Introducing topographical influences in the Extended-Klausmeier vegetation model

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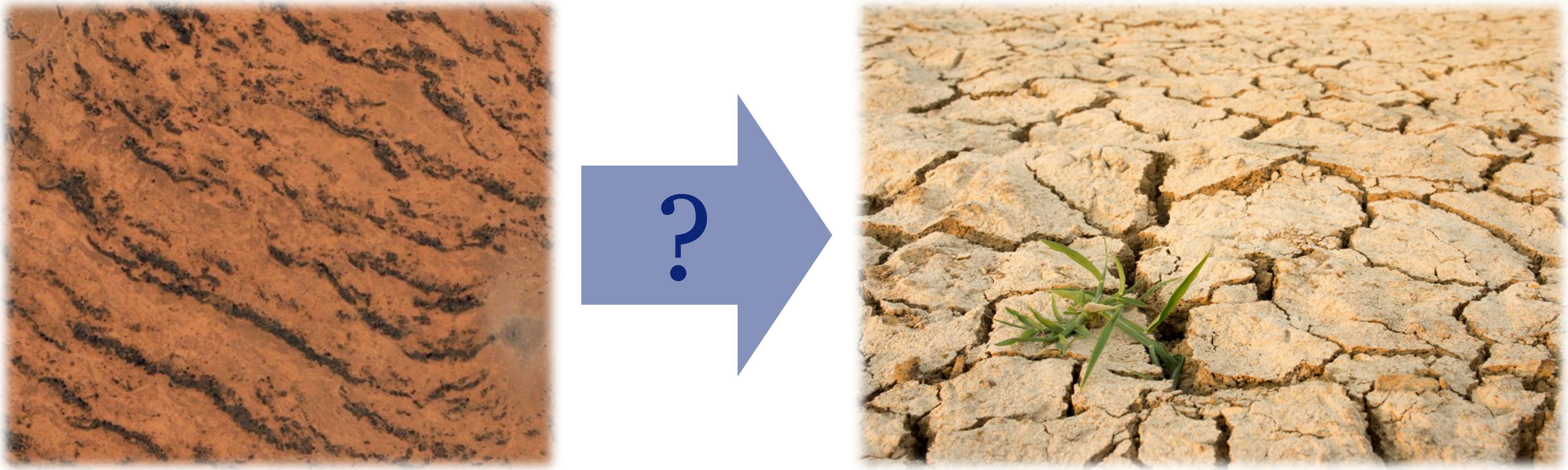
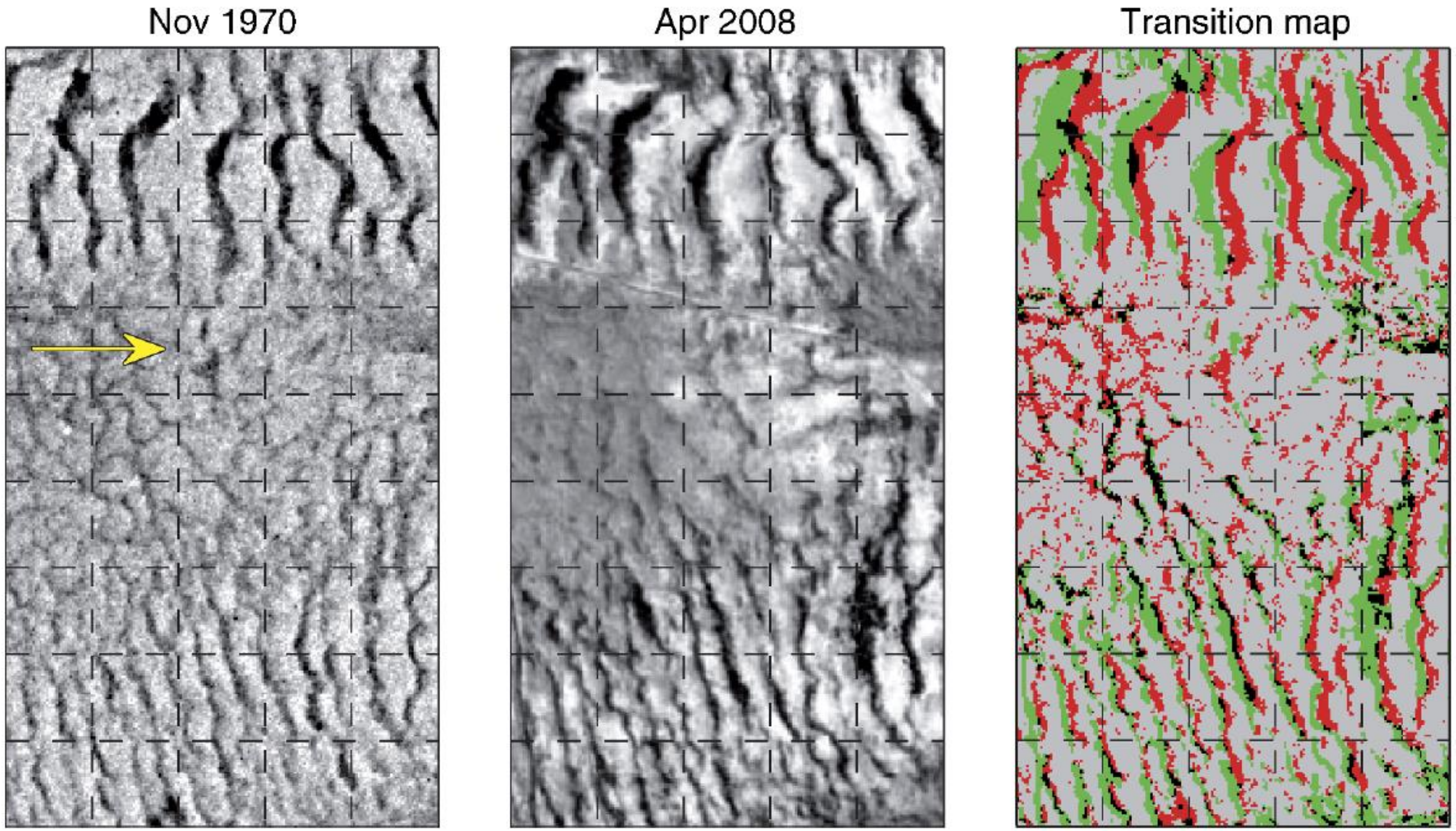
The study of desertification



