TIPES
TIPPING POINTS IN THE EARTH SYSTEM

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Tipping in Spatially Extended Systems

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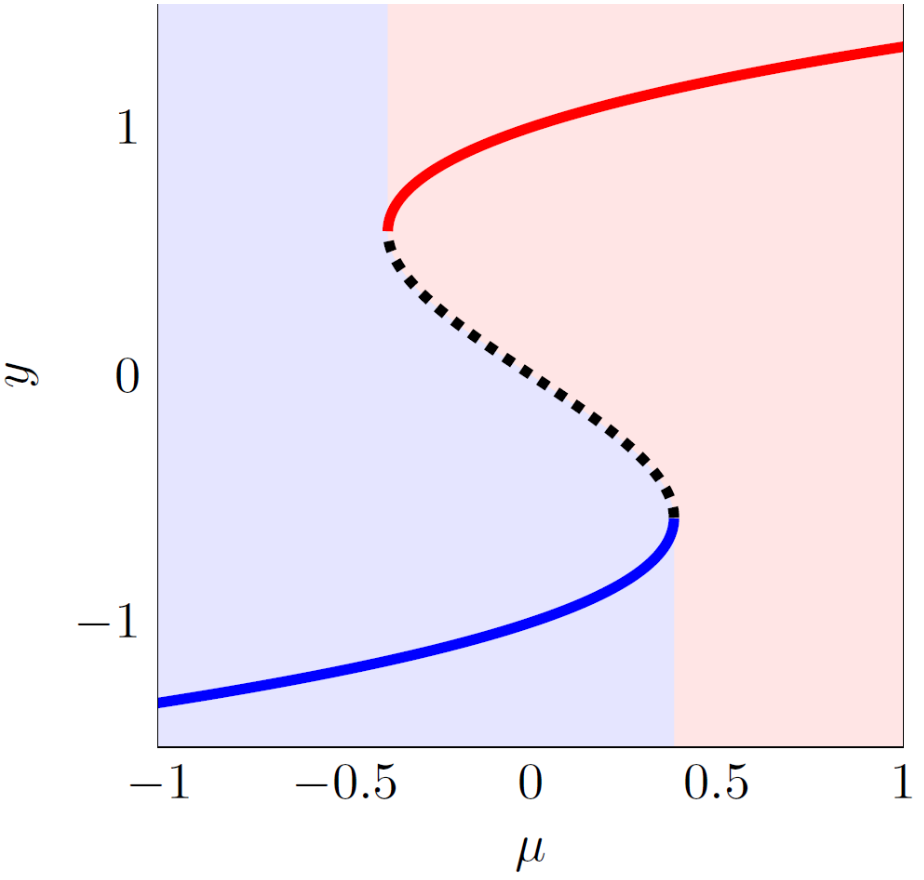
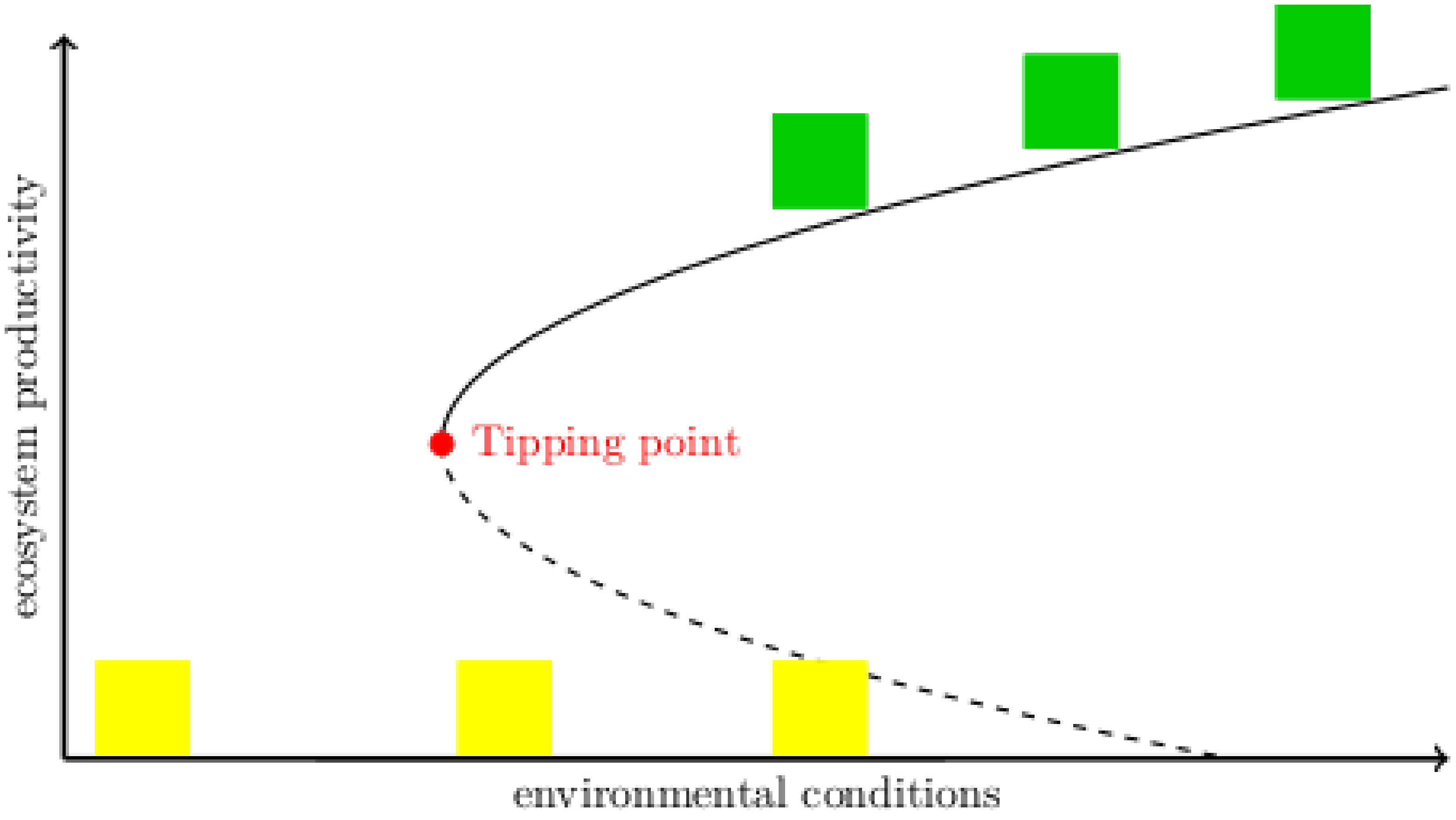
Classic Theory of Tipping





Canonical example:

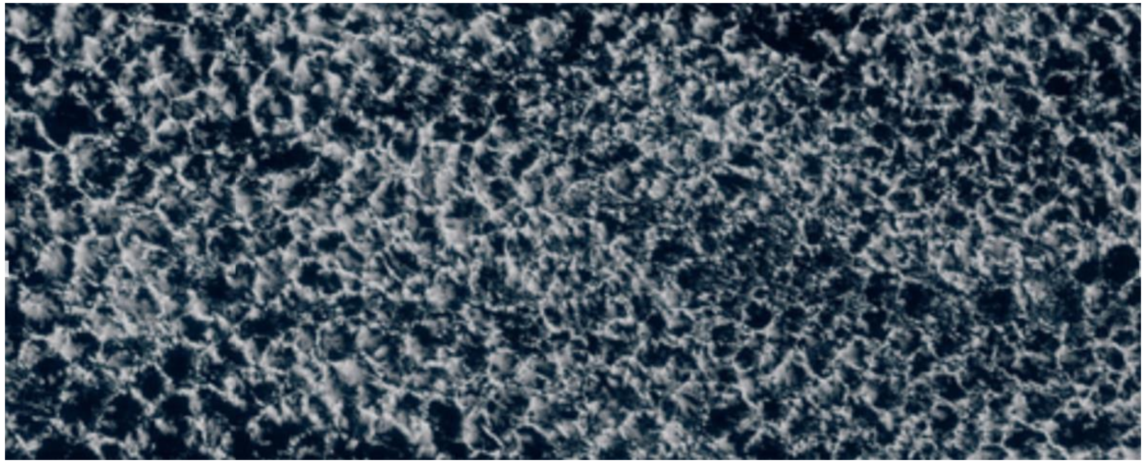
$$\frac{dy}{dt} = y(1 - y^2) + \mu$$

$$\frac{d\vec{y}}{dt} = f(\vec{y}; \mu)$$

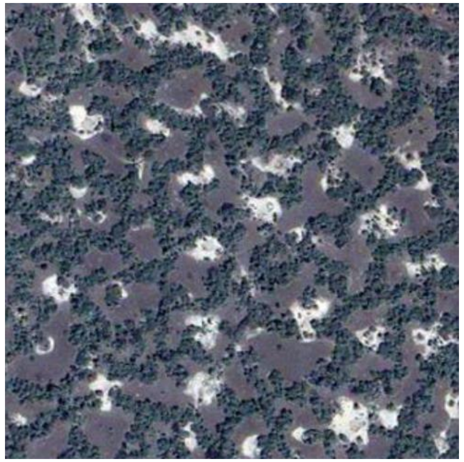
$$\begin{cases} \dfrac{du}{dt} = f(u, v; \mu) \\ \dfrac{dv}{dt} = g(u, v; \mu) \end{cases}$$

