TiPES
TIPPING POINTS IN THE EARTH SYSTEM

Co-financed by the Connecting Europe Facility of the European Union

Tipping in Spatially Extended Systems

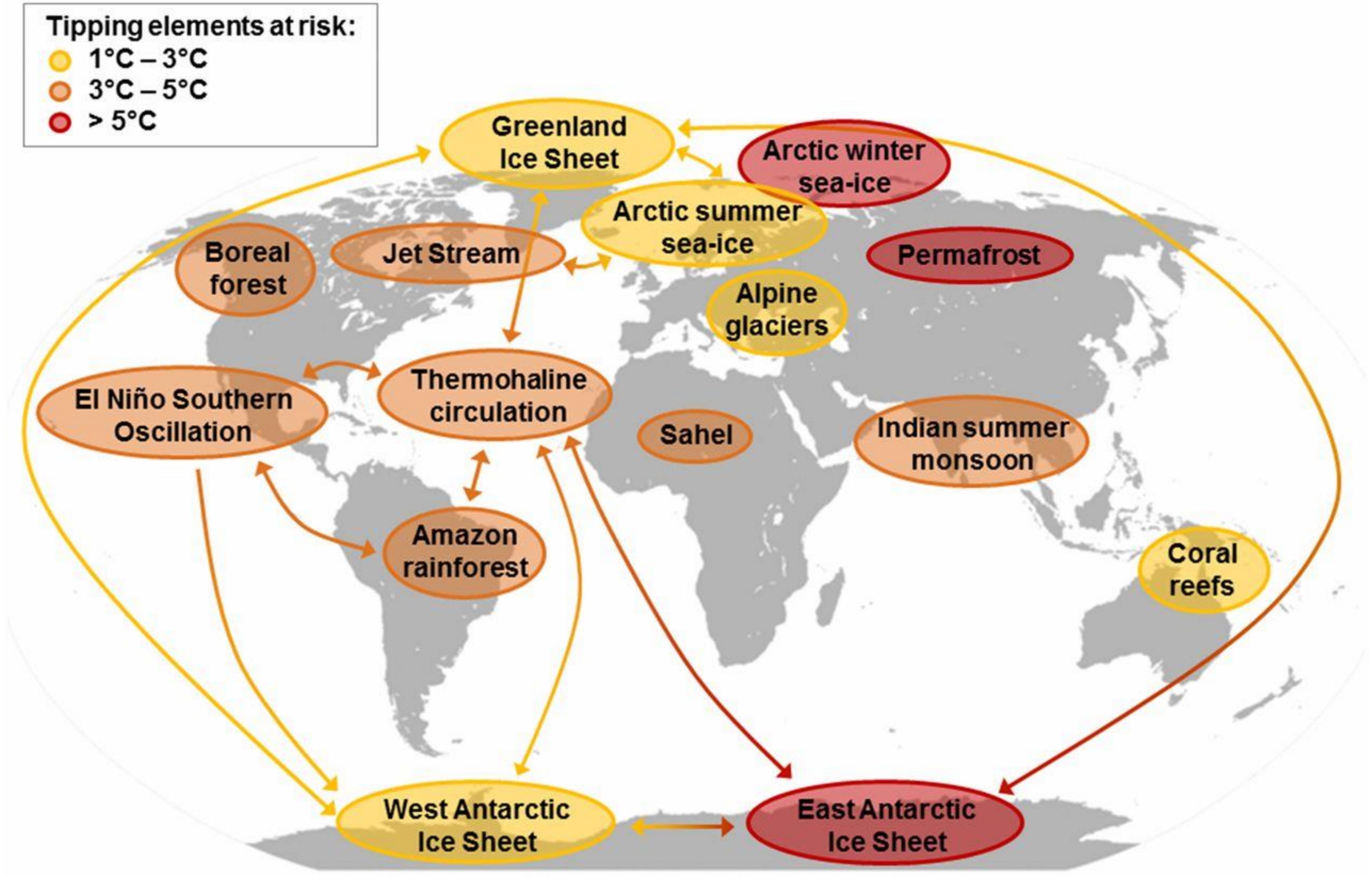
2022-03-04, VCU Biomath Seminar

Robbin Bastiaansen

UNIVERSITY OF COPENHAGEN UiT ENS VU UCL Université catholique de Louvain PIK TUM Met Office

Tipping Points

IPCC AR6 (2021) : "a critical threshold beyond which a system reorganizes, often abruptly and/or irreversibly"

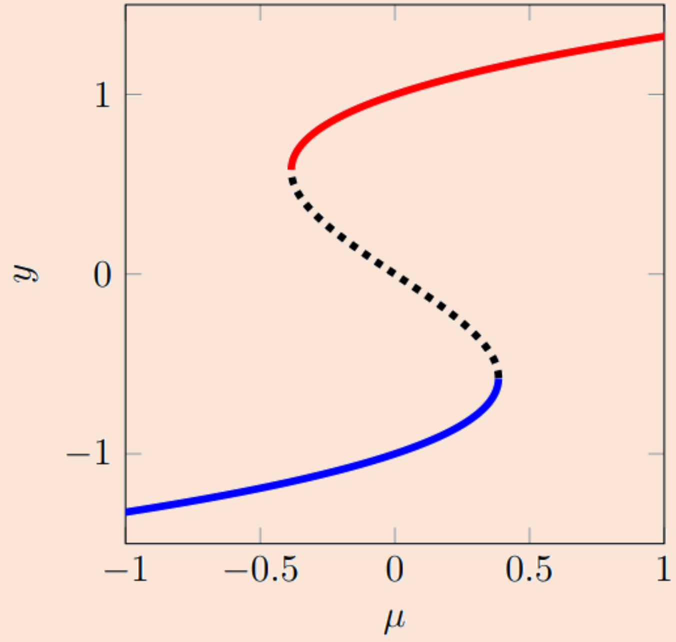


source: Steffen et al, 2018

Mathematics

Tipping points ↔ Bifurcations

$$\frac{dy}{dt} = f(y, \mu)$$



What about spatially extended systems?

Robbin Bastiaansen
r.bastiaansen@uu.nl

bastiaansen.github.io

- Background in (Applied) Mathematics

- 2015-2019:

  PhD @ Leiden University on *Pattern Formation* and *Desertification*

  (with Arjen Doelman, Martina Chirilus-Bruckner & Max Rietkerk)

- Since JAN 2020:

  PostDoc @ IMAU, Utrecht University on *Climate Sensitivity and Response*

  (with Anna von der Heydt & Henk Dijkstra)

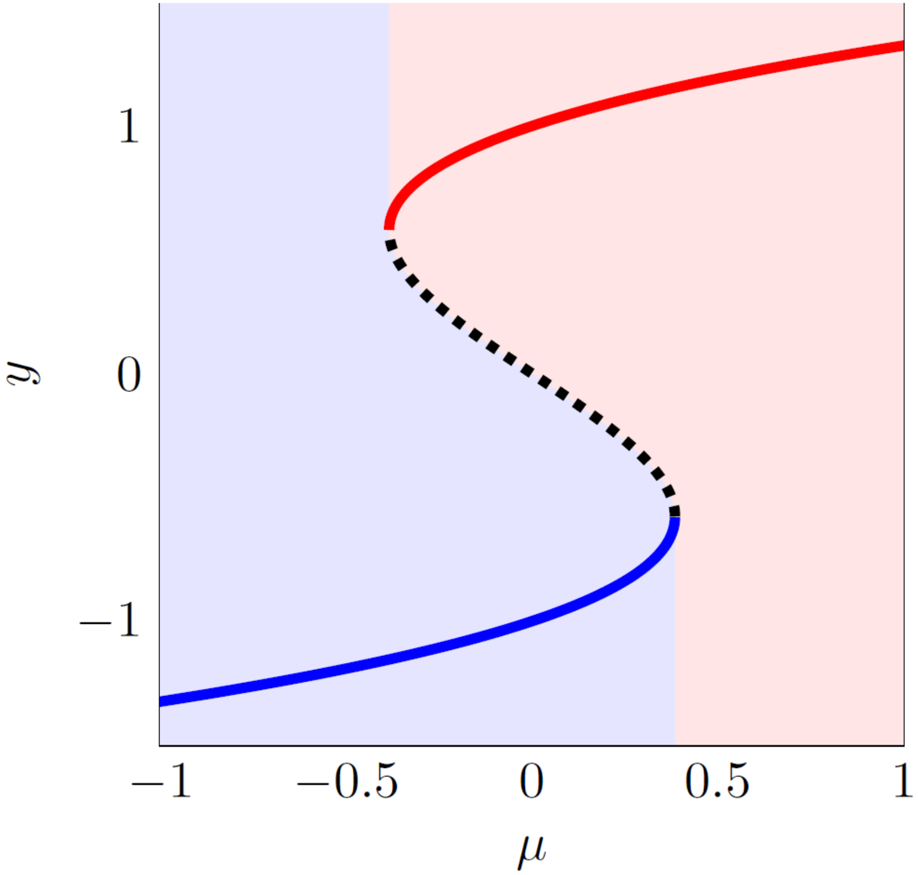
  Work within H2020 project TiPES: Tipping Points in the Earth System

Part 0: Tipping in ODEs

Tipping in ODEs (1)

Canonical example:

$$\frac{dy}{dt} = y(1 - y^2) + \mu$$



Concrete example: Global Energy Balance Model

$$\frac{dT}{dt} = Q\big(1 - \alpha(T)\big) - \varepsilon\sigma_0 T^4 + \mu$$

Classic Literature

[Holling, 1973]

[Noy-Meier, 1975]

[May, 1977]

Tipping

[Ashwin et al, 2012]

Bifurcation-Tipping : Basin disappears

Noise-Tipping : Forced outside Basin

Rate-Tipping : *(more complicated)*

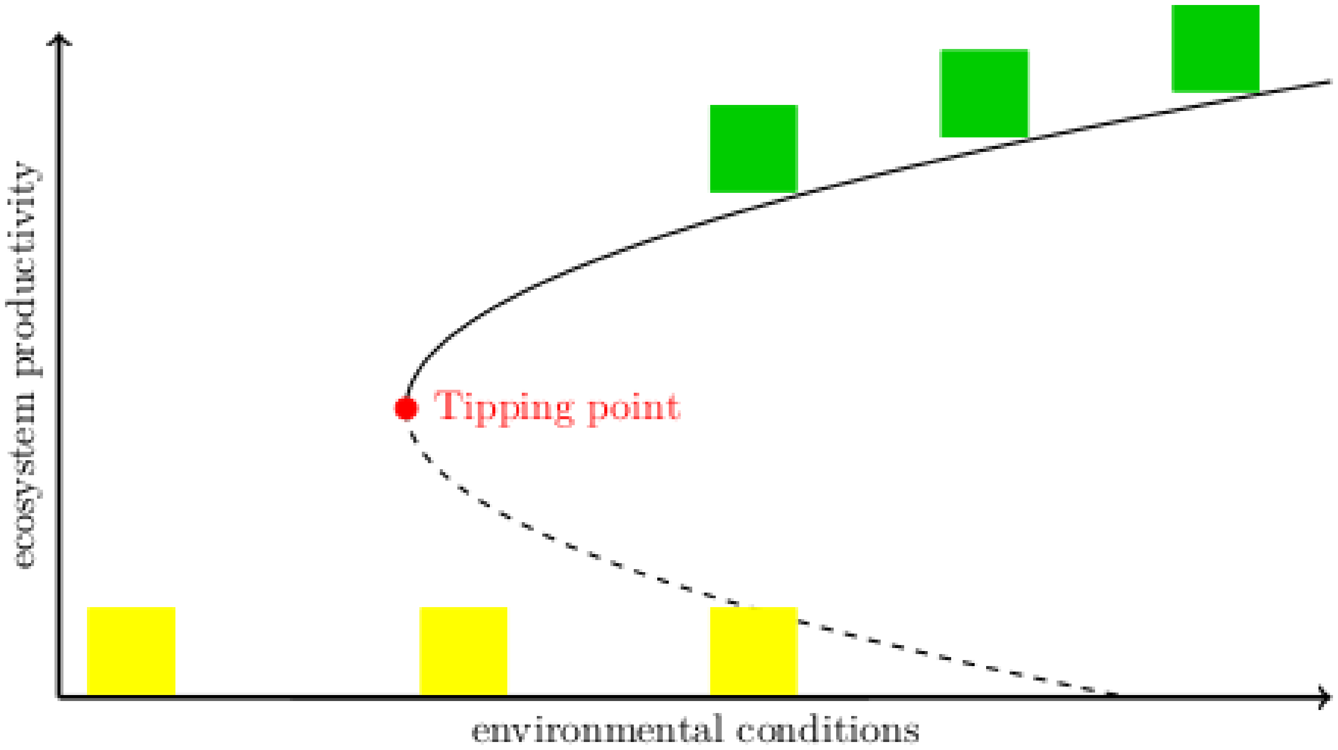
Tipping in ODEs (2)