Vegetation is not stationary



Main Question:

How to handle the effects of topography in a model?

Image Source: Vincent Deblauwe et al – Determinants and dynamics of banded vegetation pattern migration in arid climates

A simple mathematical model

extended-Klausmeier model

$$U_t = U_{xx} + (H_x\, U)_x + a - U - UV^2$$
$$V_t = D^2 V_{xx} - mV + UV^2$$

Variables:

U Water

V Vegetation

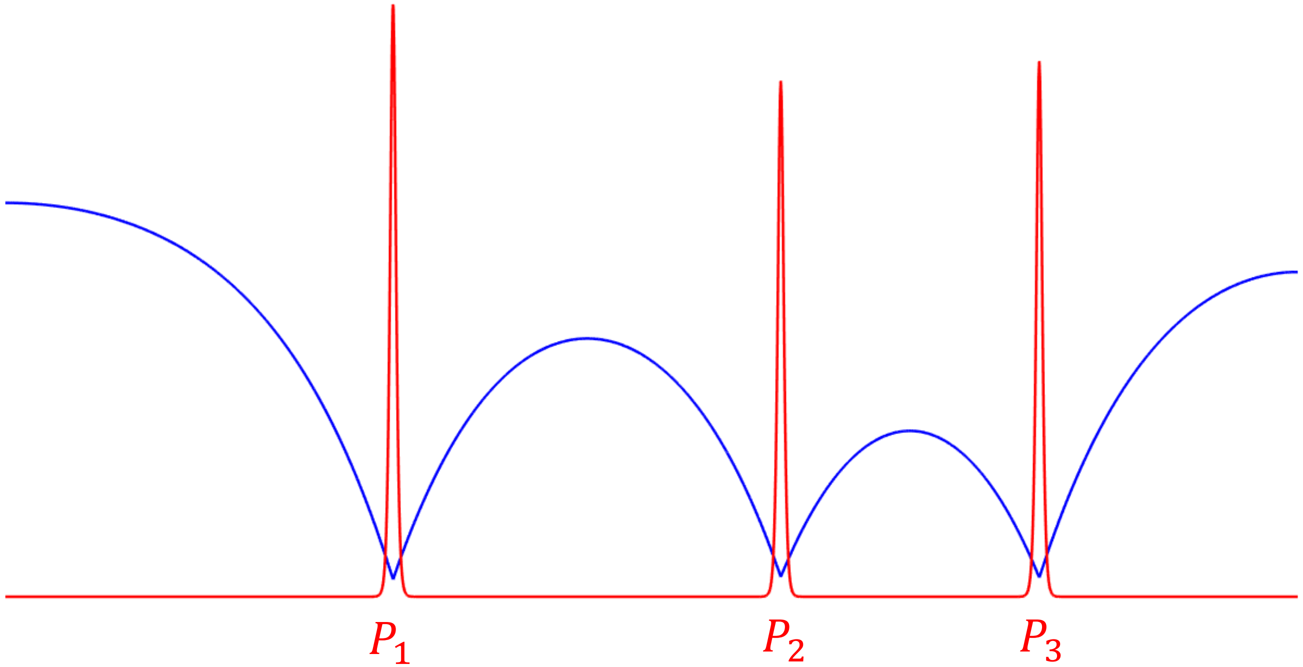
Parameters:

a Rainfall

m Mortality of plants

D Small parameter

H Height of terrain



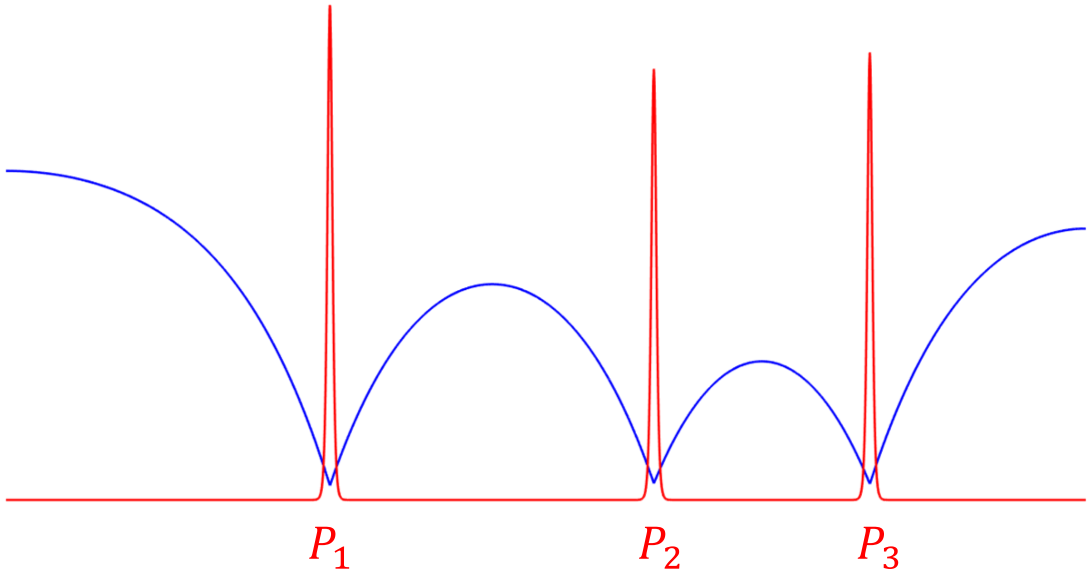
Pulse dynamics – PDE to ODE reduction

- Uses regular expansions or geometric singular perturbation theory
- Obtain pulse location ODE via formal computations (conform [W.Chen & M. Ward, 2009])

$$\frac{dP_j}{dt} = \frac{Da^2}{m\sqrt{m}} \frac{1}{6} \left[\bar{U}_x(P_j^+)^2 - \bar{U}_x(P_j^-)^2 \right] + \text{exp small terms}$$

- $\bar{U}$ satisfies

$$\begin{cases} 0 = \bar{U}_{xx} + H_x \bar{U}_x + H_{xx} \bar{U} + 1 - \bar{U} \\ \bar{U}(P_j) \approx 0 \end{cases}$$