Examples of spatial patterning – regular patterns

mussel beds

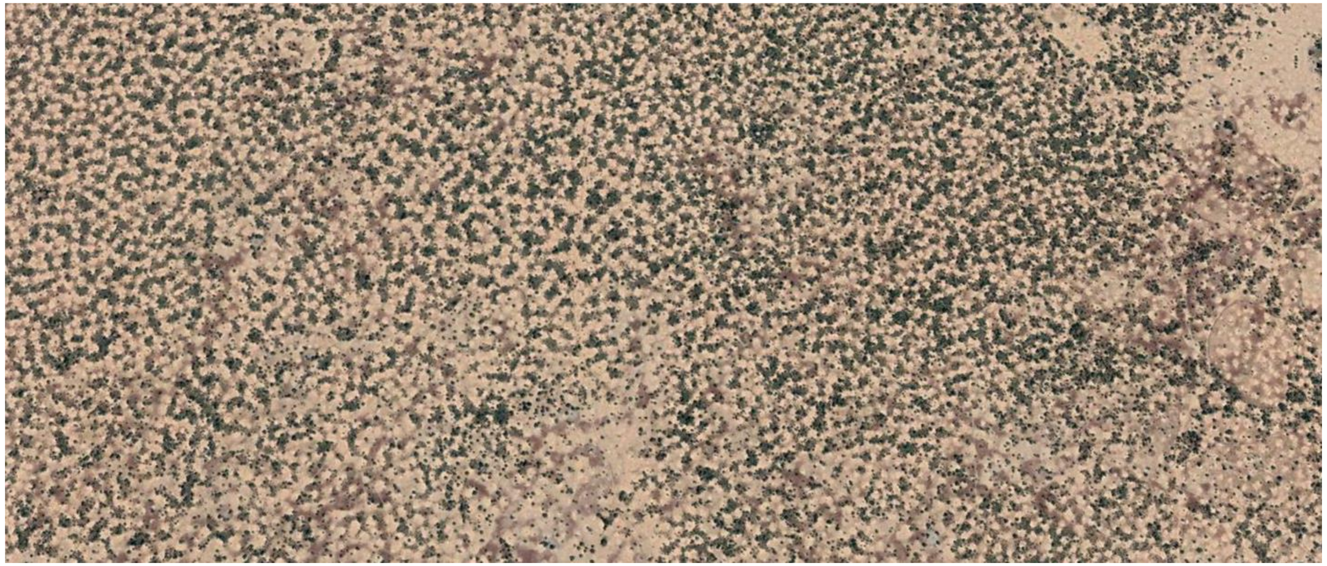


clouds



savannas

melt ponds

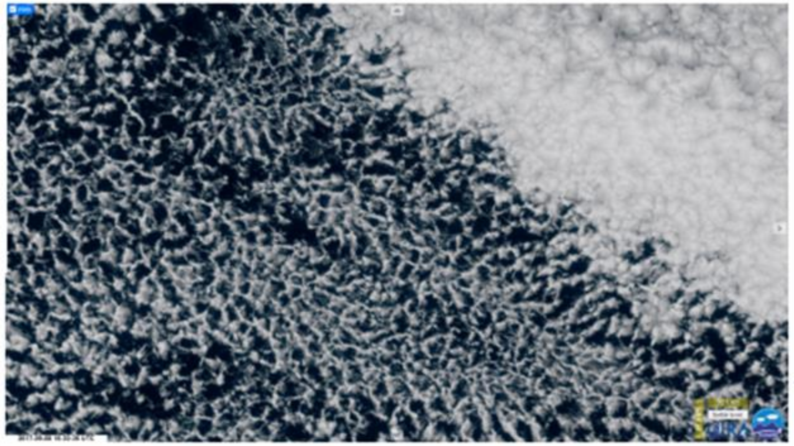


drylands

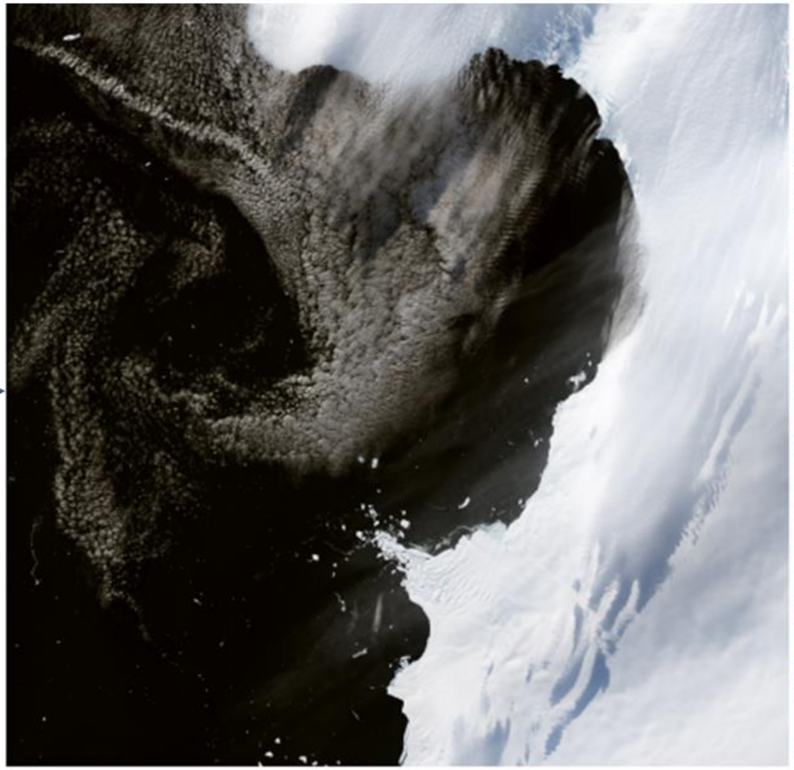
Examples of spatial patterning – spatial interfaces

tropical forest
& savanna
ecosystems
[Google Earth]

types of
stratocumulus
clouds
[RAMMB/CIRA SLIDER]



sea-ice & water
at Eltanin Bay
[NASA's Earth observatory]



algae bloom
in Lake St. Clair
[NASA's Earth observatory]

Part 1: Turing Patterns

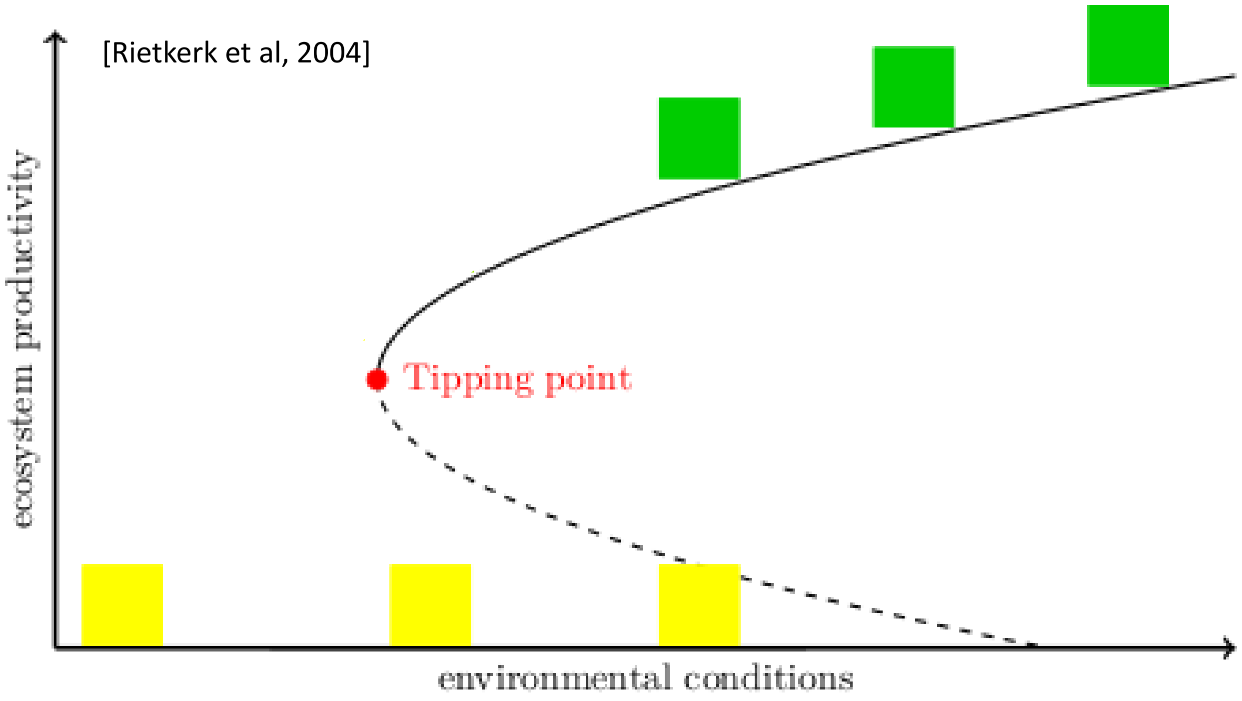
image courtesy: National Science Foundation

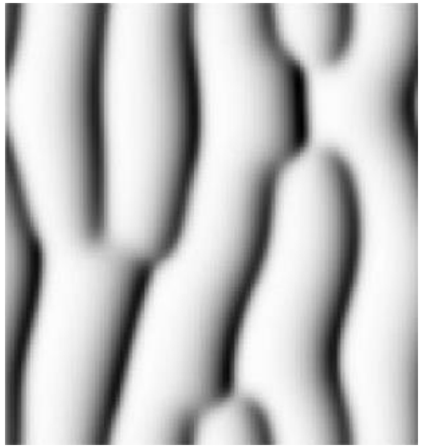
Patterns in models

Add spatial transport:

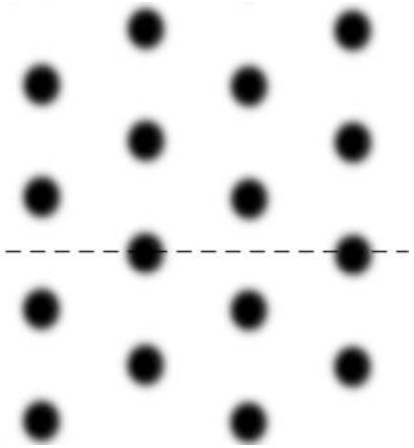
Reaction-Diffusion equations:

$$\begin{cases} \dfrac{du}{dt} = f(u,v) + D_u\,\Delta u \\ \dfrac{dv}{dt} = g(u,v) + D_v\,\Delta v \end{cases}$$

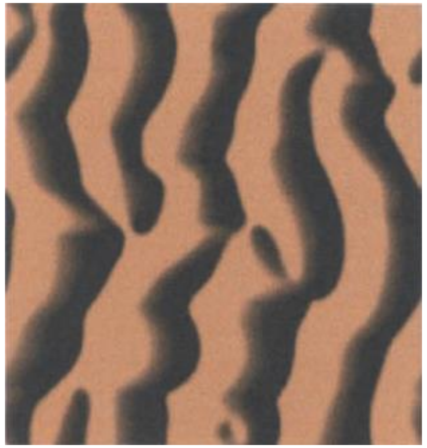




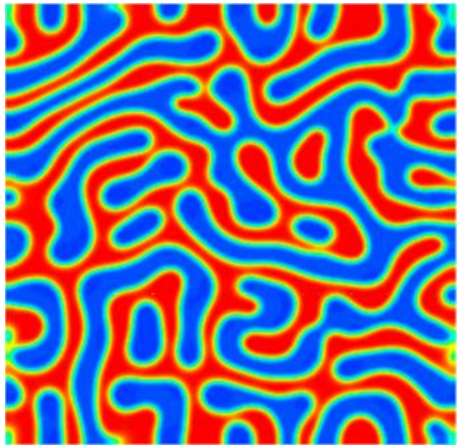
[Klausmeier, 1999]



[Gilad et al, 2004]

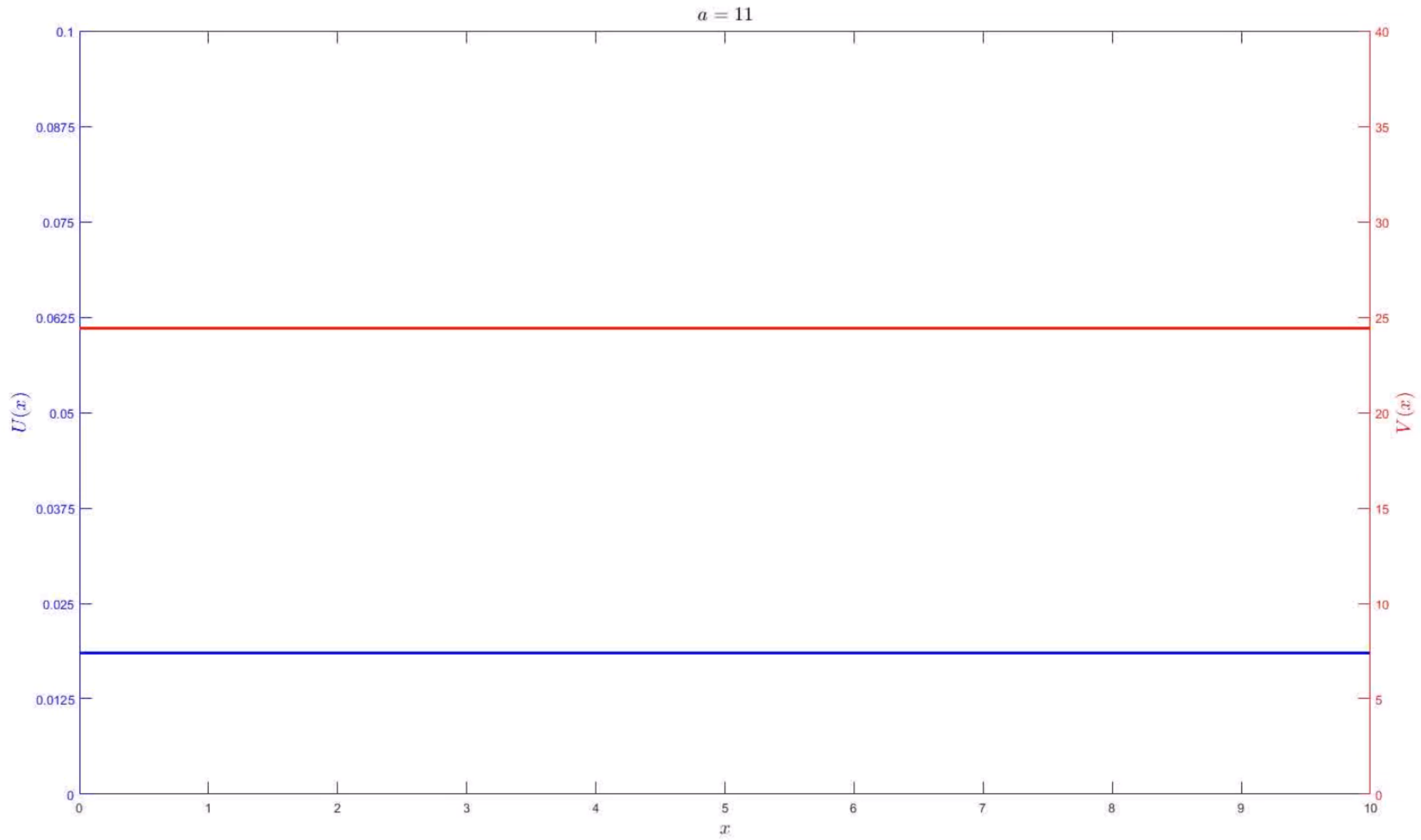


[Rietkerk et al, 2002]



[Liu et al, 2013]

Behaviour of PDEs



Turing patterns

$$\begin{cases} \frac{du}{dt} = f(u,v) + D_u \Delta u \\ \frac{dv}{dt} = g(u,v) + D_v \Delta v \end{cases}$$

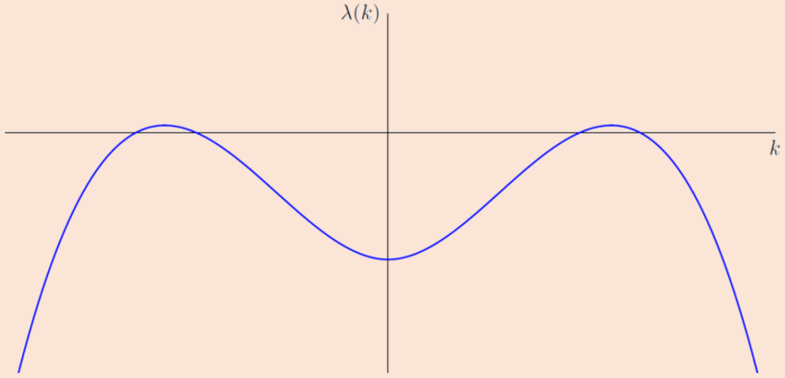
Turing bifurcation

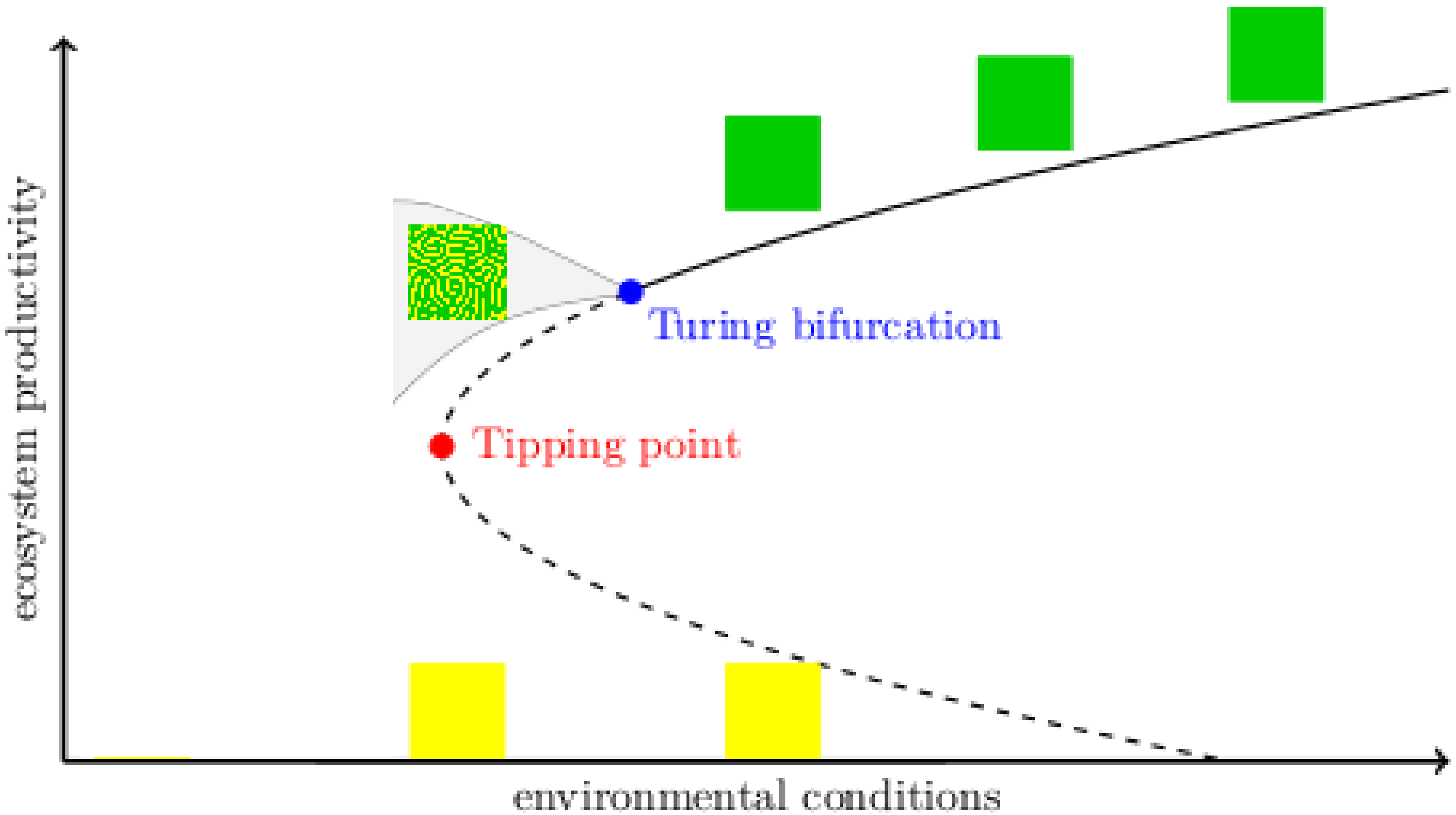
Instability to non-uniform perturbations