Two components:

$$\begin{cases} \dfrac{du}{dt} = f(u,v) \\ \dfrac{dv}{dt} = g(u,v) \end{cases}$$

includes common models:

- Predetor-Prey
- Activator-Inhibitor



Examples of tipping in ODEs include:

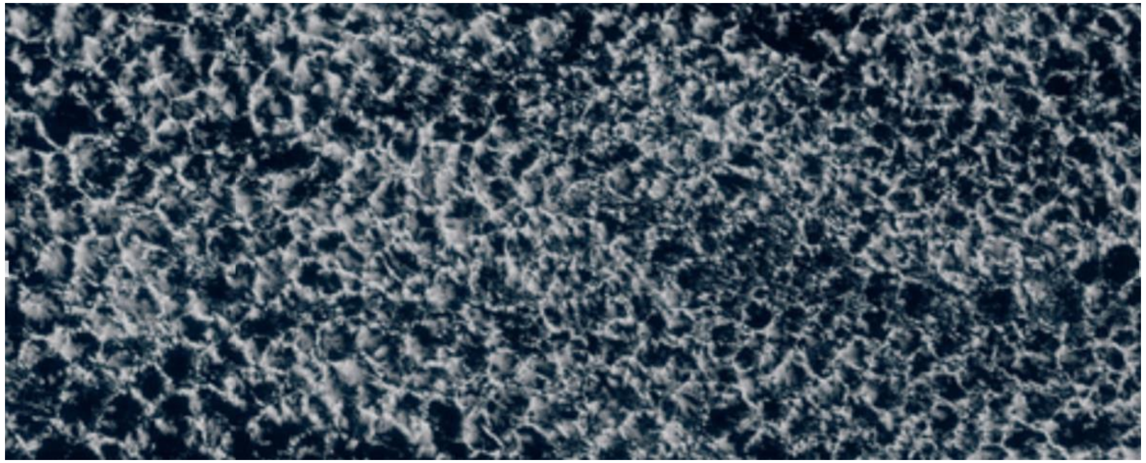
- Forest-Savanna bistability
- Deep ocean exchange
- Cloud formation
- Ice sheet melting
- Turbidity in shallow lakes

Part 1:
Turing Patterns

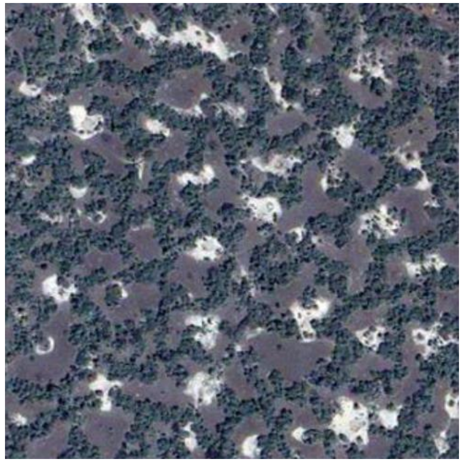
image courtesy: National Science Foundation

Examples of spatial Patterning

mussel beds

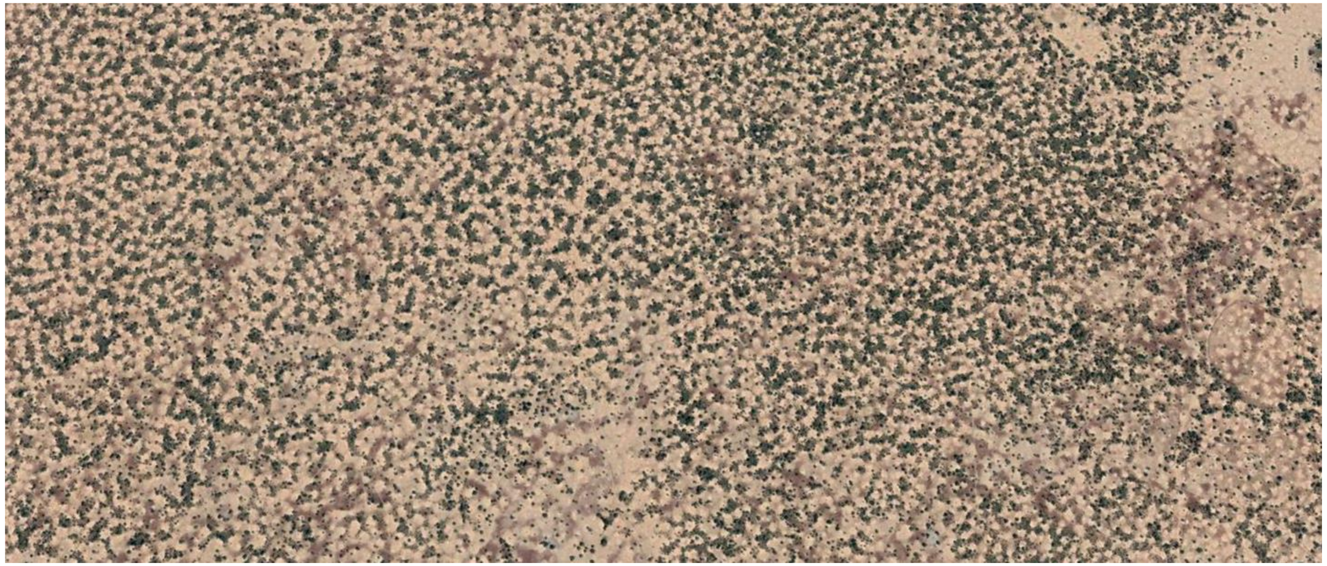


clouds



savannas

melt ponds



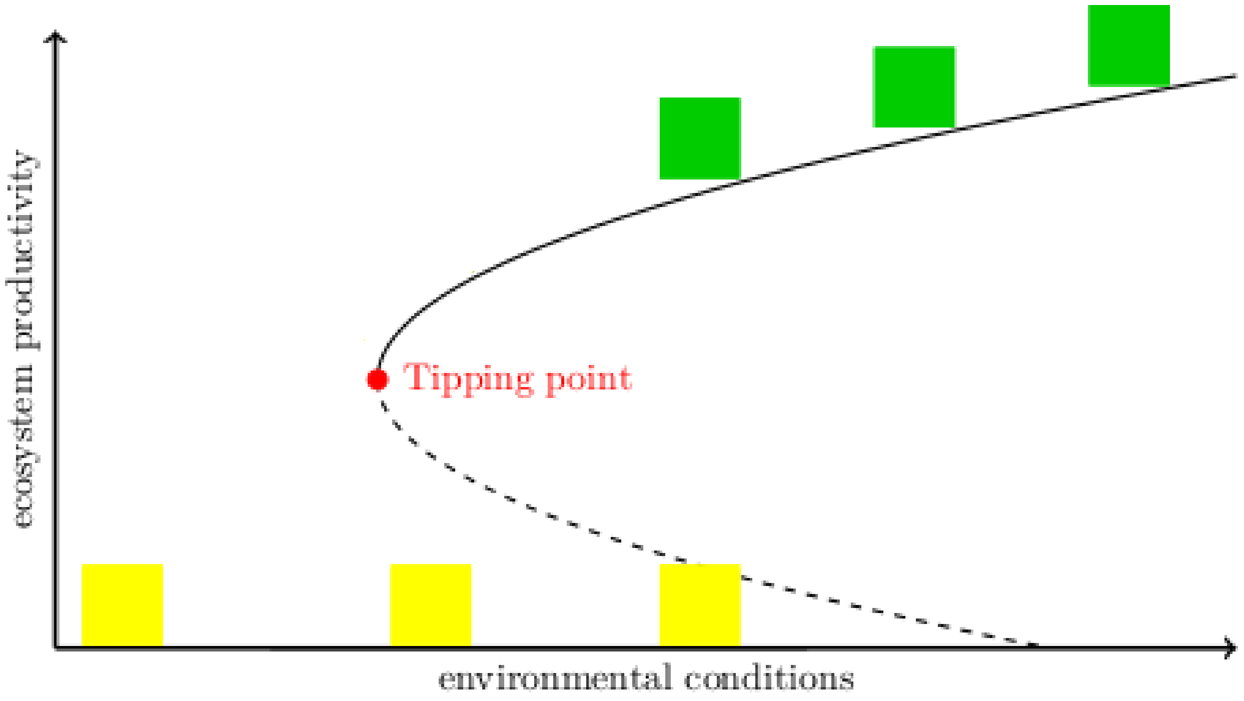
drylands

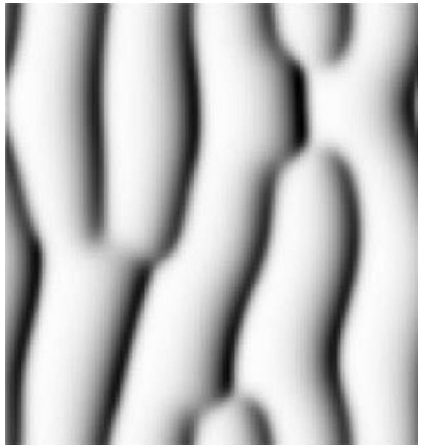
Patterns in models

Add spatial transport:

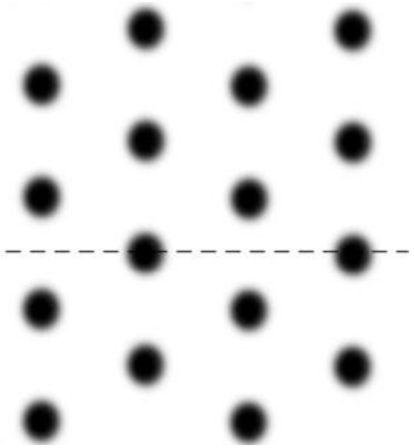
Reaction-Diffusion equations:

$$\begin{cases} \dfrac{du}{dt} = f(u,v) + D_u\, \Delta u \\ \dfrac{dv}{dt} = g(u,v) + D_v\, \Delta v \end{cases}$$

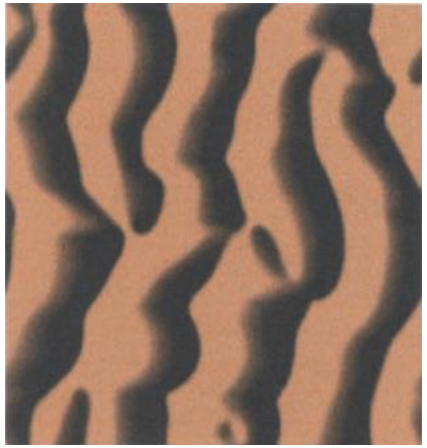




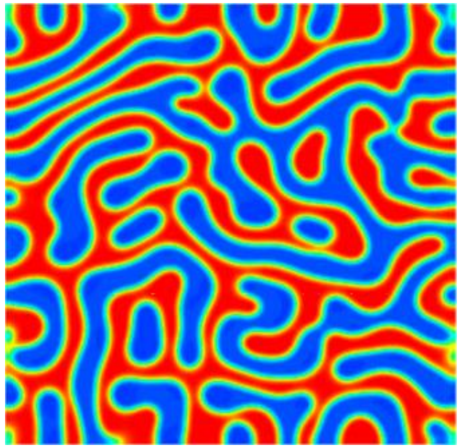
[Klausmeier, 1999]