→ Movement determined through water availability ←

Focus on 1 pulse

fixed slope → autonomous ODE system

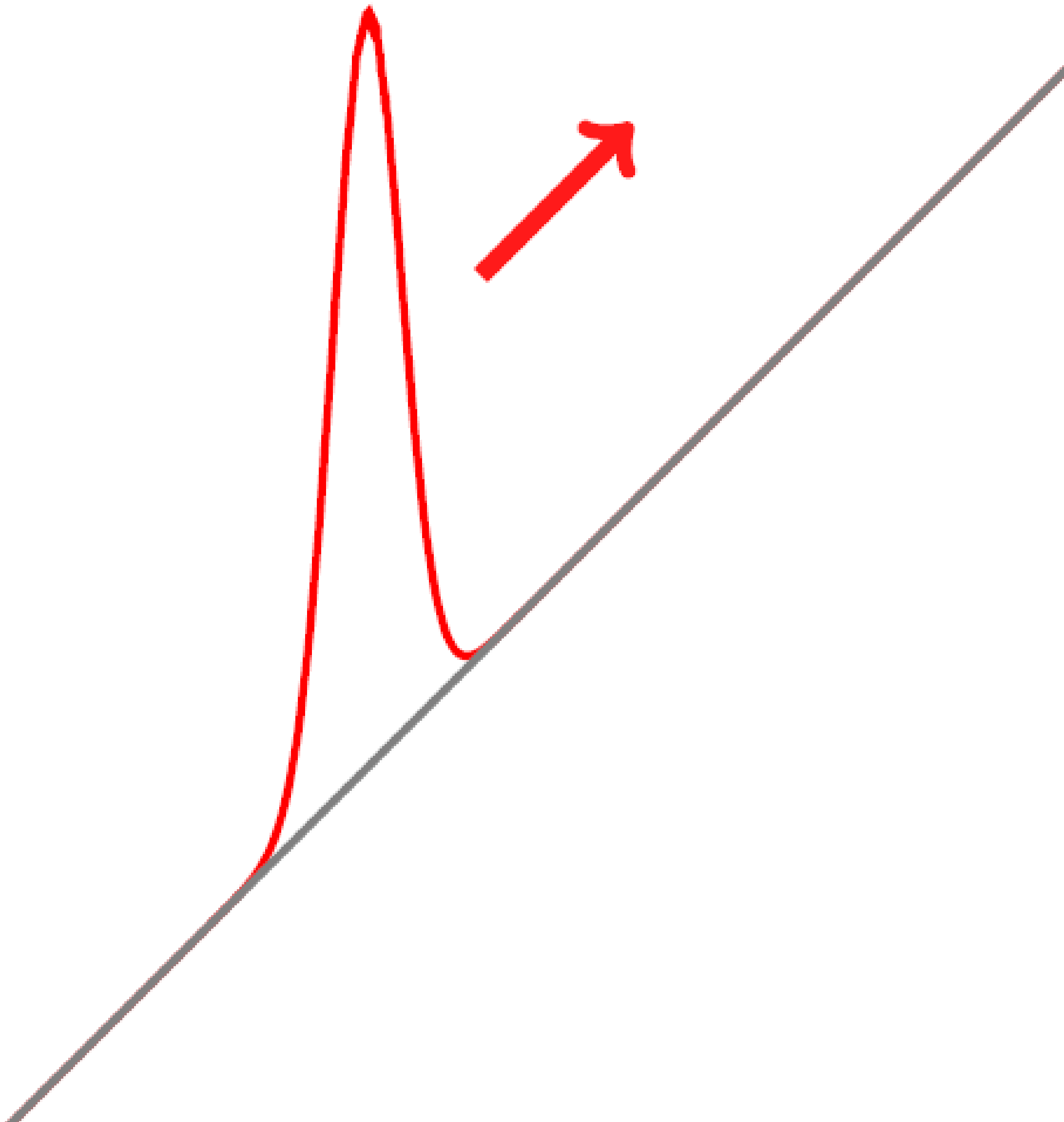
• uphill movement

• speed increases with slope