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u_* \\ v_* \end{pmatrix} + e^{\lambda t} e^{ikx} \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix}$$

→ Dispersion relation

$$\lambda(k) = \cdots$$





Weakly non-linear analysis

Ginzburg-Landau equation / Amplitude Equation

& Eckhaus/Benjamin-Feir-Newel criterion

[Eckhaus, 1965; Benjamin & Feir, 1967; Newell, 1974]

Busse balloon

$$\begin{cases} \frac{du}{dt} = f(u,v) + D_u \Delta u \\ \frac{dv}{dt} = g(u,v) + D_v \Delta v \end{cases}$$

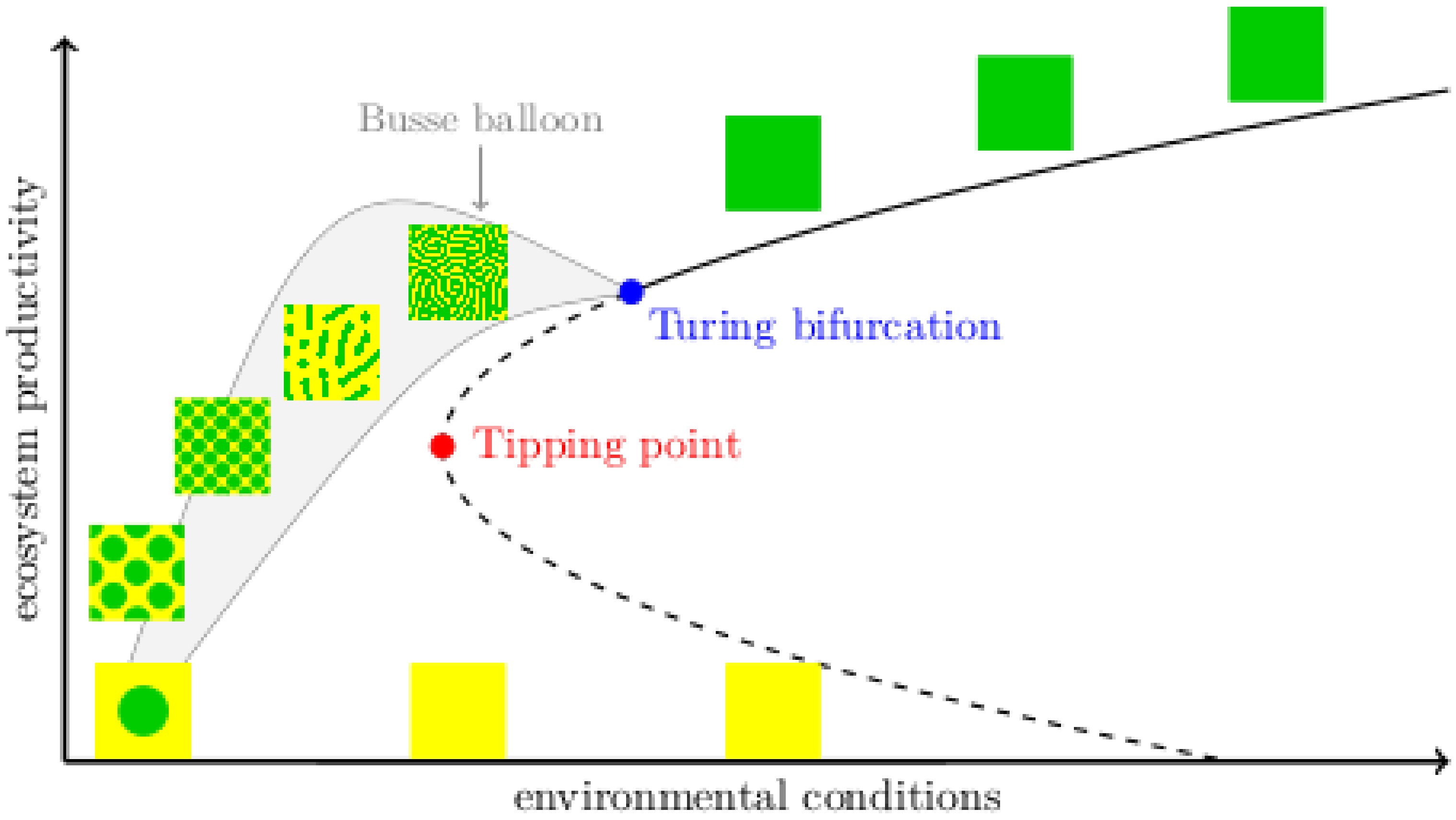
Busse balloon

A model-dependent shape in *(parameter, observable)* space that indicates all stable patterned solutions to the PDE.