[Gilad et al, 2004]

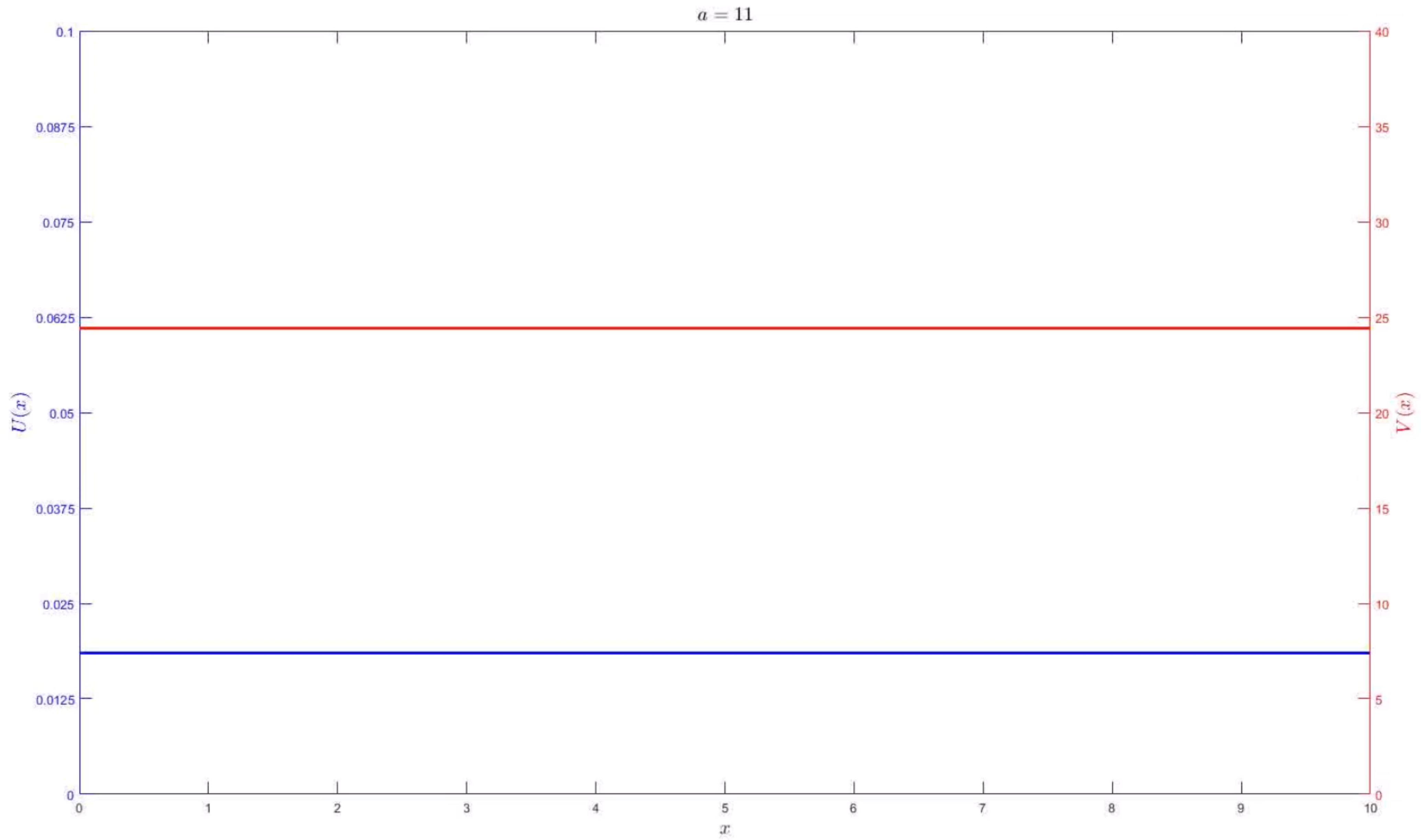


[Rietkerk et al, 2002]



[Liu et al, 2013]

Behaviour of PDEs



Turing patterns

$$\begin{cases} \frac{du}{dt} = f(u,v) + D_u \, \Delta u \\ \frac{dv}{dt} = g(u,v) + D_v \, \Delta v \end{cases}$$

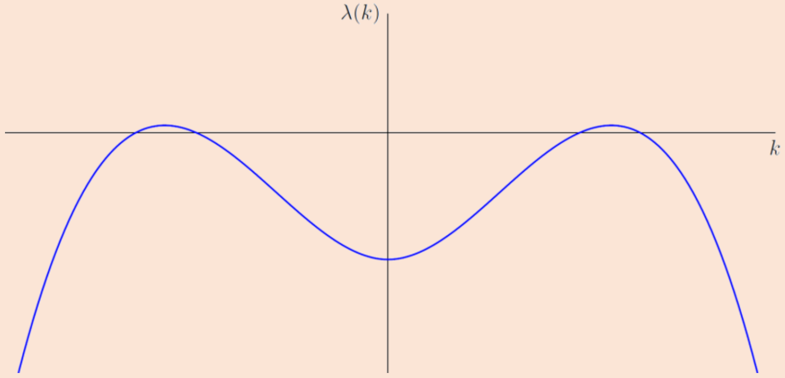
Turing bifurcation

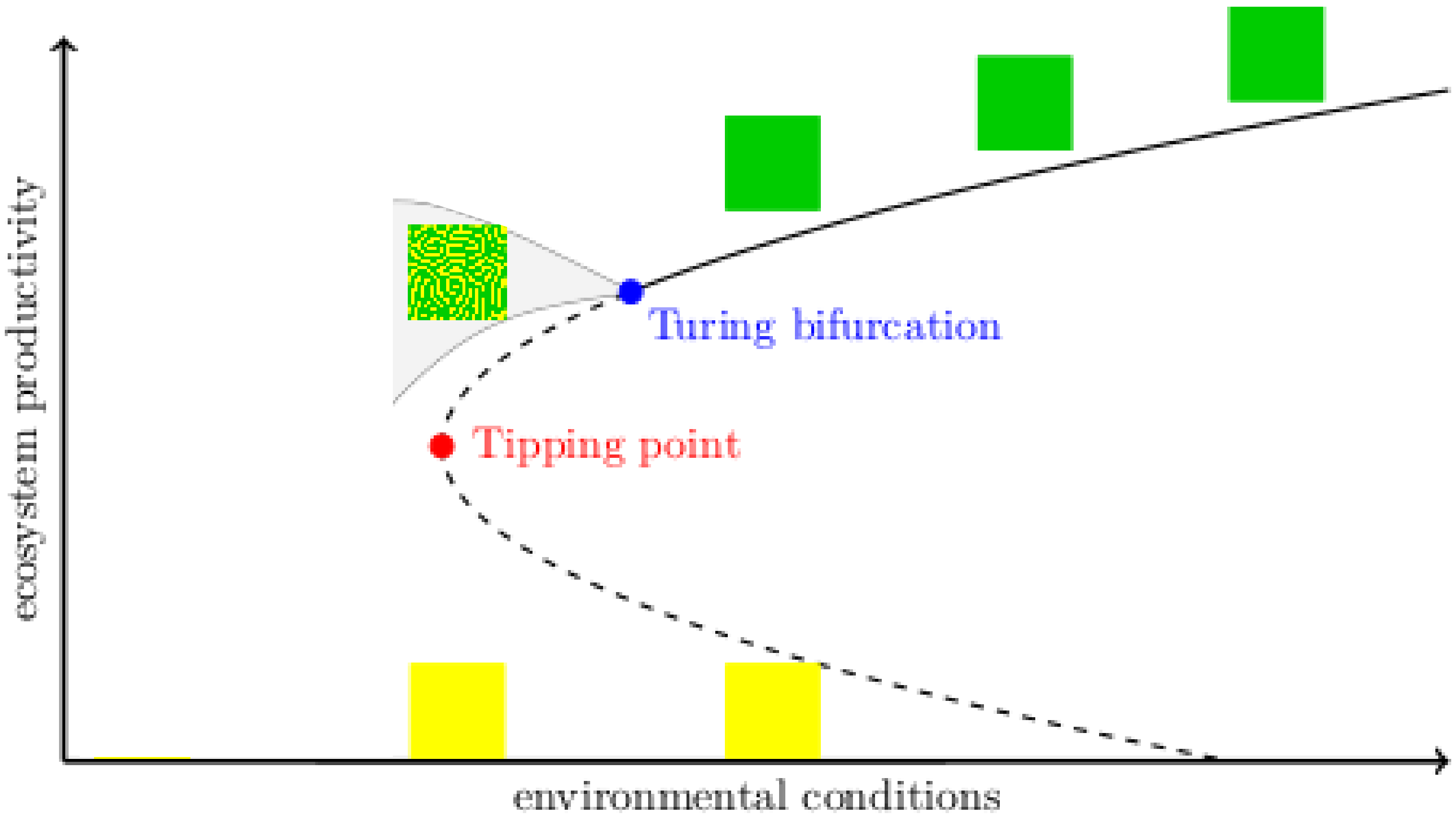
Instability to non-uniform perturbations

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u_* \\ v_* \end{pmatrix} + e^{\lambda t} e^{ikx} \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix}$$

→ Dispersion relation

$$\lambda(k) = \cdots$$





Weakly non-linear analysis

Ginzburg-Landau equation / Amplitude Equation

& Eckhaus/Benjamin-Feir-Newel criterion

[Eckhaus, 1965; Benjamin & Feir, 1967; Newell, 1974]

Busse balloon

$$\begin{cases} \dfrac{du}{dt} = f(u,v) + D_u\,\Delta u \\ \dfrac{dv}{dt} = g(u,v) + D_v\,\Delta v \end{cases}$$

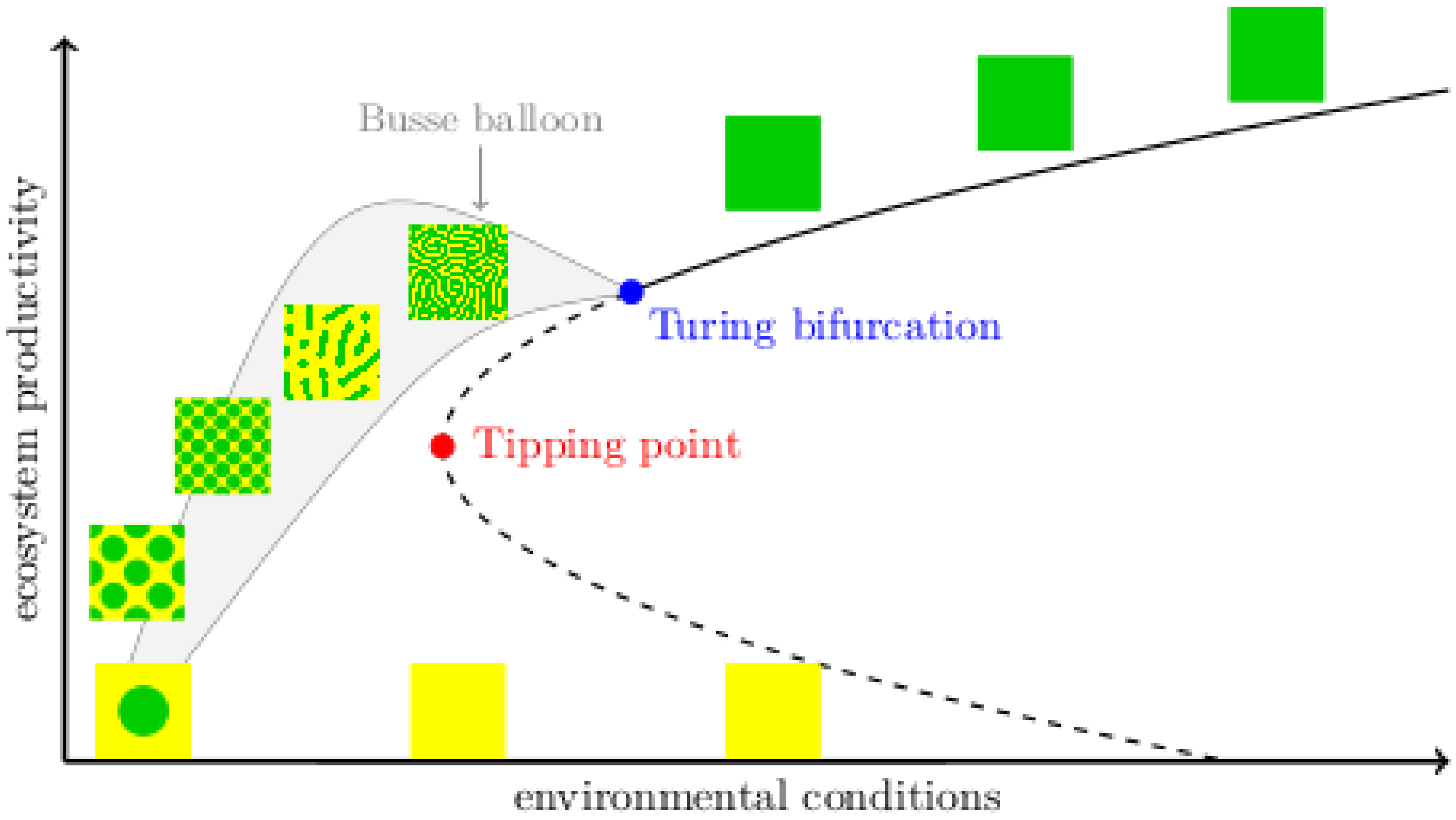
Busse balloon

A model-dependent shape in *(parameter, observable)* space that indicates all stable patterned solutions to the PDE.