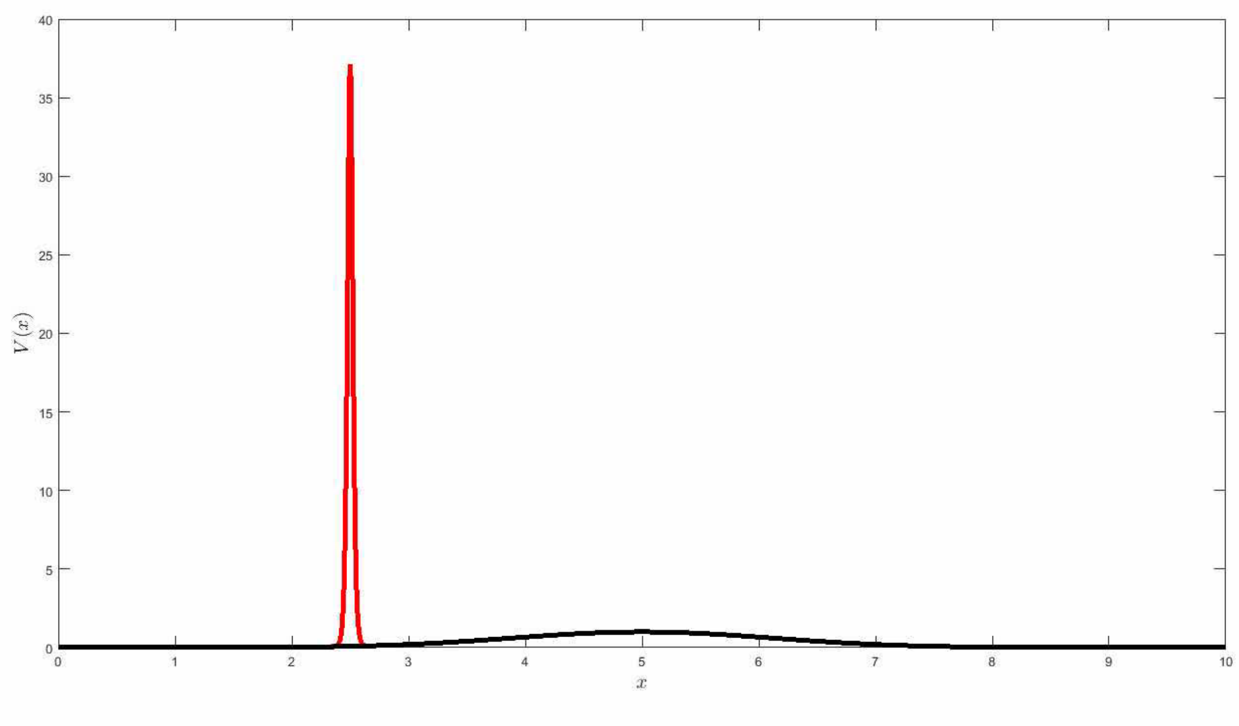
[L. Sewalt & A. Doelman, 2017]

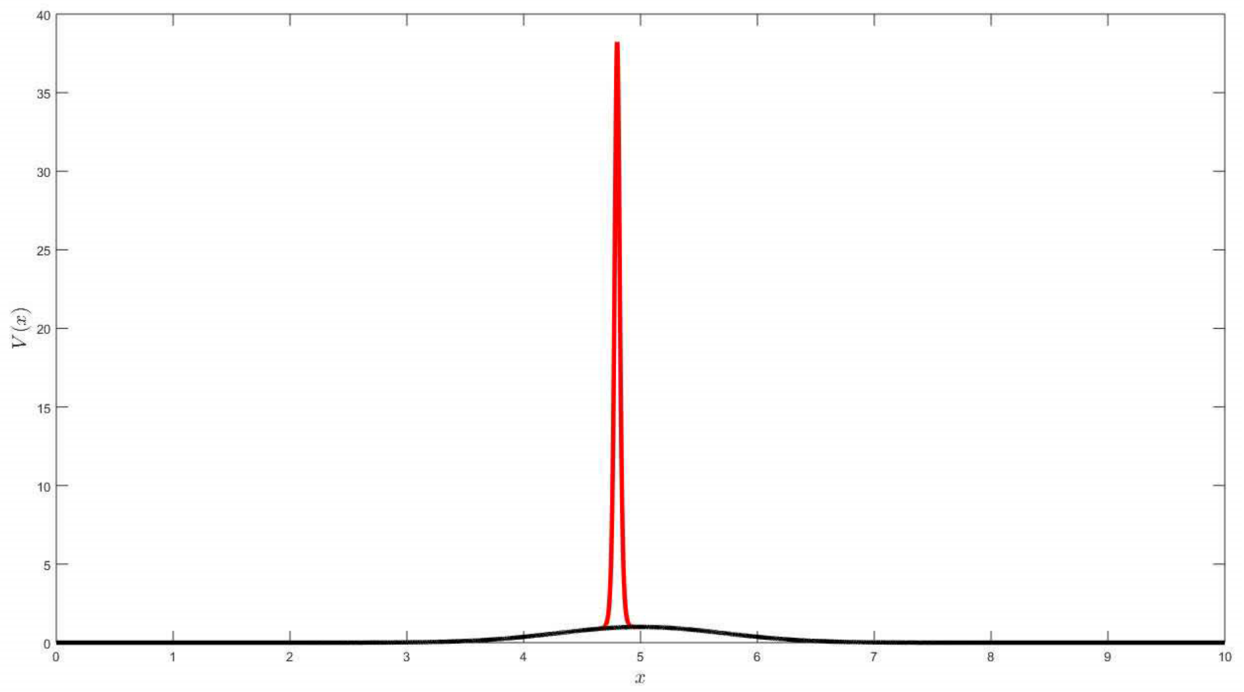
$$c(H_x) = \frac{Da^2}{m\sqrt{m}} \frac{H_x \sqrt{H_x^2 + 4}}{6}$$

