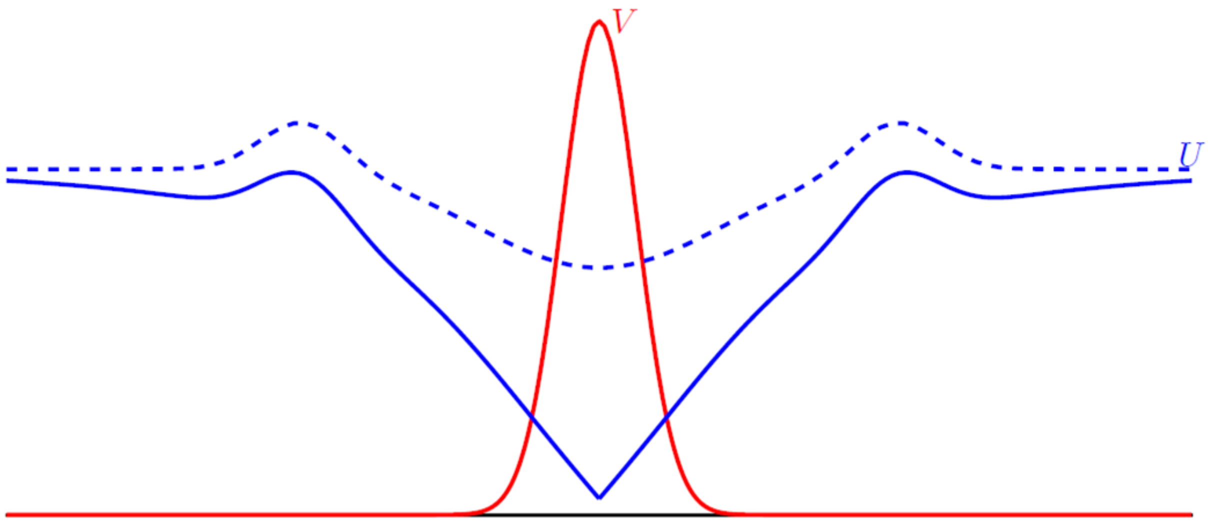
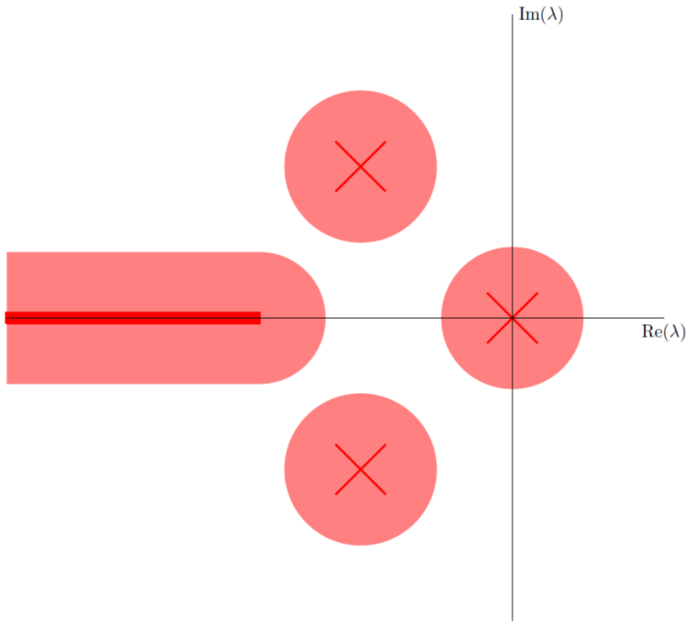
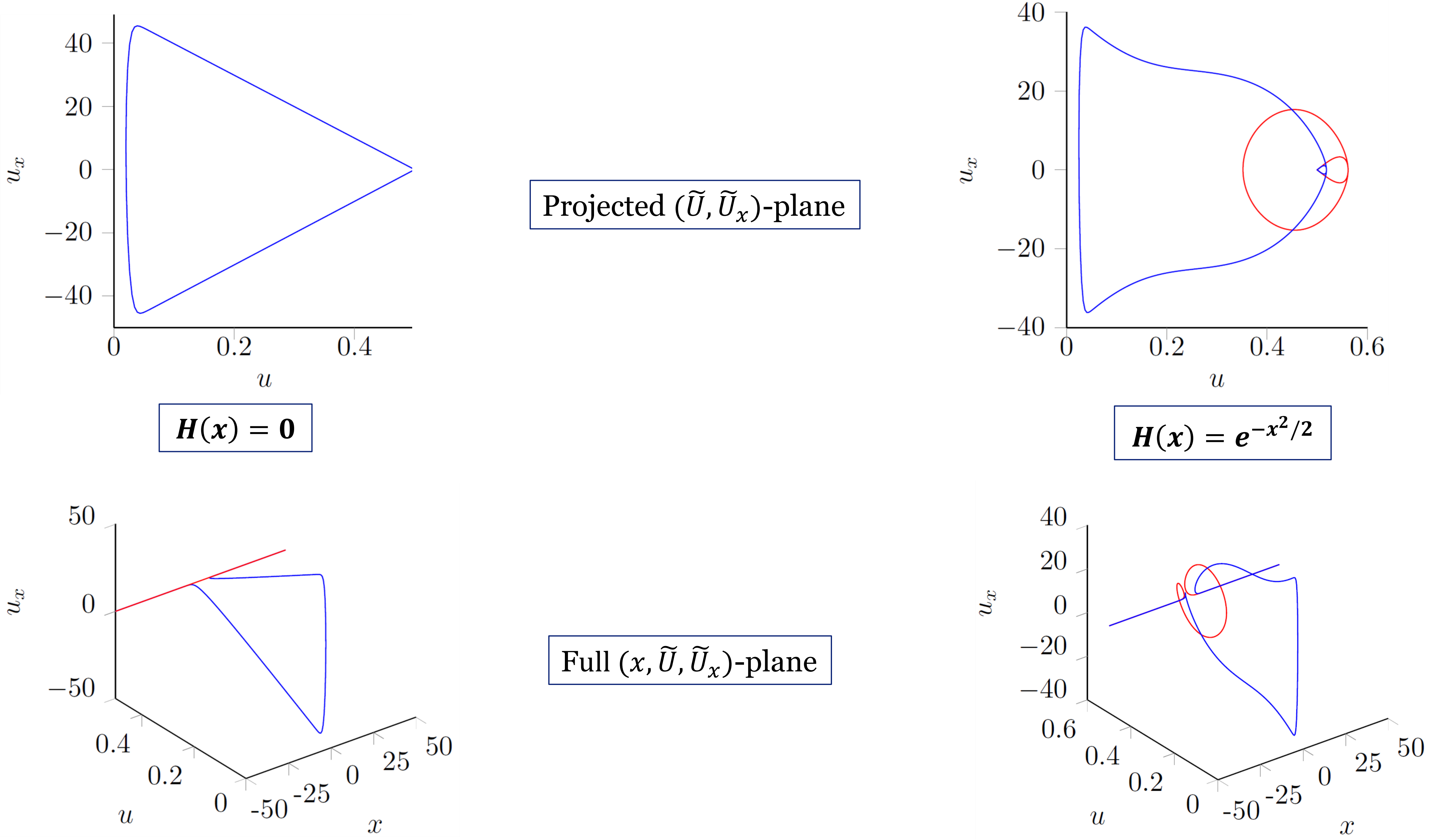