Construction Busse balloon

Via numerical continuation

few general results on the shape of Busse balloon

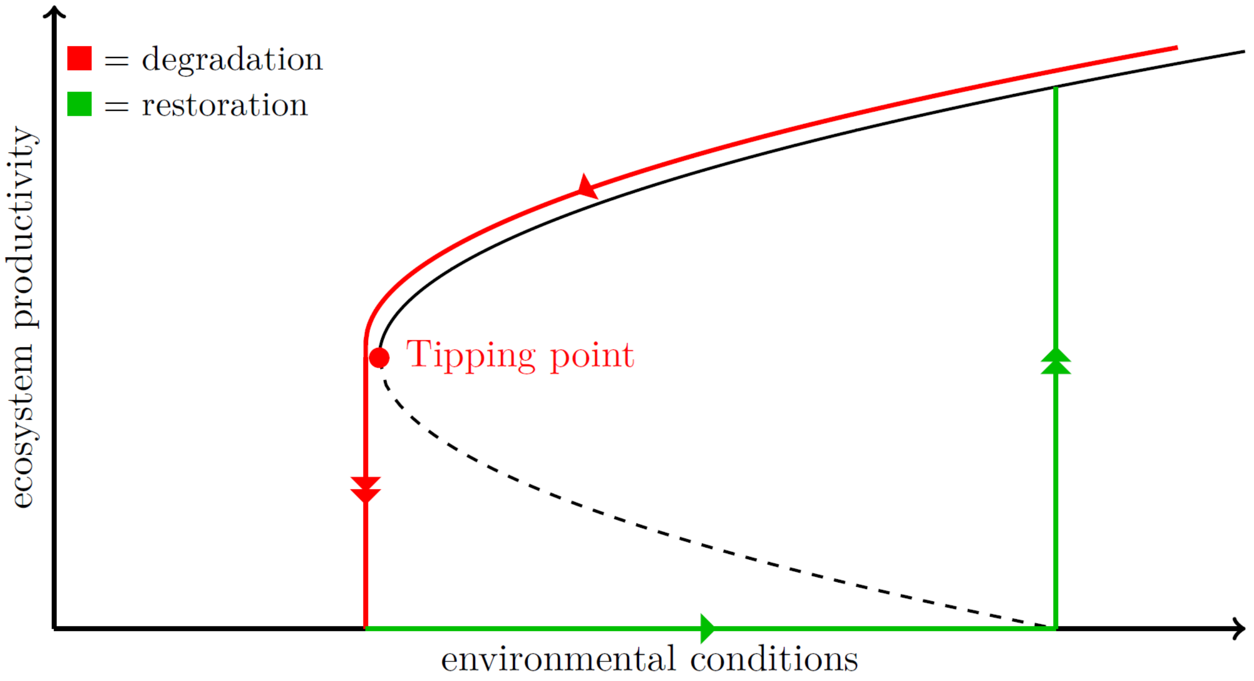


Busse balloon

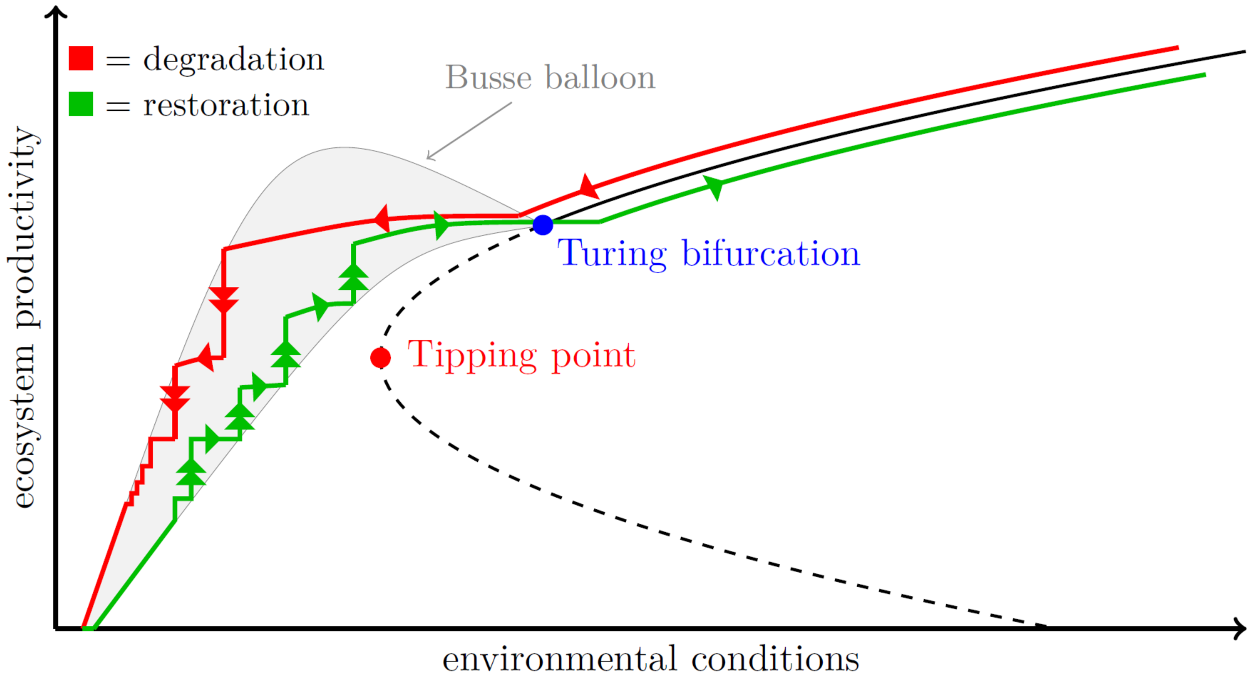
Idea originates from thermal convection

[Busse, 1978]