(follows via ODE or rigorous computations)



A pulse on a hill





Vegetation pulses may move downhill

Rigorous mathematics: existence of a pulse

Existence theorem

If $H(x)$ is symmetric in $x = 0$ and $\delta := \sup_{x\in\mathbb{R}} \sqrt{H_x(x)^2 + H_{xx}(x)^2} < \frac{\sqrt{2}-1}{8}$

then

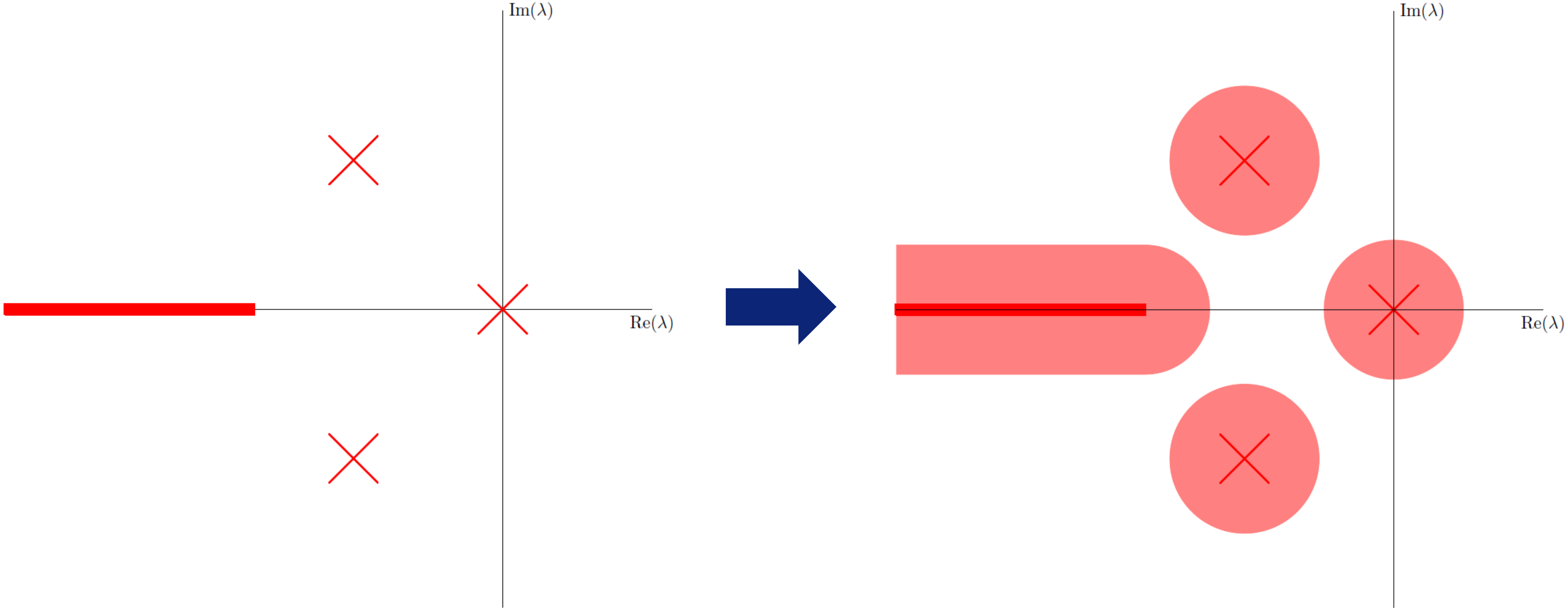
a stationary symmetric one-pulse solution to the PDE exists