Projected $(\widetilde{U}, \widetilde{U}_x)$-plane
$H(x) = 0$
$H(x) = e^{-x^2/2}$
Full $(x, \widetilde{U}, \widetilde{U}_x)$-plane
u_x
u
x

The small eigenvalue

Normally: translational invariance $\longleftrightarrow$ eigenvalue $\lambda = 0$

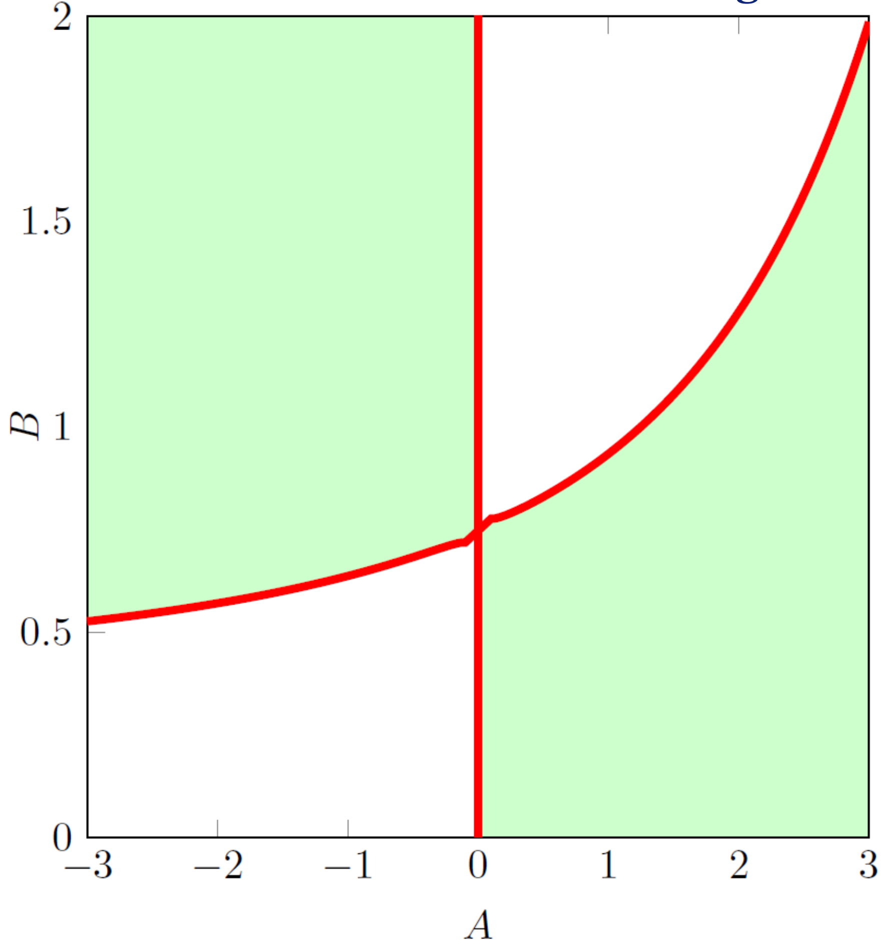
Adding terrain: eigenvalue gets perturbed