Construction Busse balloon

Via numerical continuation

few general results on the shape of Busse balloon

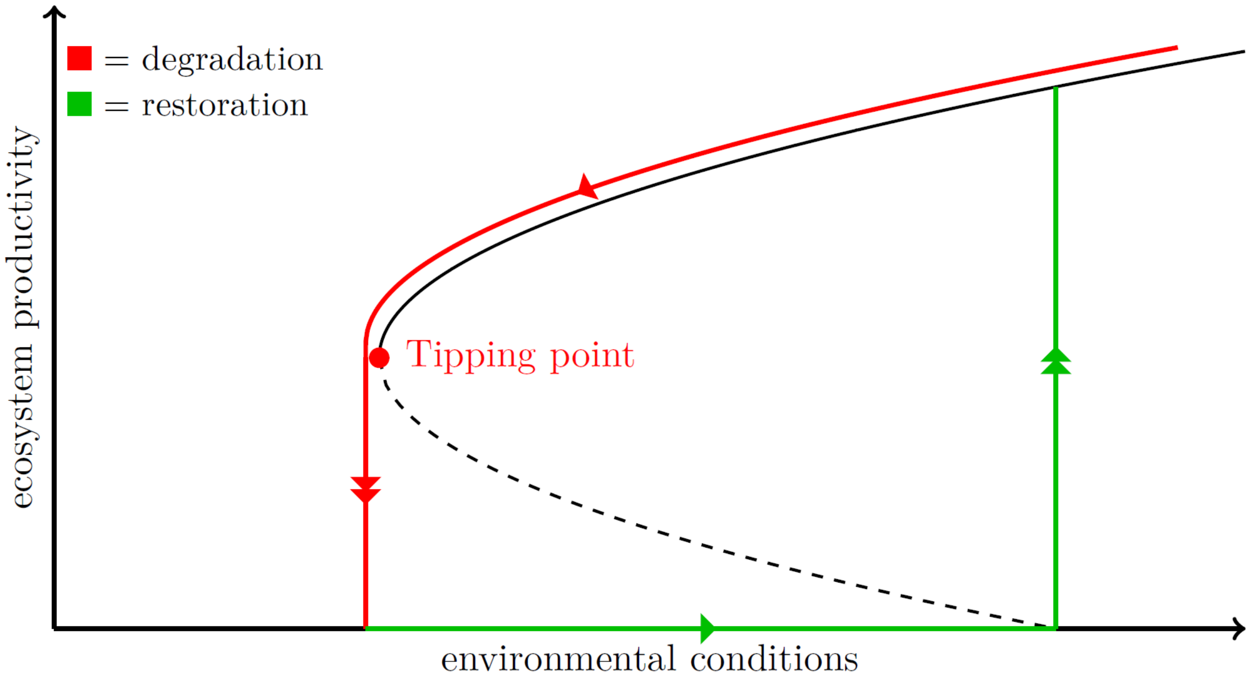


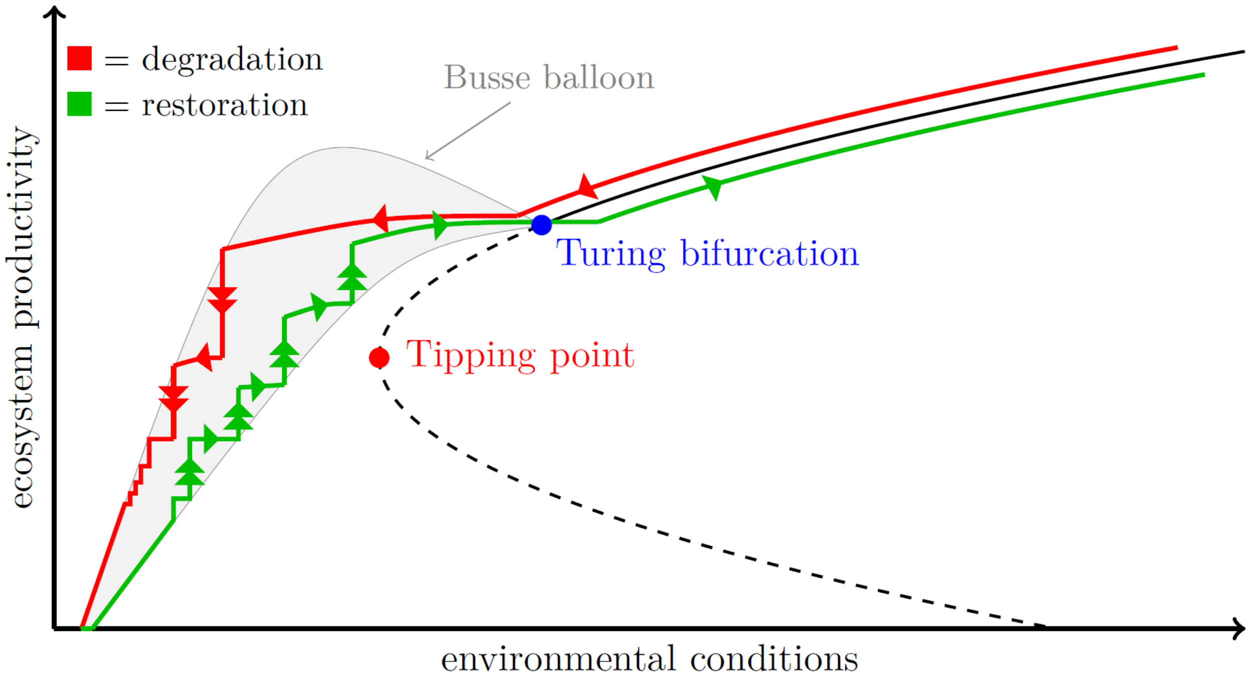
Busse balloon

Idea originates from thermal convection

[Busse, 1978]

Tipping of (Turing) patterns

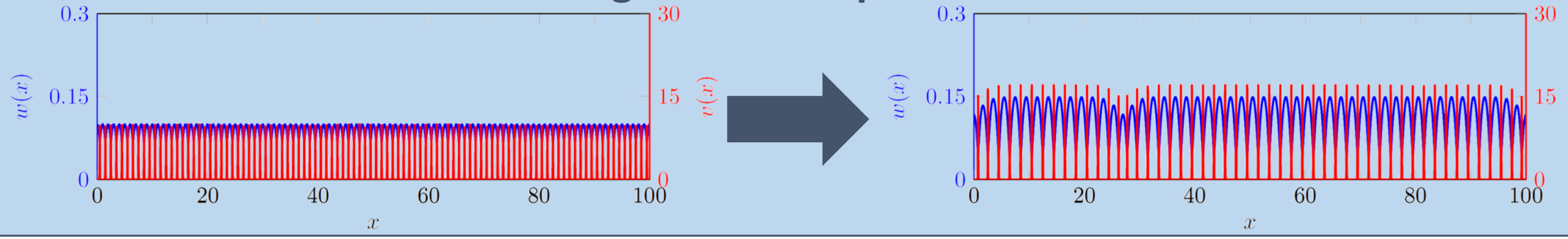




Classic tipping

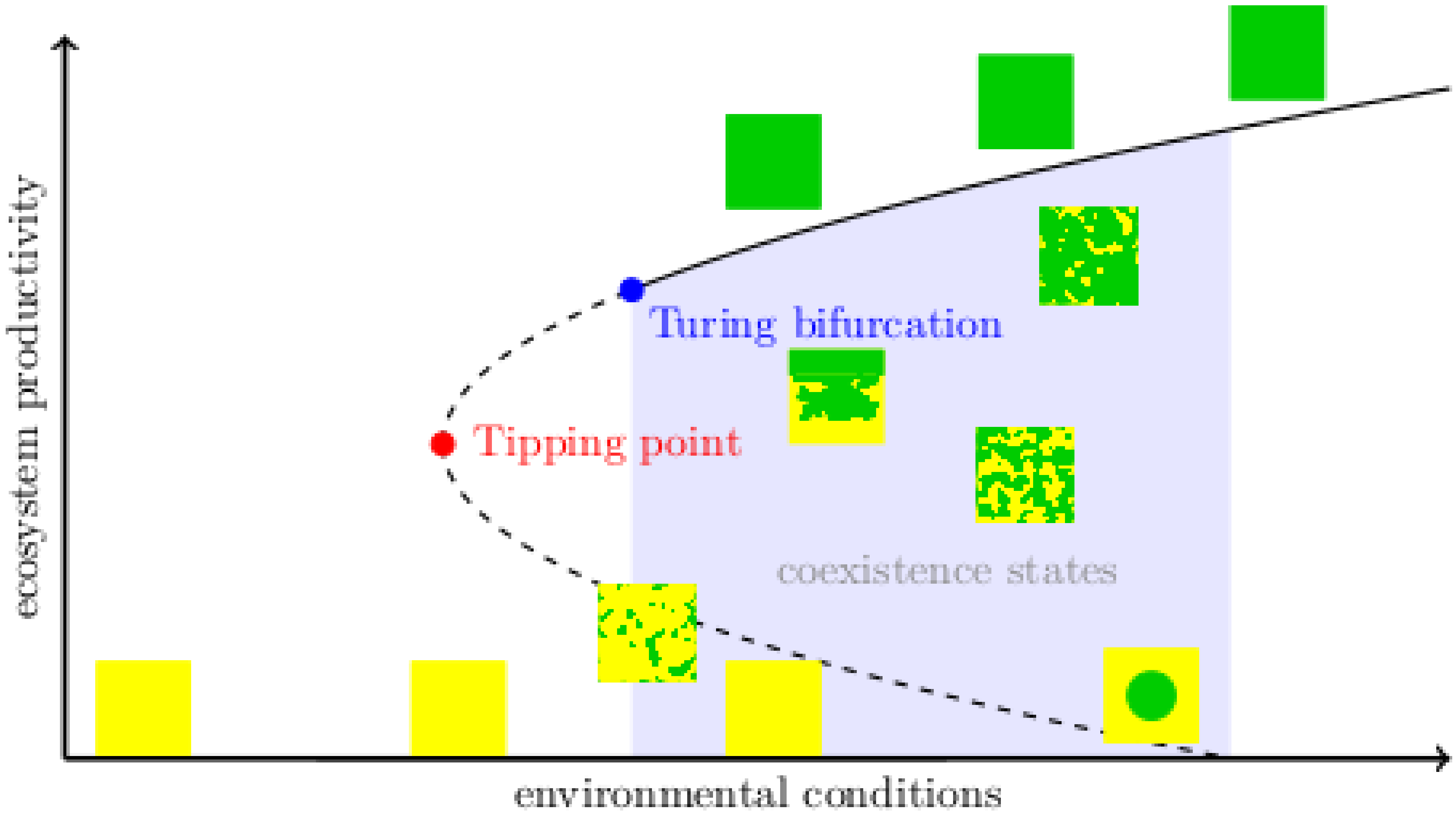
Tipping of patterns

Degradation of patterns



Part 2: Coexistence States

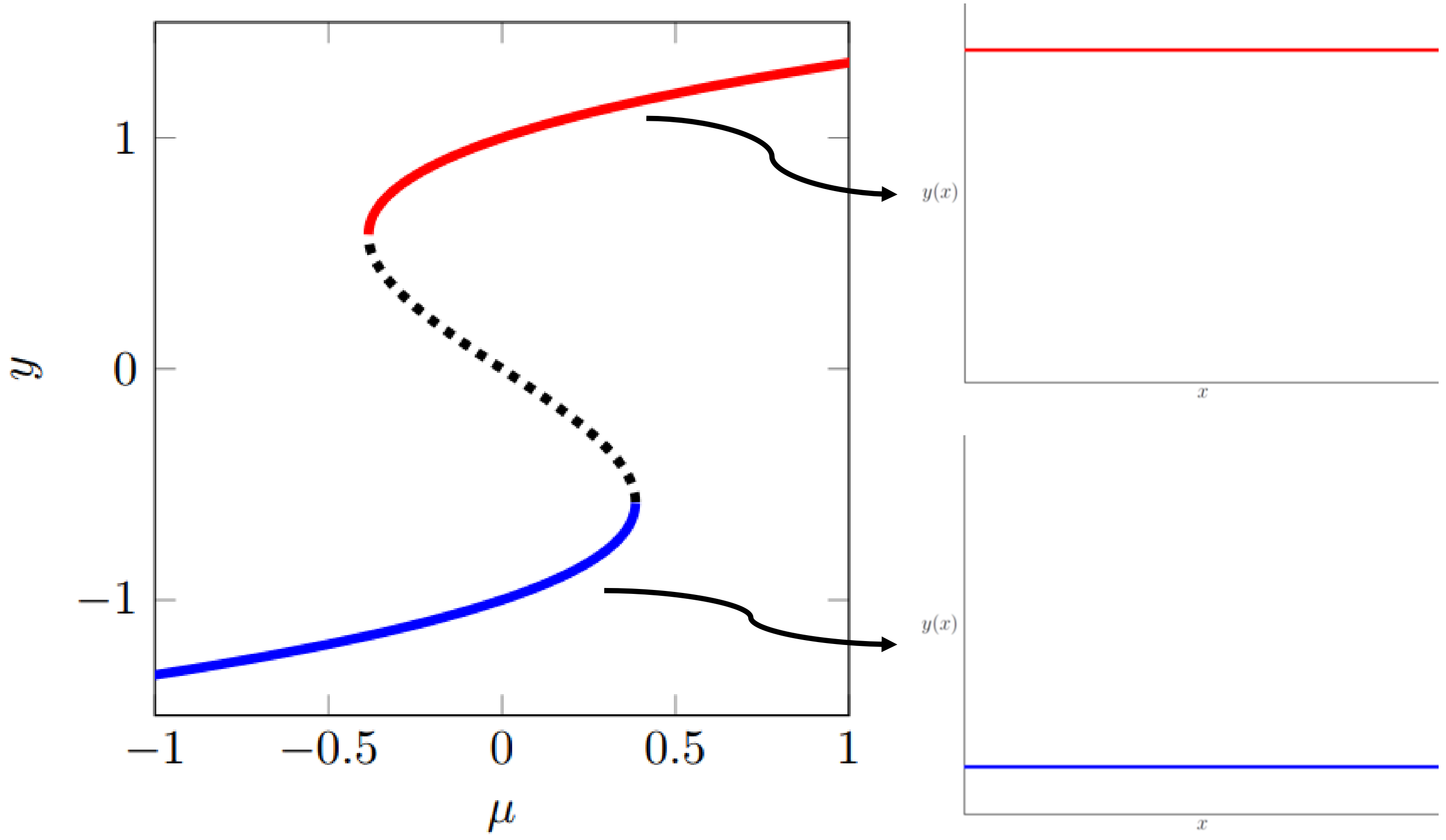
Coexistence states in bifurcation diagram