(under the standard Gray-Scott magnitude assumptions on the parameters)

Proof techniques:

- (Standard) geometric singular perturbation theory
- Uses roughness of exponential dichotomies
  - Gives bounds on stable and unstable manifolds of ‘fixed points’

Rigorous mathematics: stability



Autonomous ($H_x(x) \equiv 0$)

[A. Doelman, R.A. Gardner, T.J. Kaper, 1998]

Non-Autonomous

($\delta \coloneqq \sup_{x\in\mathbb{R}} \sqrt{H_x(x)^2 + H_{xx}(x)^2} < \delta_c$)

The small eigenvalue

Normally: translational invariance $\longleftrightarrow$ eigenvalue $\lambda = 0$

Adding terrain: eigenvalue gets perturbed

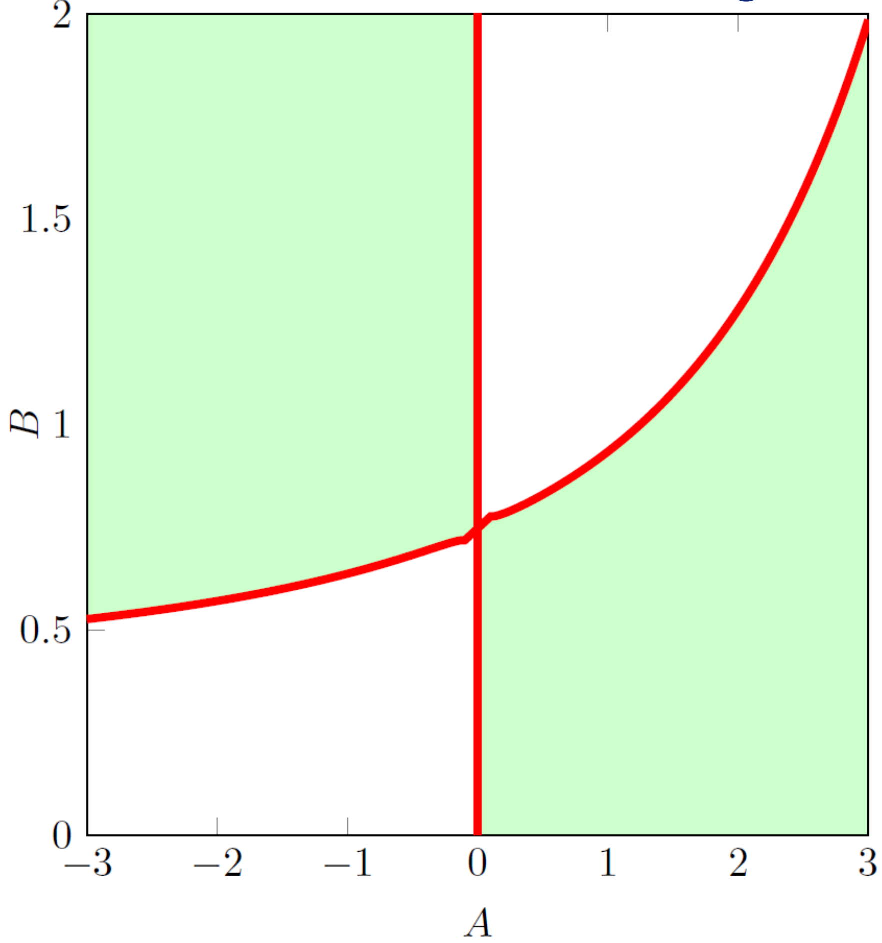
Perturbed eigenvalue $\longleftrightarrow$ eigenvalue of ODE

For terrains with small slope and curvature:

(with rigorous computations)

$$\lambda = \delta\, C \left[H(0) + \int_0^\infty H(x)[1 - 4e^{-x}]e^{-x}\, dx \right]$$