Perturbed eigenvalue $\longleftrightarrow$ eigenvalue of ODE

For terrains with small slope and curvature:

(with rigorous computations)

Bifurcation diagram

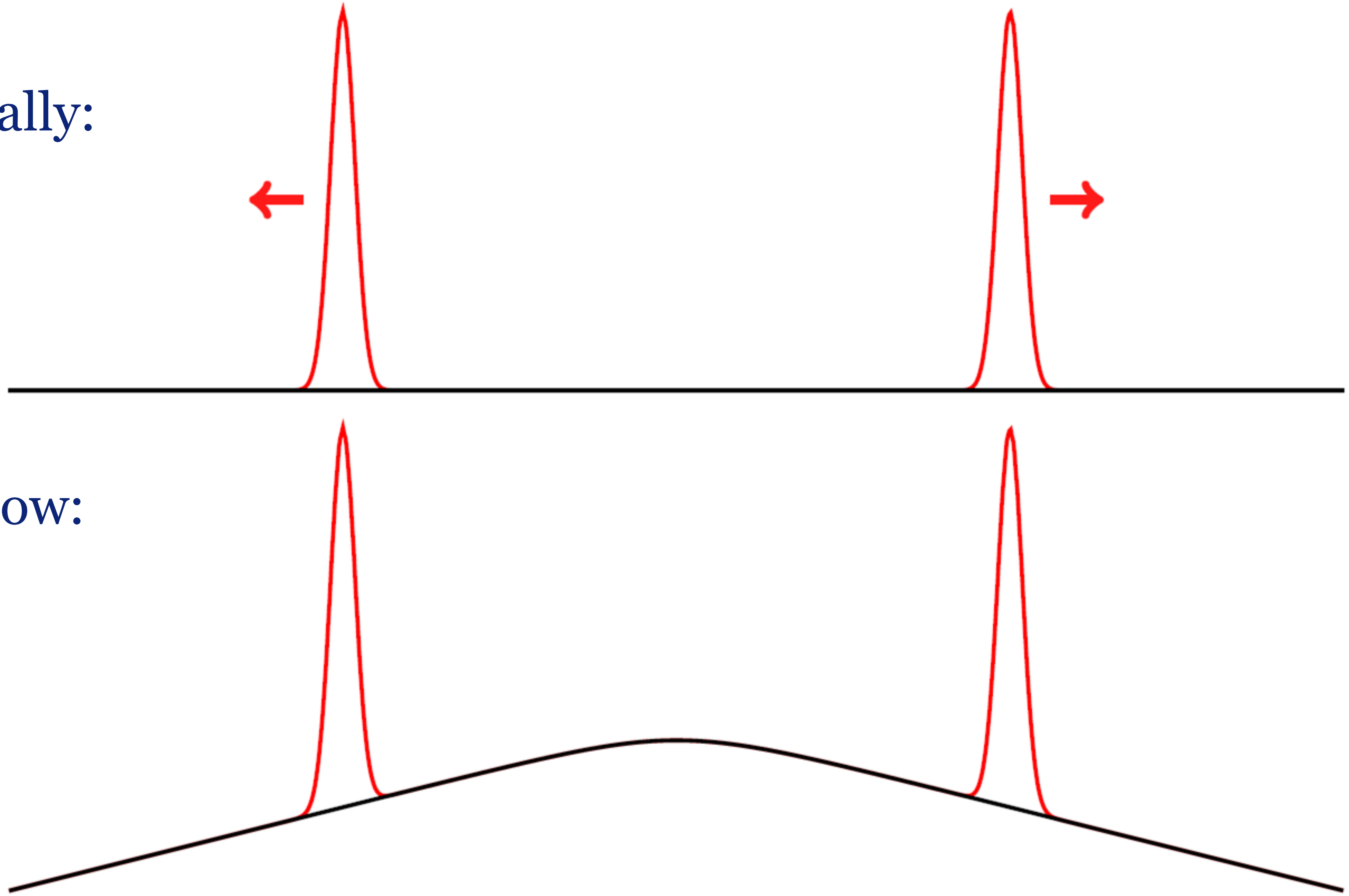


$$H(x) = Ae^{-Bx^2}$$

Adding more pulses

Normally:

Now:



Stationary two-pulse solutions exist!