Tipping of (Turing) patterns

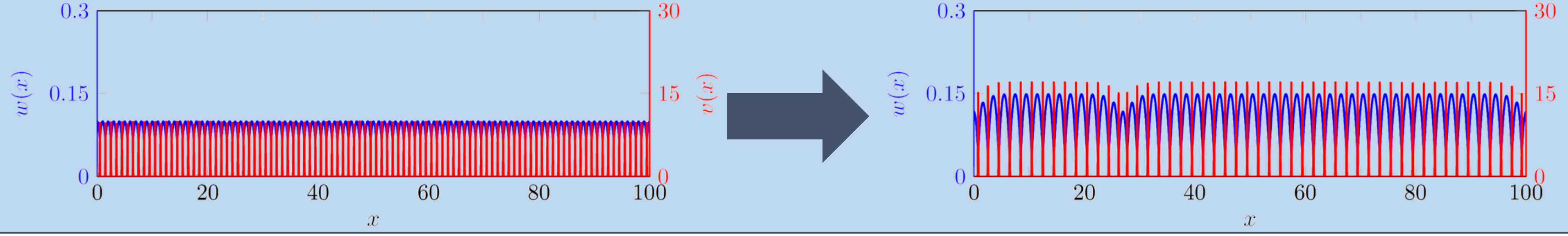


Classic tipping



Tipping of patterns

Degradation of patterns



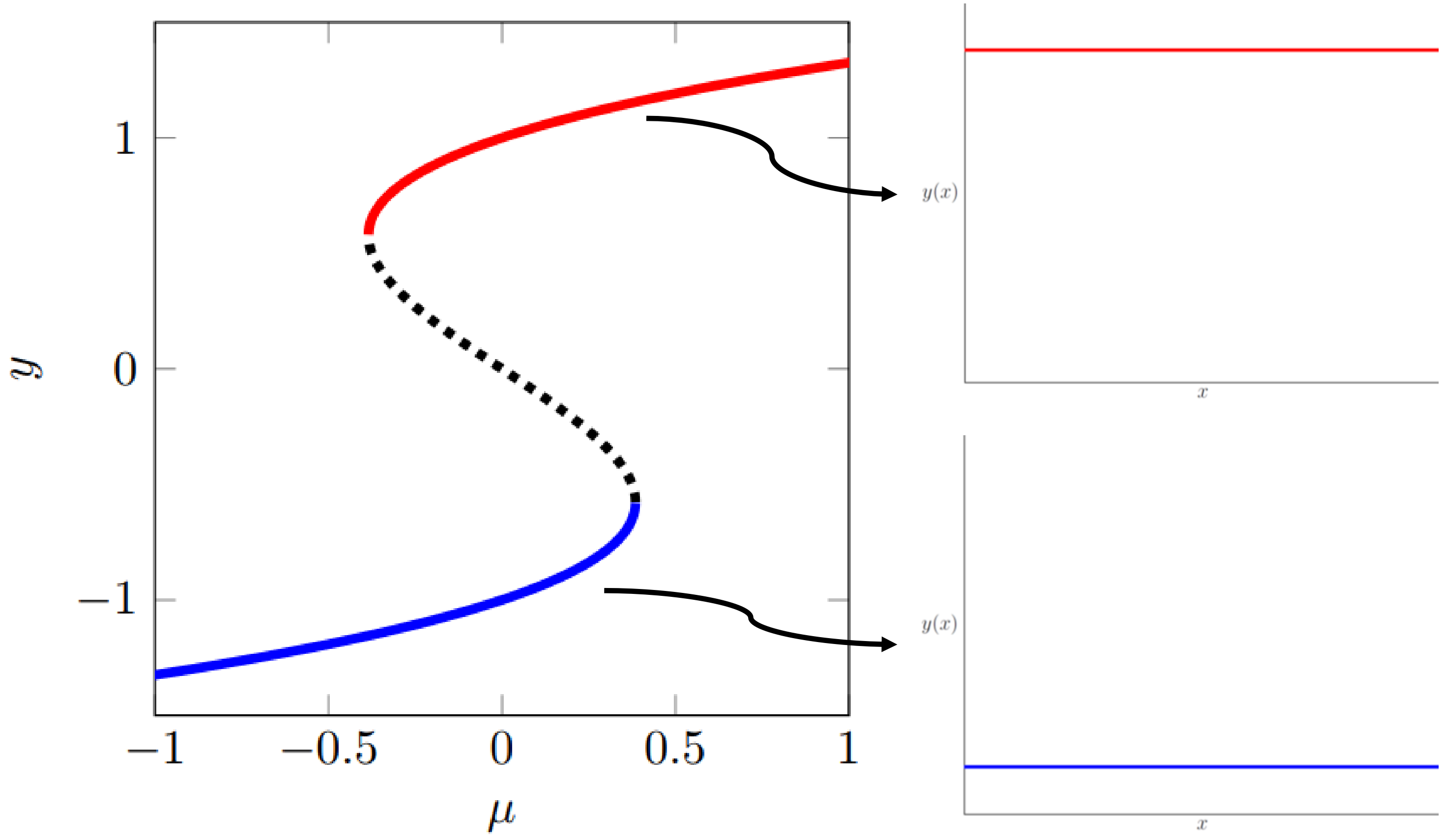
Part 2: Coexistence States

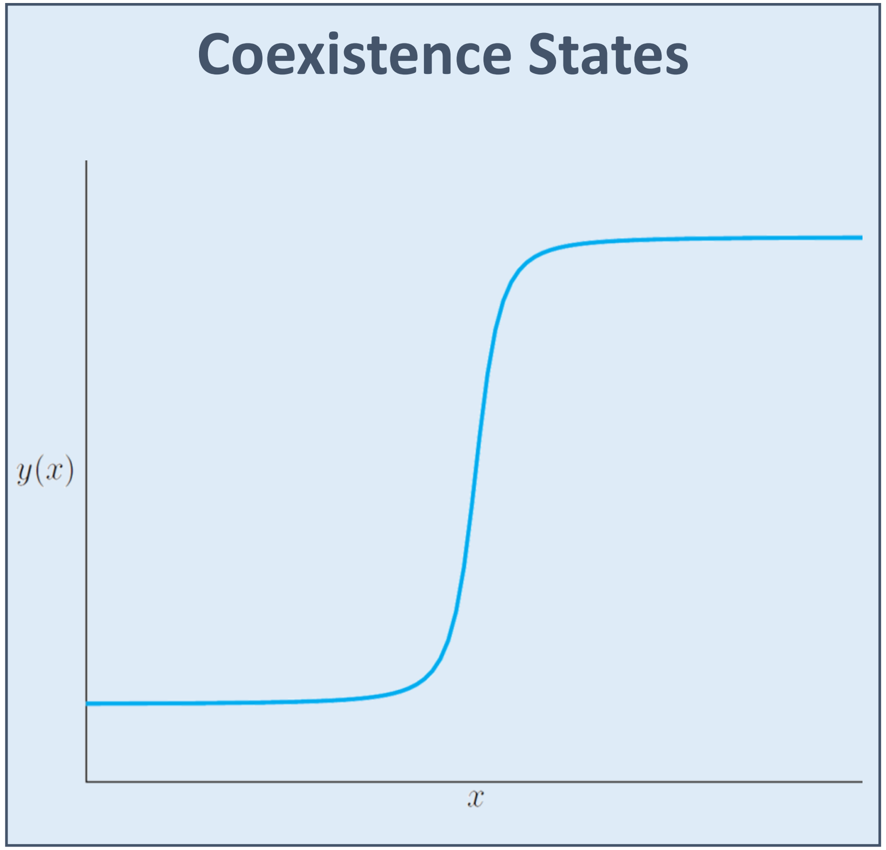
and spatial heterogeneities

Coexistence states

Bistable (Allen-Cahn/Nagumo) equation:

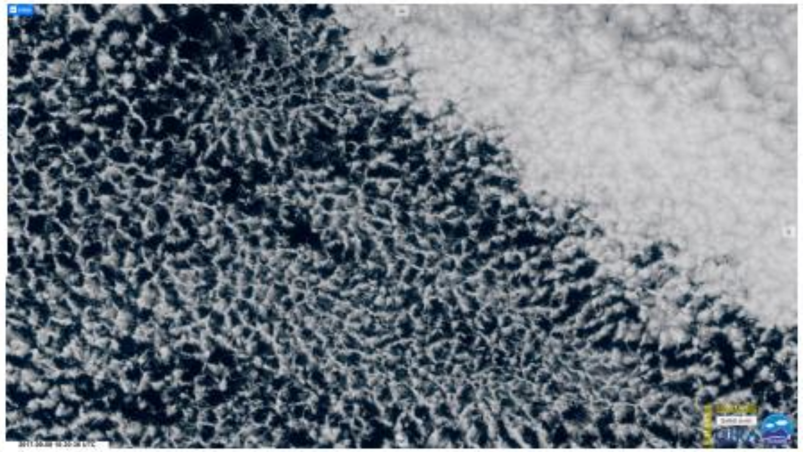
$$\frac{\partial y}{\partial t} = y(1 - y^2) + \mu + D\,\frac{\partial^2 y}{\partial x^2}$$





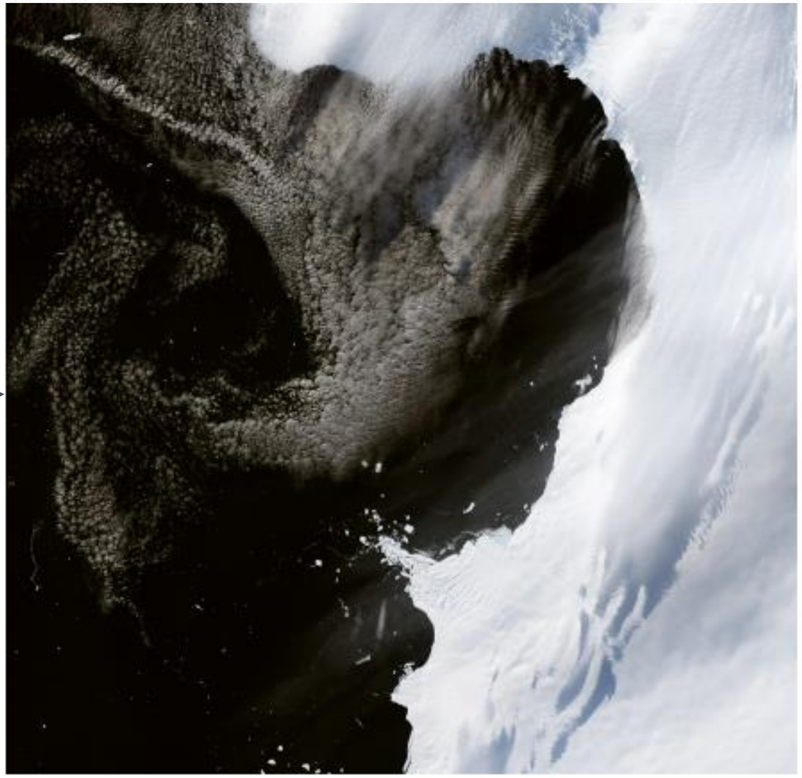
Examples of Coexistence States

tropical forest
& savanna
ecosystems
[Google Earth]



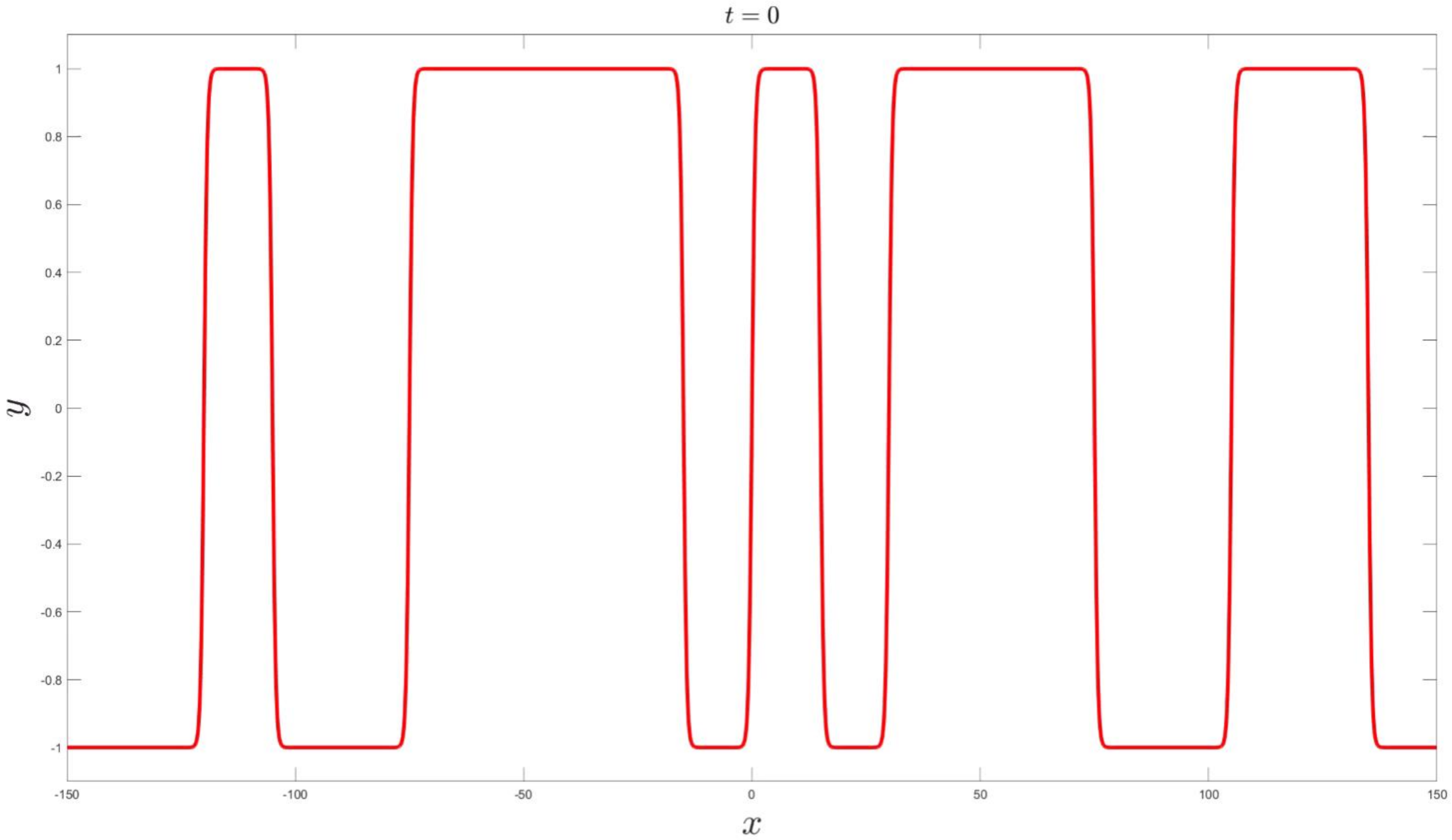
types of
stratocumulus
clouds
[RAMMB/CIRA SLIDER]

sea-ice & water
at Eltanin Bay
[NASA's Earth observatory]



algae bloom
in Lake St. Clair
[NASA's Earth observatory]

Dynamics of $\frac{\partial y}{\partial t} = D\ \frac{\partial^2 y}{\partial x^2} + y(1 - y^2) + \mu$