Bifurcation diagram

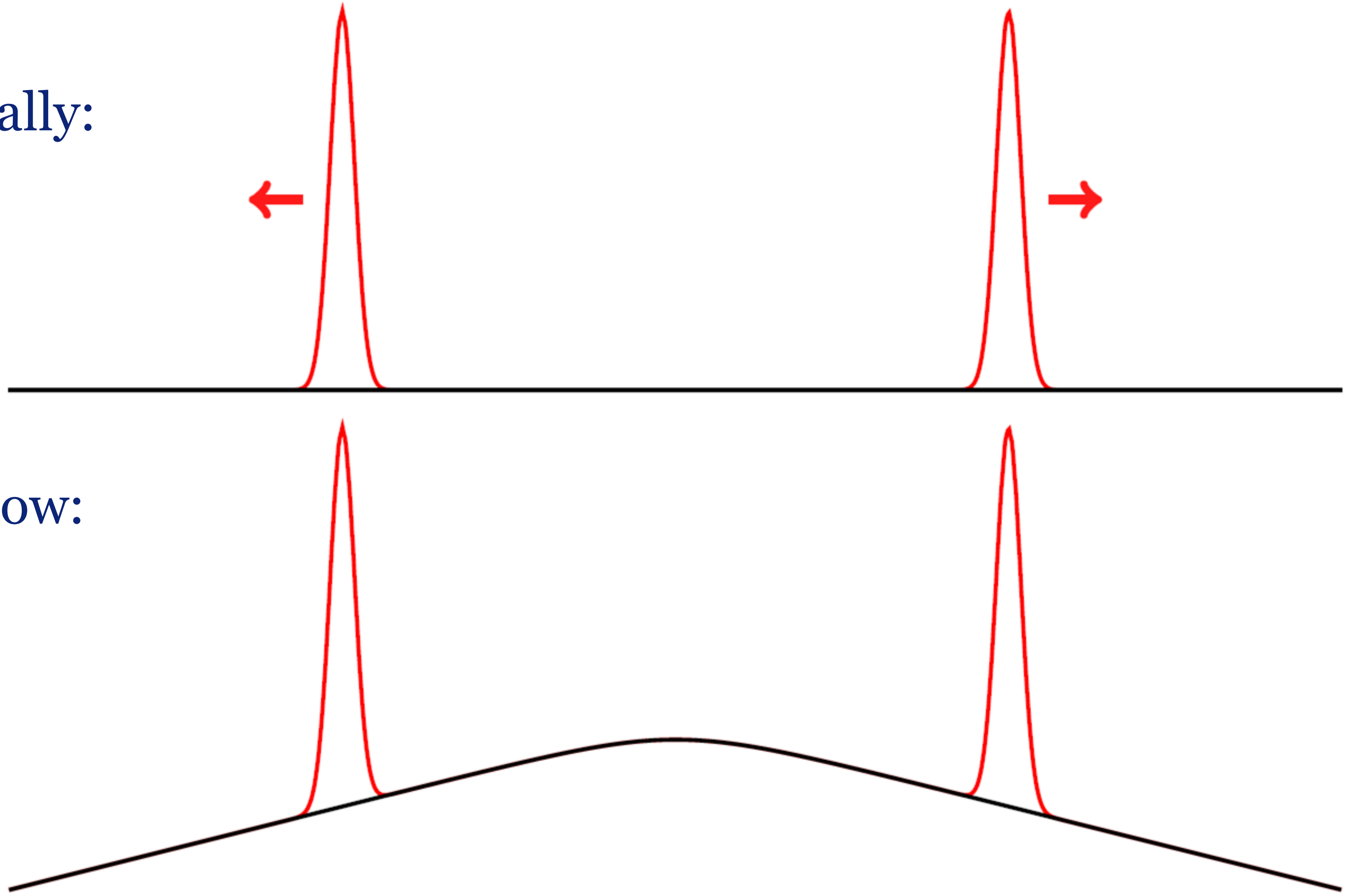


$H(x) = Ae^{-Bx^2}$

Adding more pulses

Normally:

Now:



Stationary two-pulse solutions exist!

Summary

- Topography *influences* water availability *influences* pulse movement

- Topography changes the rules of thumb
  - Downhill movement of vegetation
  - Stationary two-pulse solutions