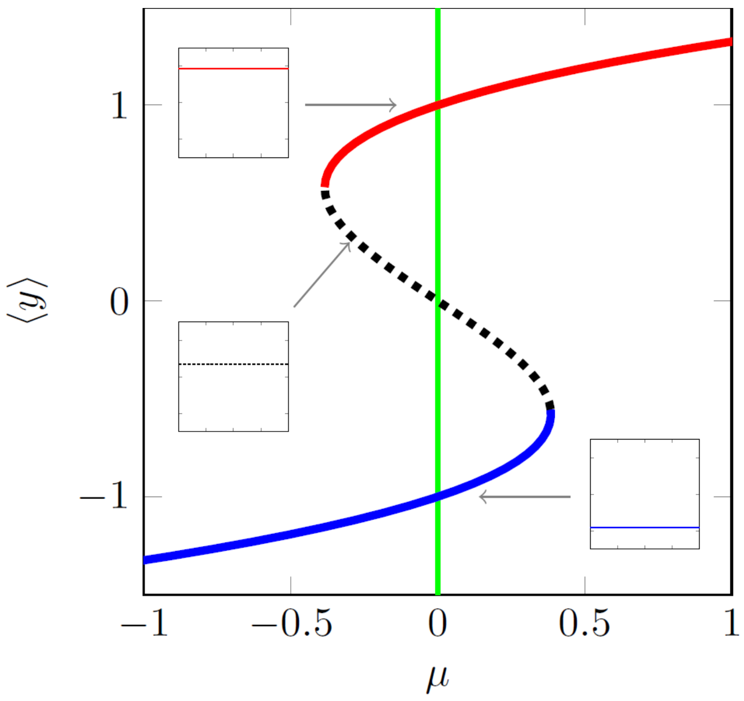


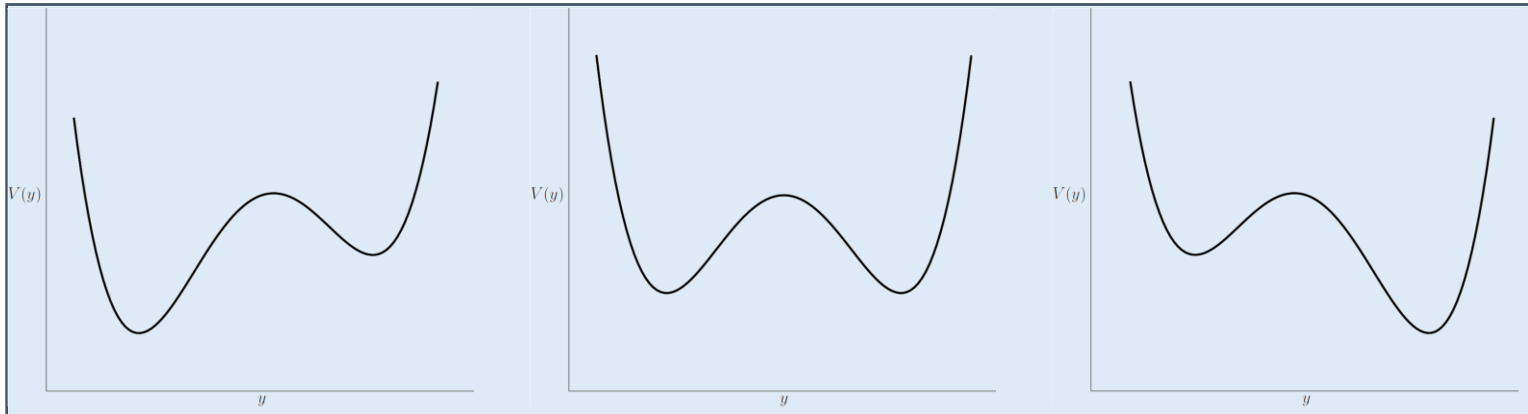
Front Dynamics

$$\frac{\partial y}{\partial t} = D\,\frac{\partial^2 y}{\partial x^2} + f(y;\mu)$$

Potential function $V(y;\mu)$:

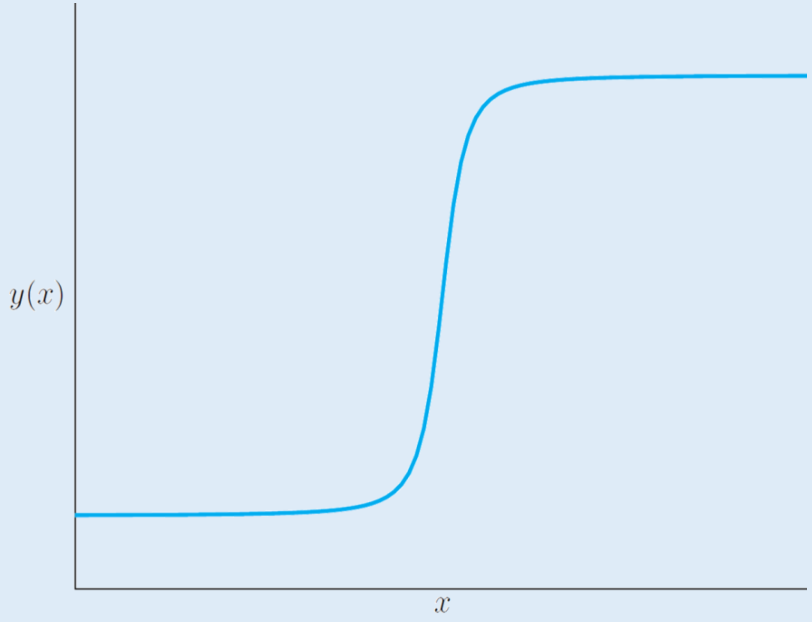
$$\frac{\partial V}{\partial y}(y;\mu) = -f(y;\mu)$$





moves left | stationary | moves right

Maxwell Point $\mu_{maxwell}$



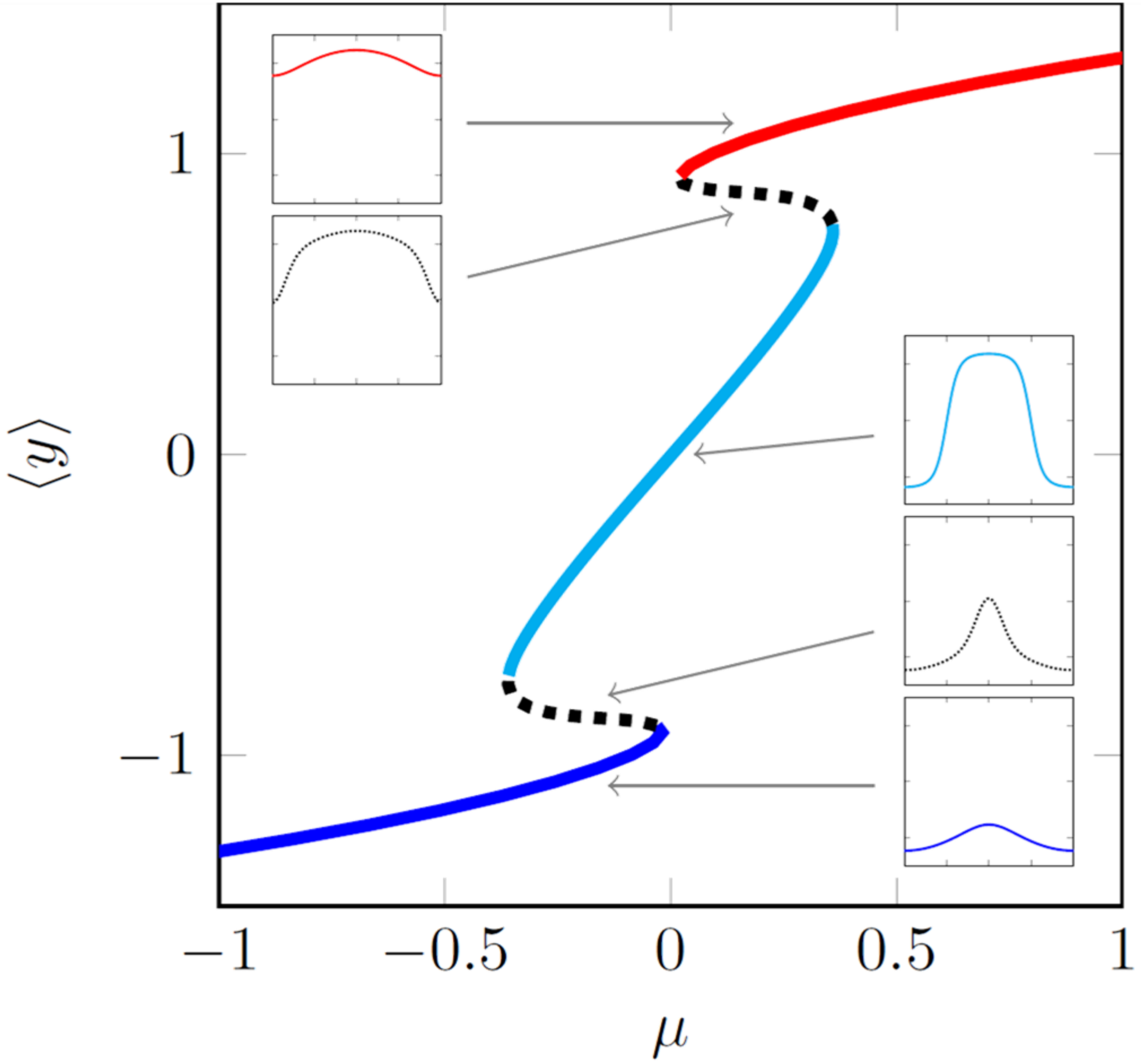
Adding Spatial Heterogeneity

$$\frac{\partial y}{\partial t} = D\frac{\partial^2 y}{\partial x^2} + f(y, x; \mu)$$

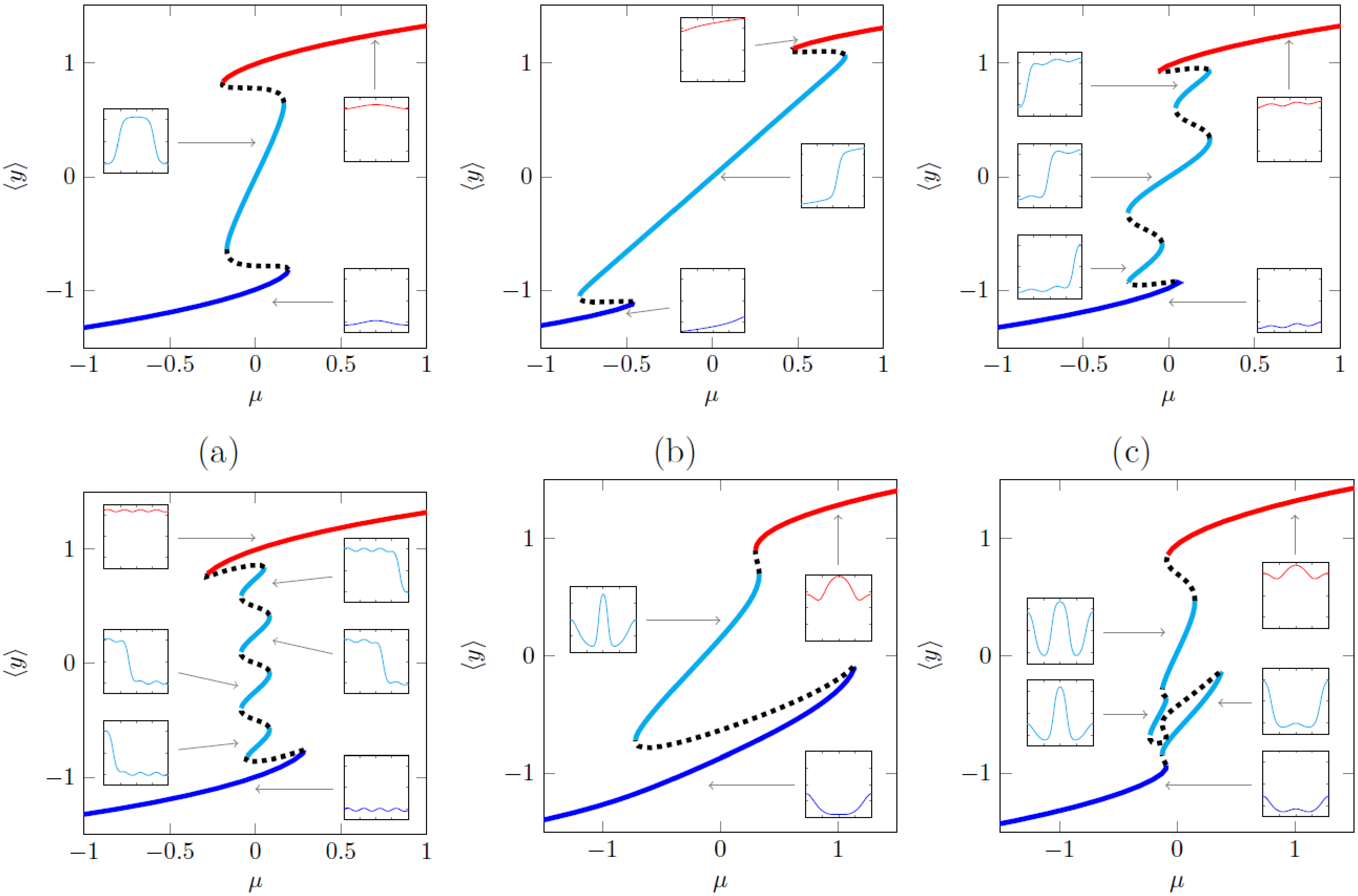
Now, the **local** difference in potentials determines the front movement

New behaviour:

- Multi-fronts can be stationary
- Maxwell point is smeared out

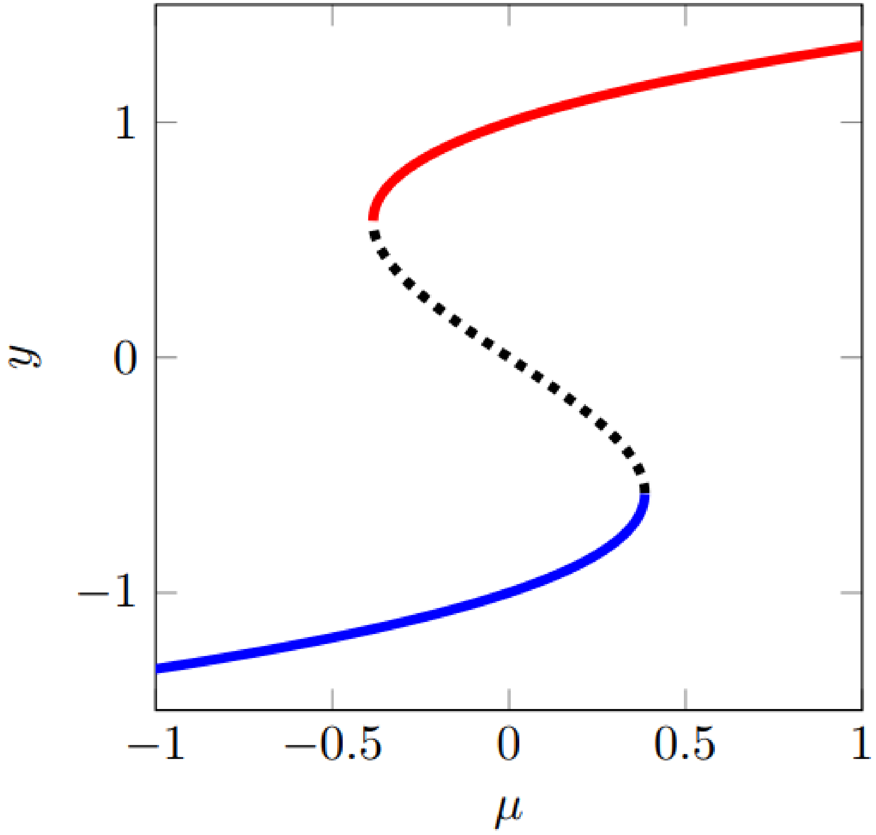


Other Spatial Heterogeneities



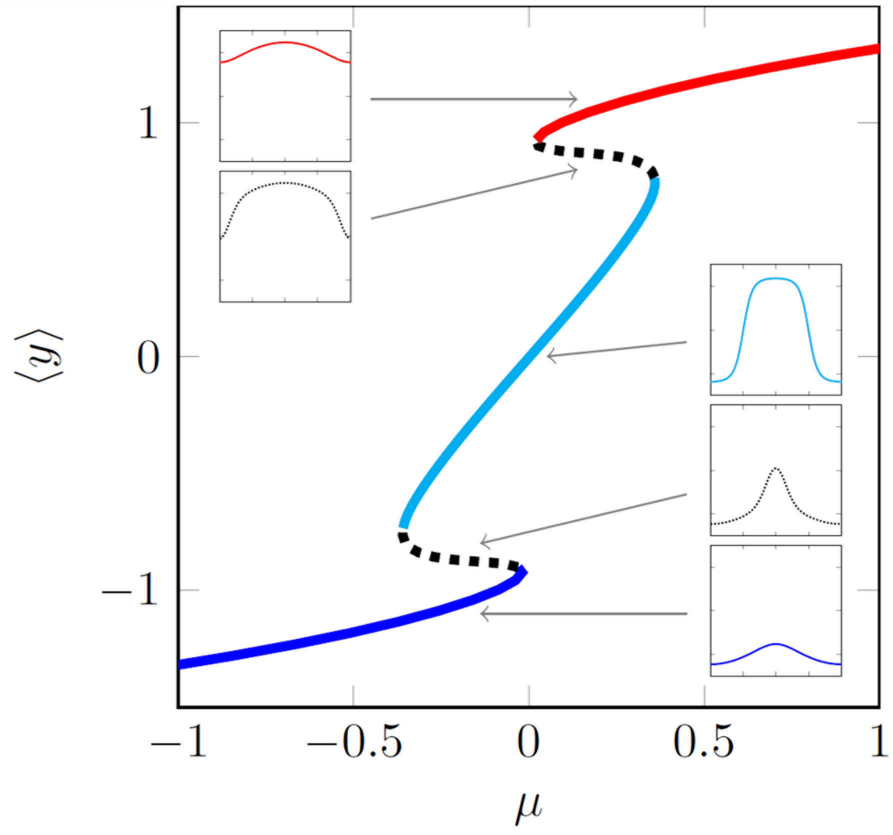
Fragmented Tipping

Classic (ODE)



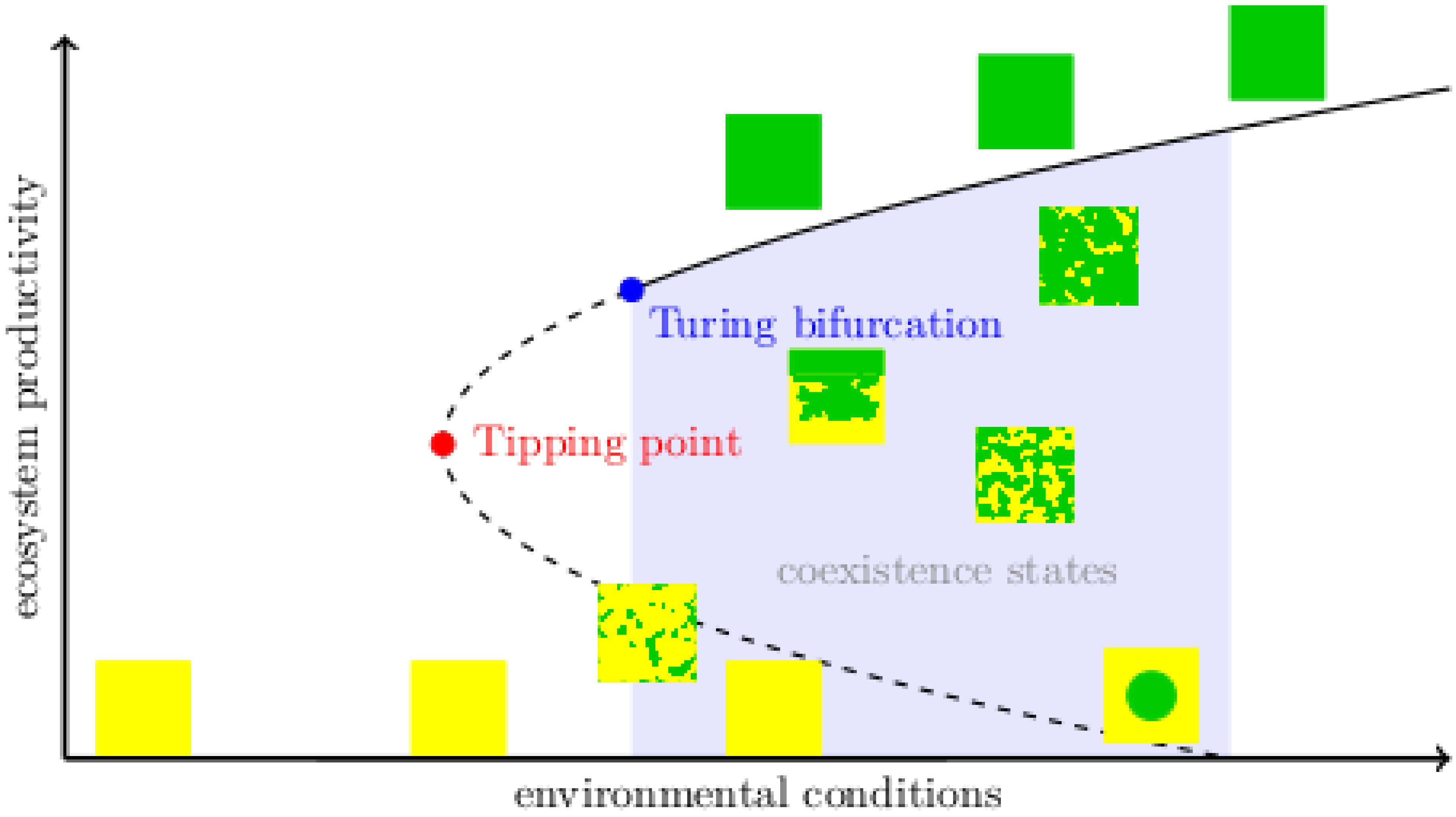
Tipping leads to full reorganisation

Heterogeneous PDE



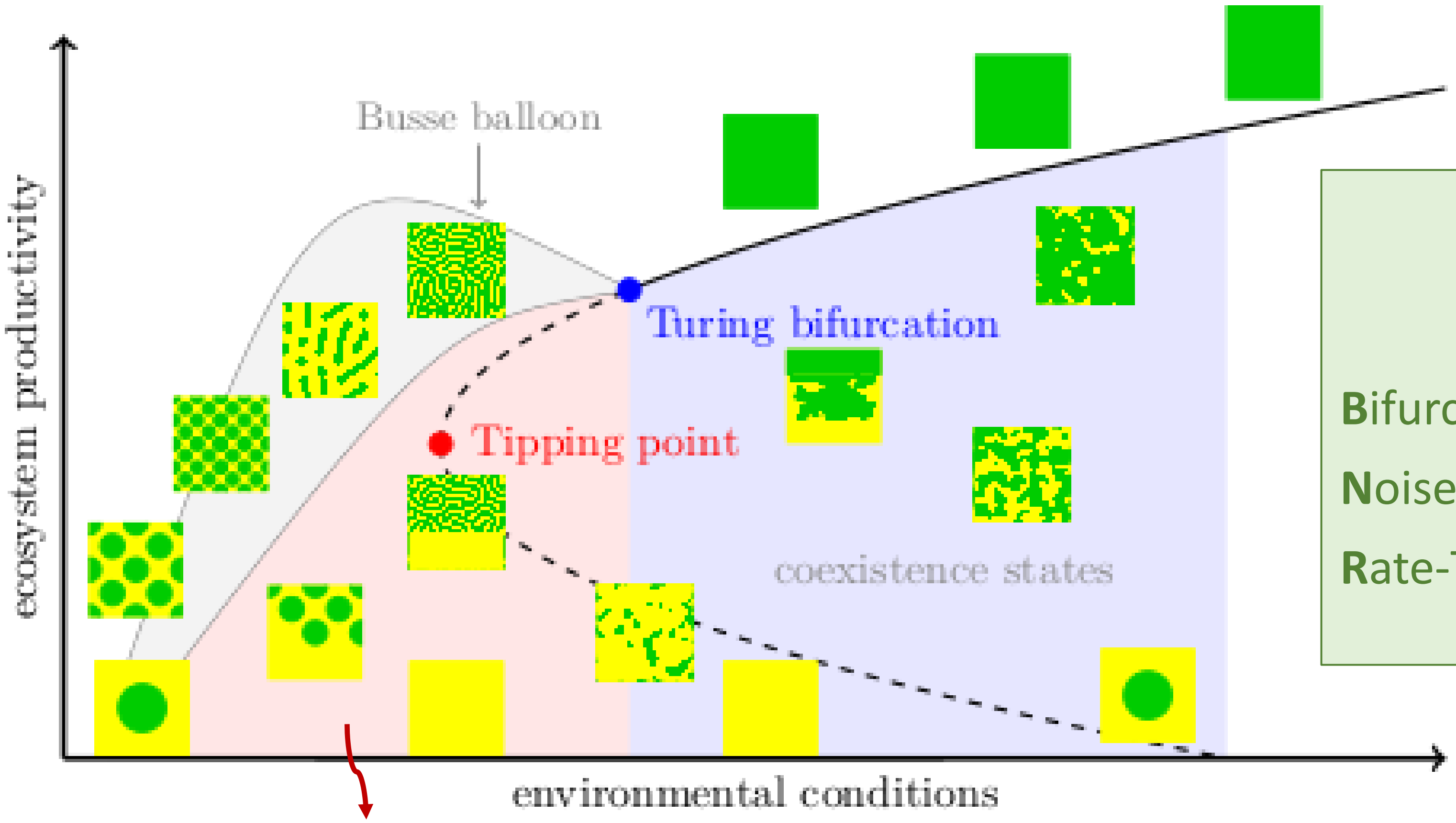
Partial tipping events possible:
Only part of the domain reorganises

Coexistence states in bifurcation diagram



Part 3:
Tipping in Spatially Extended Systems

“Bifurcation Diagram” for spatially extended systems



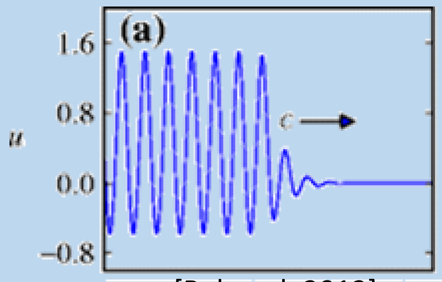
Tipping
[Ashwin et al, 2012]

Bifurcation-Tipping : Basin disappears

Noise-Tipping : Forced outside Basin

Rate-Tipping : *(more complicated)*

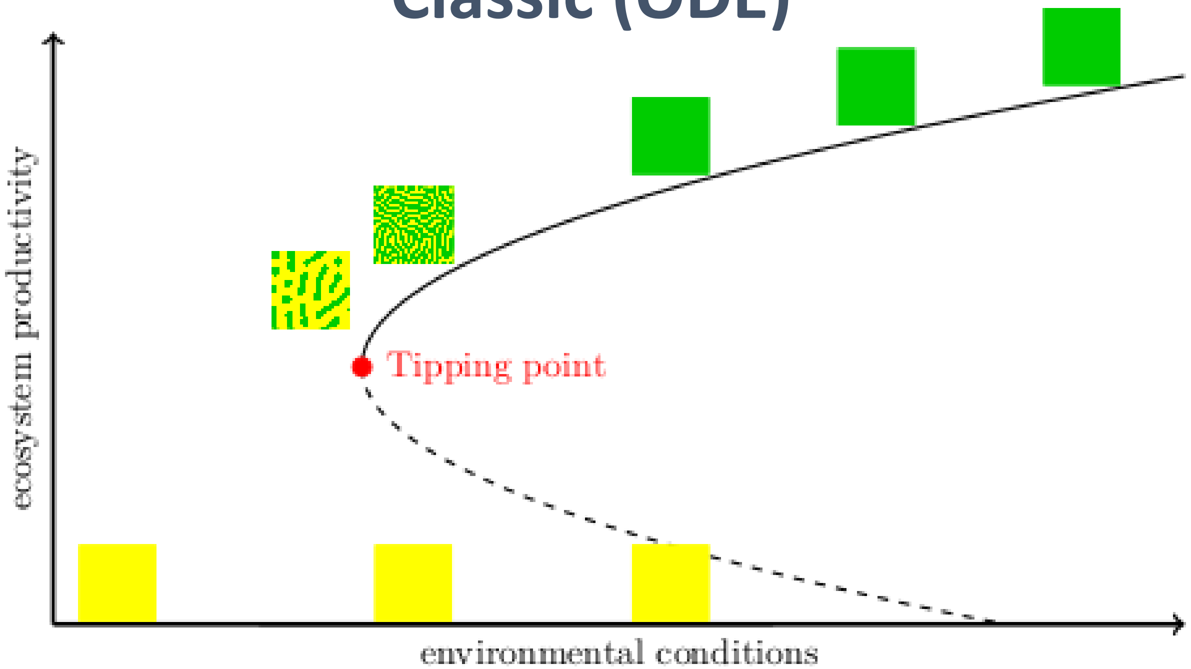
Coexistence states between patterned and uniform states also exist



[Bel et al, 2012]

What if the system tips?

Classic (ODE)



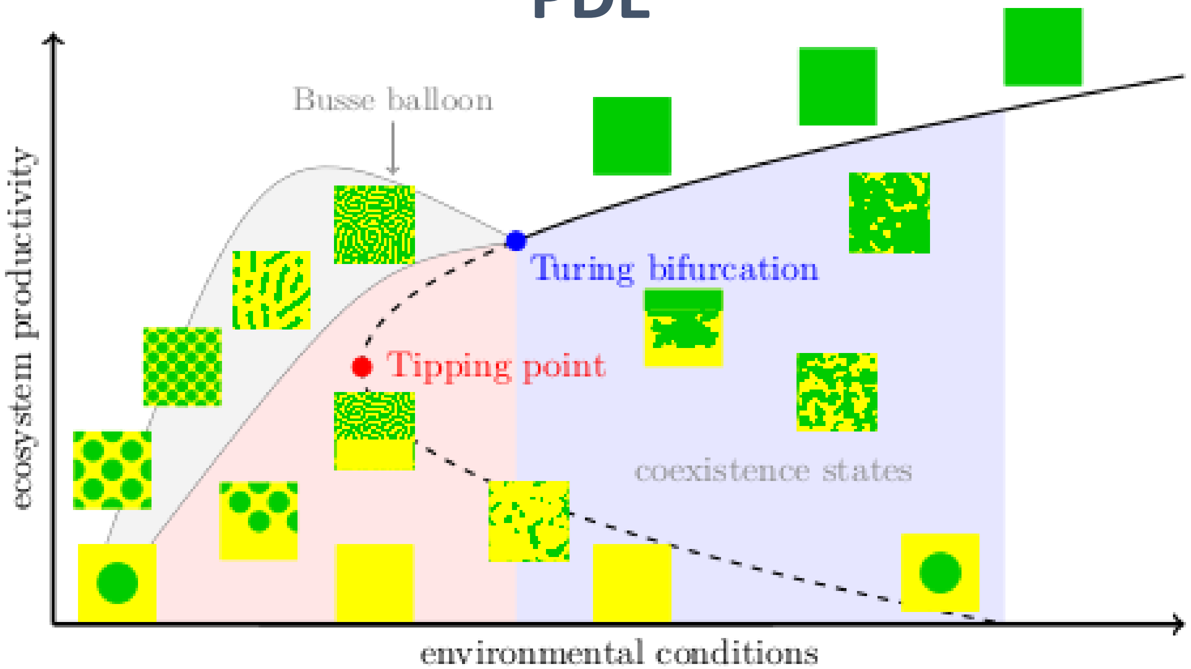
Crossing a Tipping Point:

→ Always full reorganization

Early Warning Signals

signal for <u>WHEN</u>

PDE



Crossing a bifurcation:

Now also possible:

→ Spatial reorganization (Turing patterns)

→ Fragmented tipping (coexistence states)

Early Warning Signals

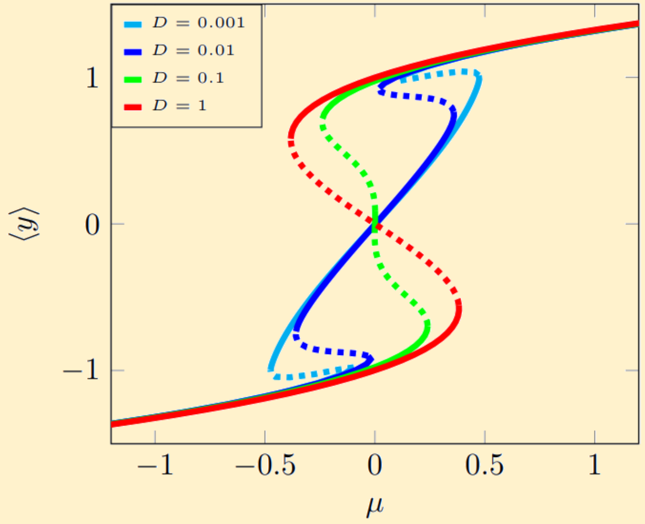
need to signal <u>WHEN</u> & <u>WHAT</u>

Do systems always behave like this? (a.k.a. the small print)

No.

Well-mixed systems

Spatially confined systems



→ Such systems (again) behave like ODEs ←

But even in other systems terms & conditions apply:
System-specific knowledge is required!

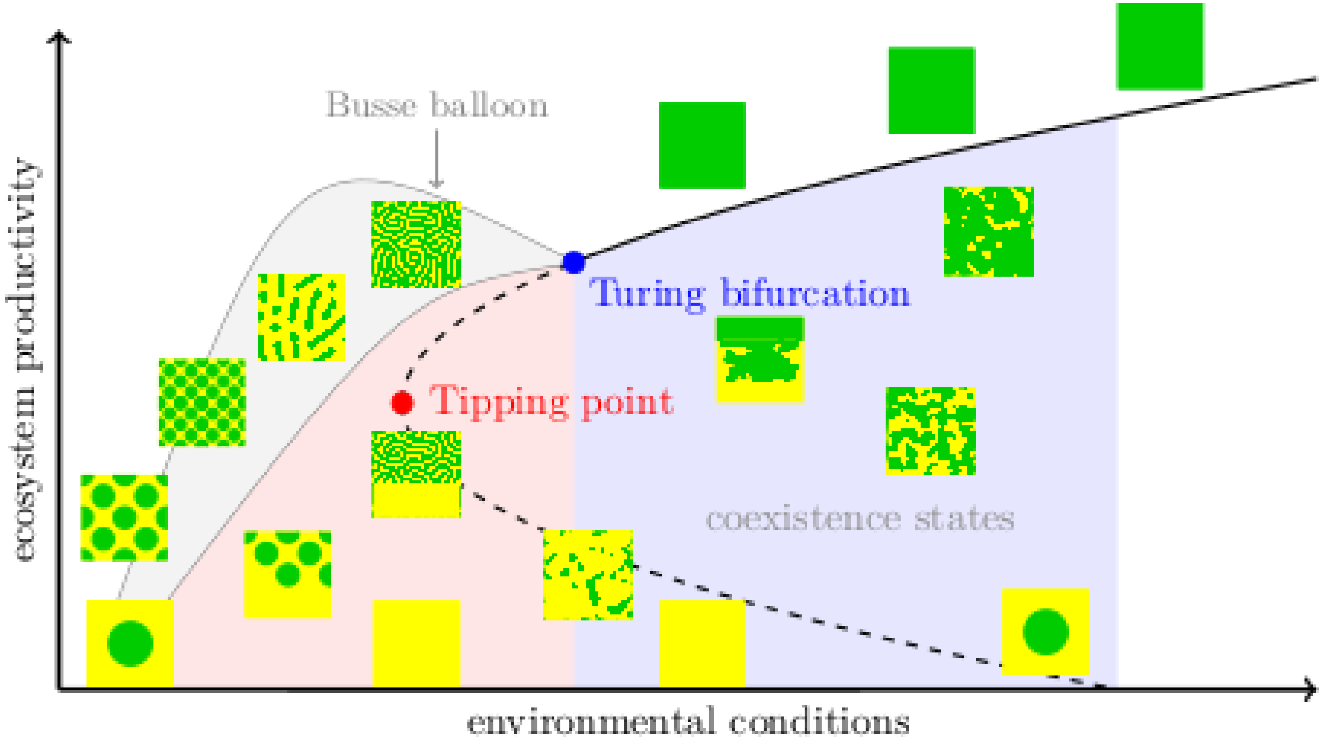
Summary

PDE dynamics richer:

- Turing Patterns
- Coexistence States

Tipping can be more subtle:

- Spatial reorganization
- Fragmented Tipping



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Bastiaansen, R., Dijkstra, H. A., & von der Heydt, A. S. (2021). Partial tipping in a spatially heterogeneous world. *arXiv preprint arXiv:2111.15566*.