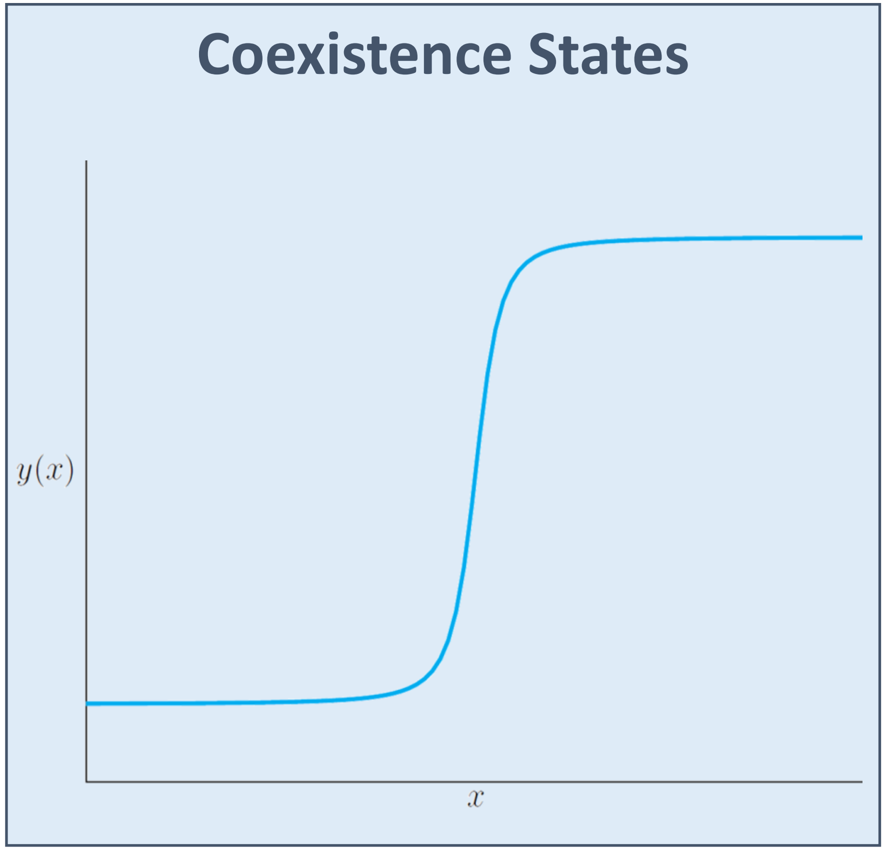


Coexistence states

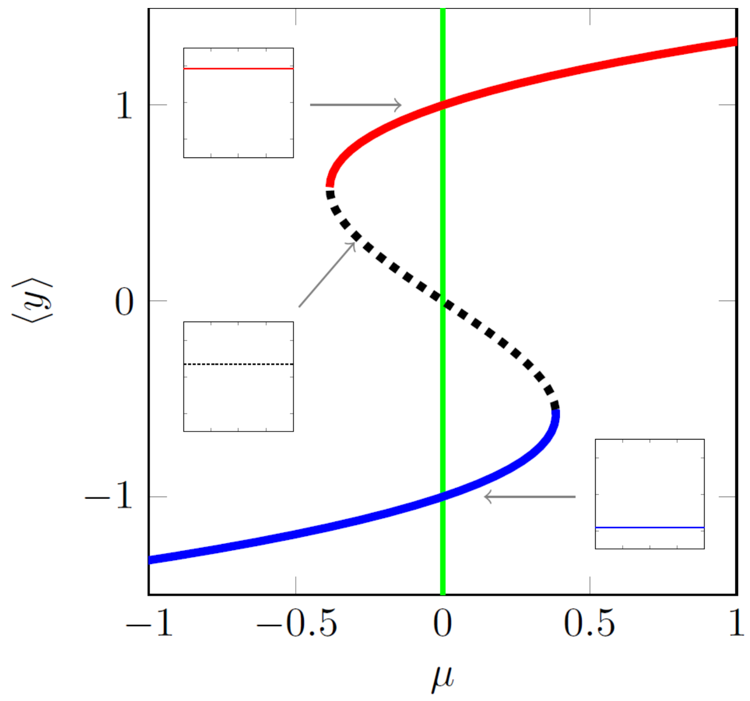
Bistable (Allen-Cahn/Nagumo) equation:

$$\frac{\partial y}{\partial t} = y(1 - y^2) + \mu + D\,\frac{\partial^2 y}{\partial x^2}$$





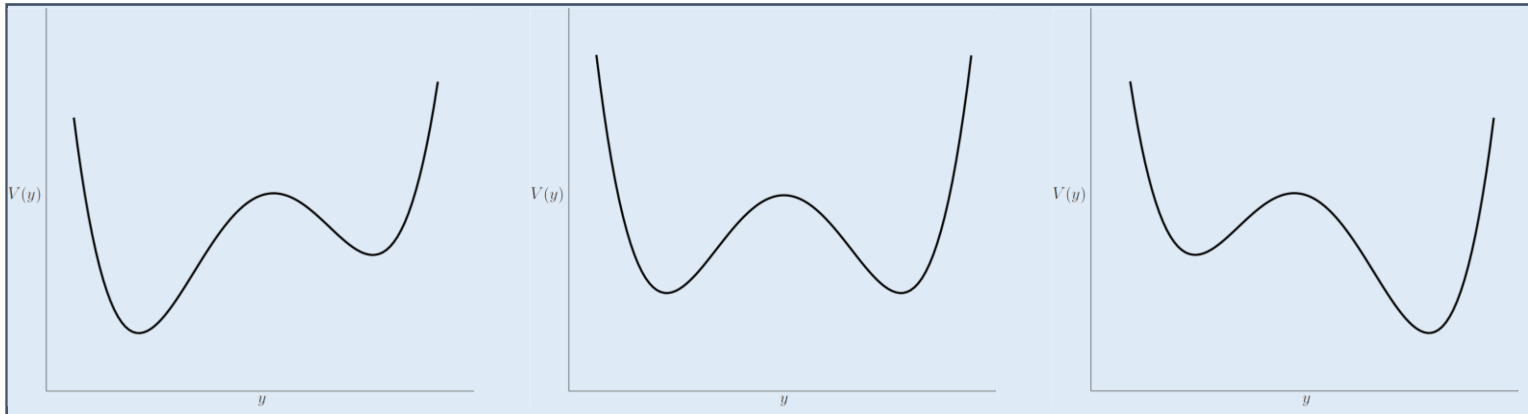
Front Dynamics



$$\frac{\partial y}{\partial t} = D\,\frac{\partial^2 y}{\partial x^2} + f(y;\mu)$$

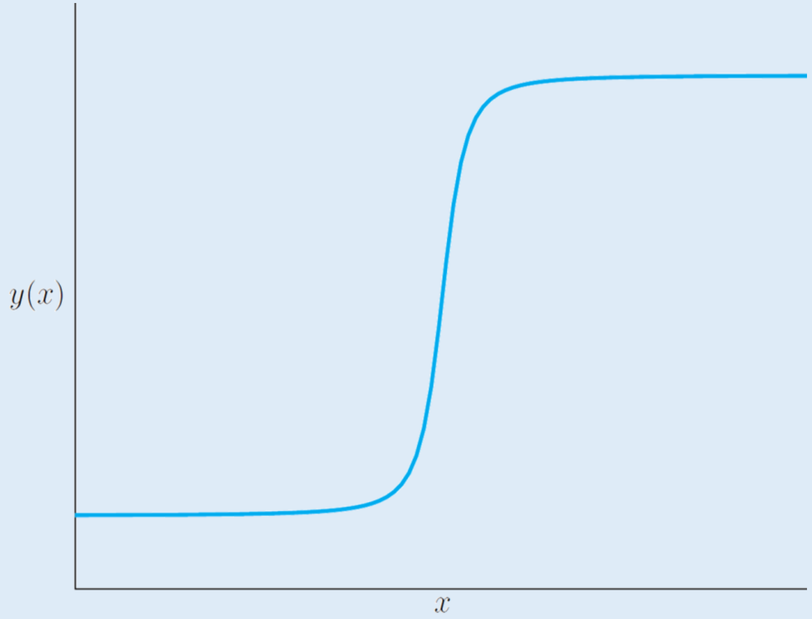
Potential function $V(y;\mu)$:

$$\frac{\partial V}{\partial y}(y;\mu) = -f(y;\mu)$$



moves right | stationary | moves left

Maxwell Point $\mu_{maxwell}$



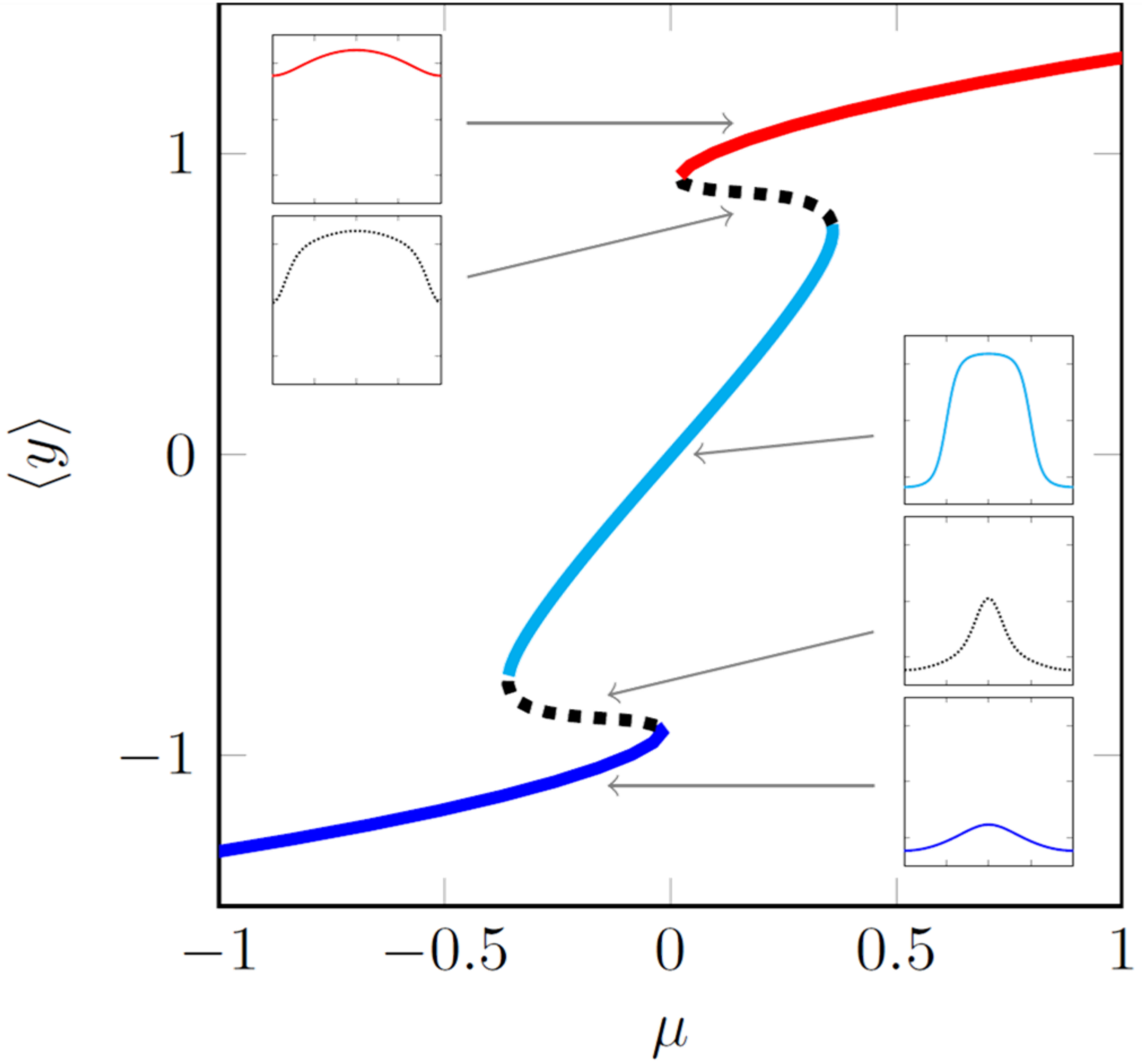
Adding Spatial Heterogeneity

$$\frac{\partial y}{\partial t} = D\frac{\partial^2 y}{\partial x^2} + f(y, x; \mu)$$

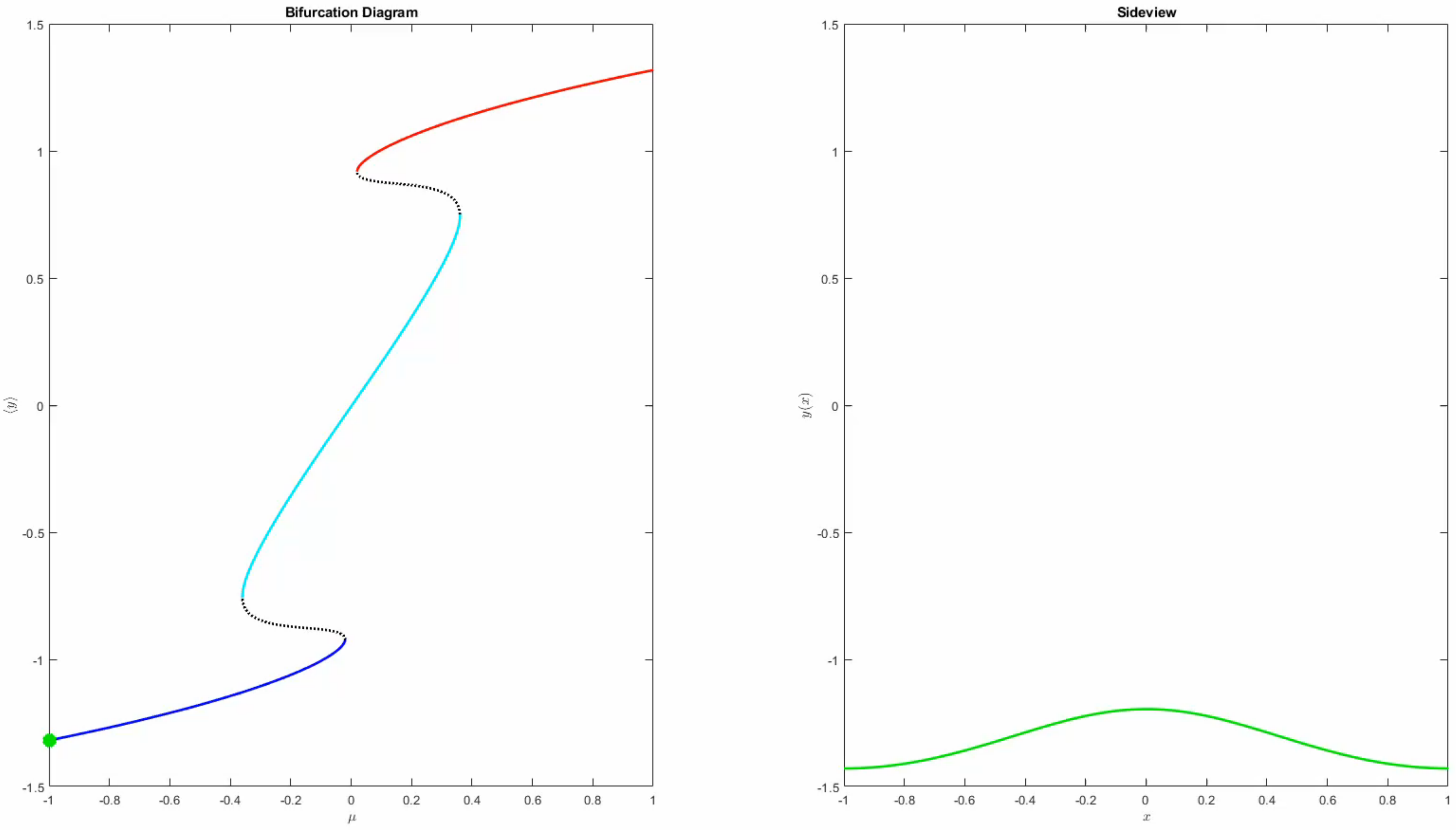
Now, the **local** difference in potentials determines the front movement

New behaviour:

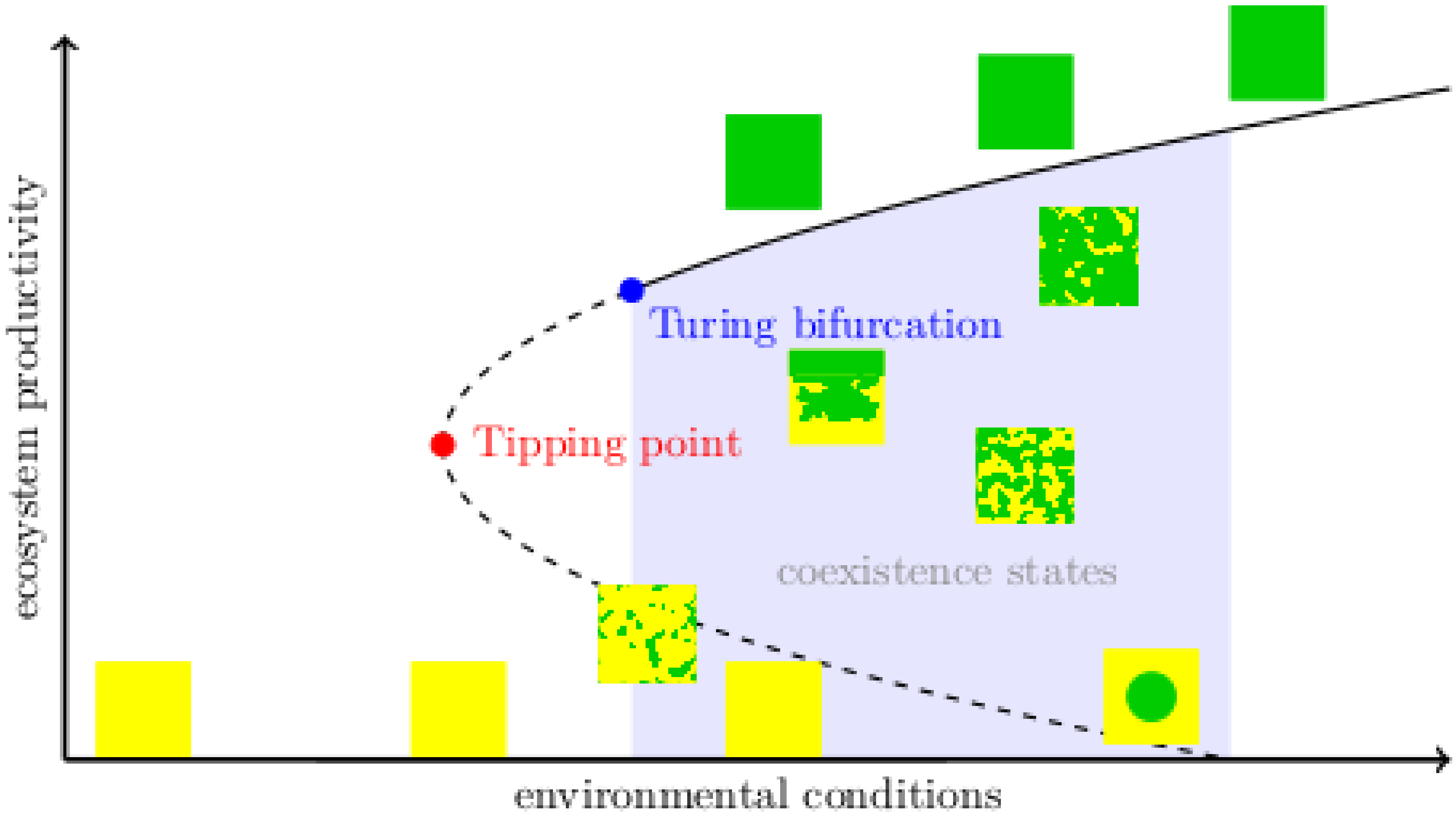
- Multi-fronts can be stationary
- Maxwell point is smeared out



Fragmented Tipping

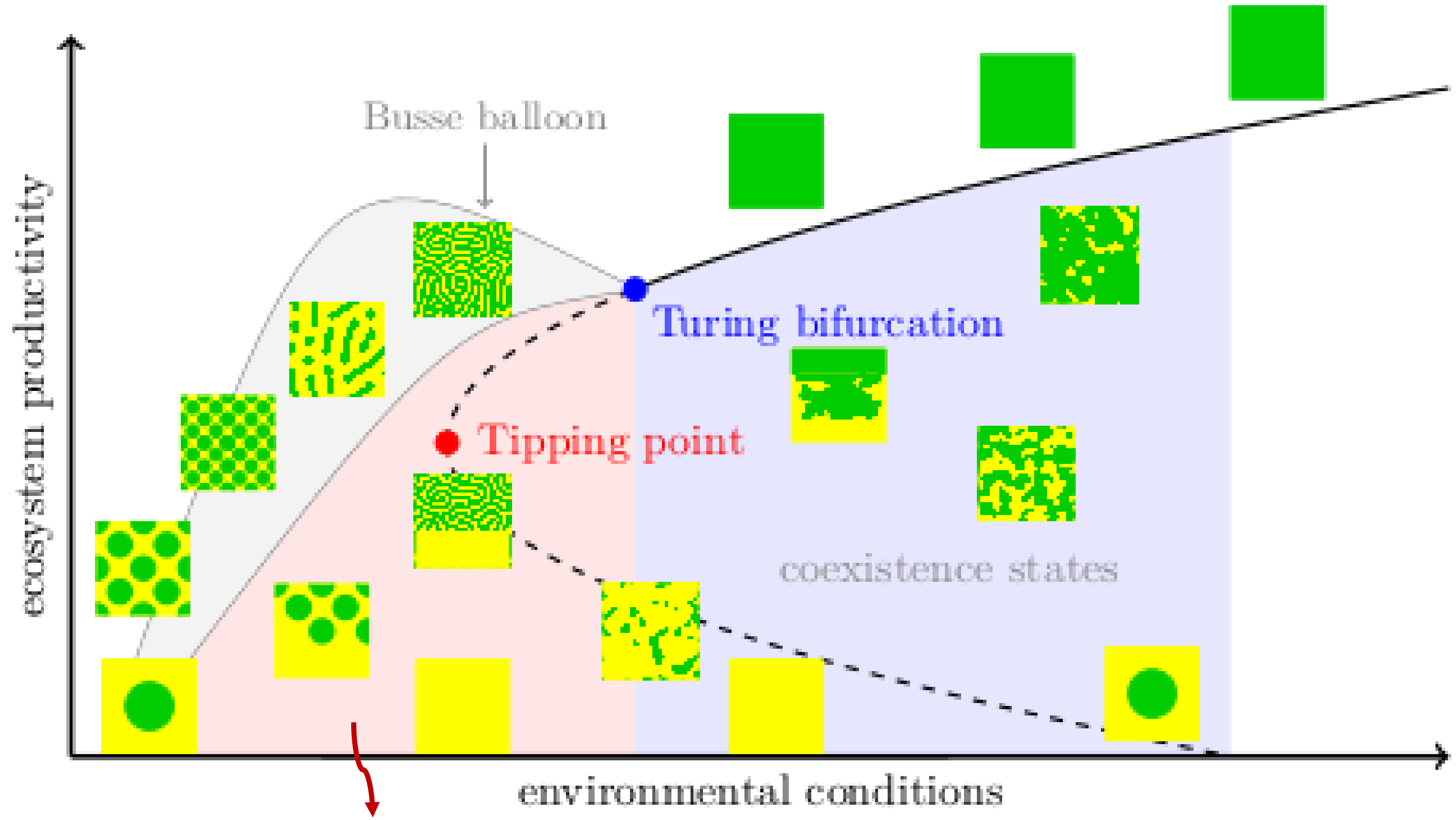


Coexistence states in bifurcation diagram

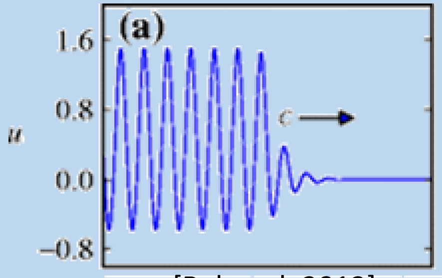


Part 3:
Tipping in Spatially Extended Systems

"Bifurcation Diagram" for spatially extended systems



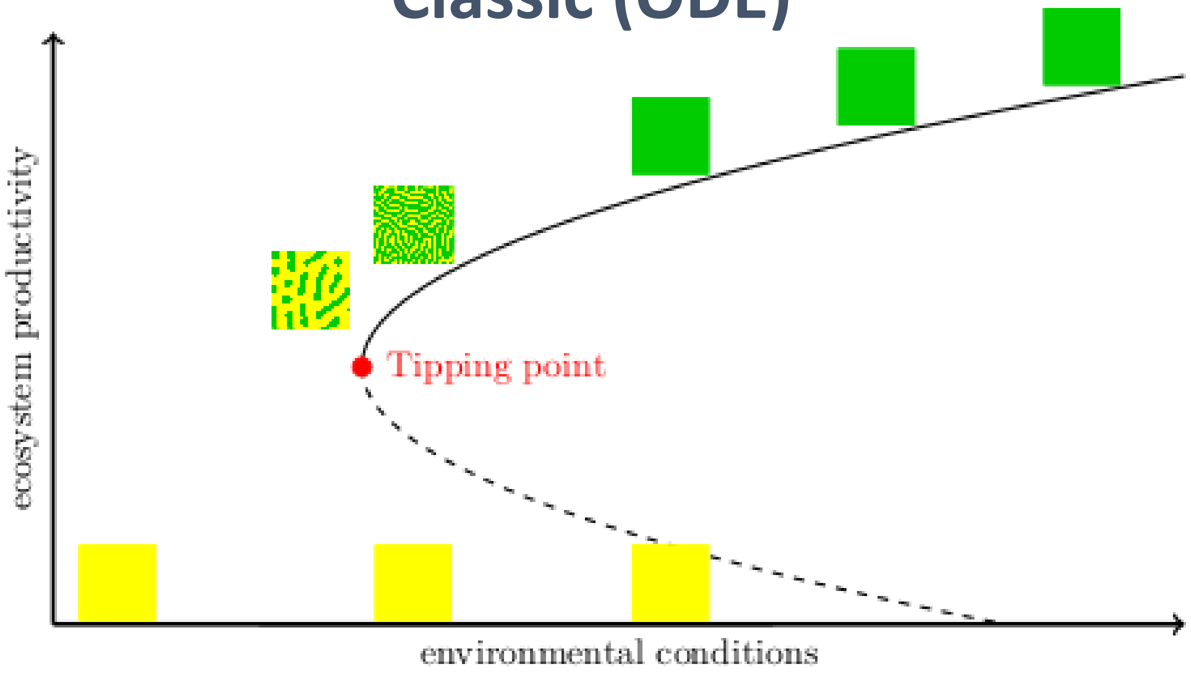
Coexistence states
between patterned and
uniform states also exist



[Bel et al, 2012]

What if the system tips?

Classic (ODE)



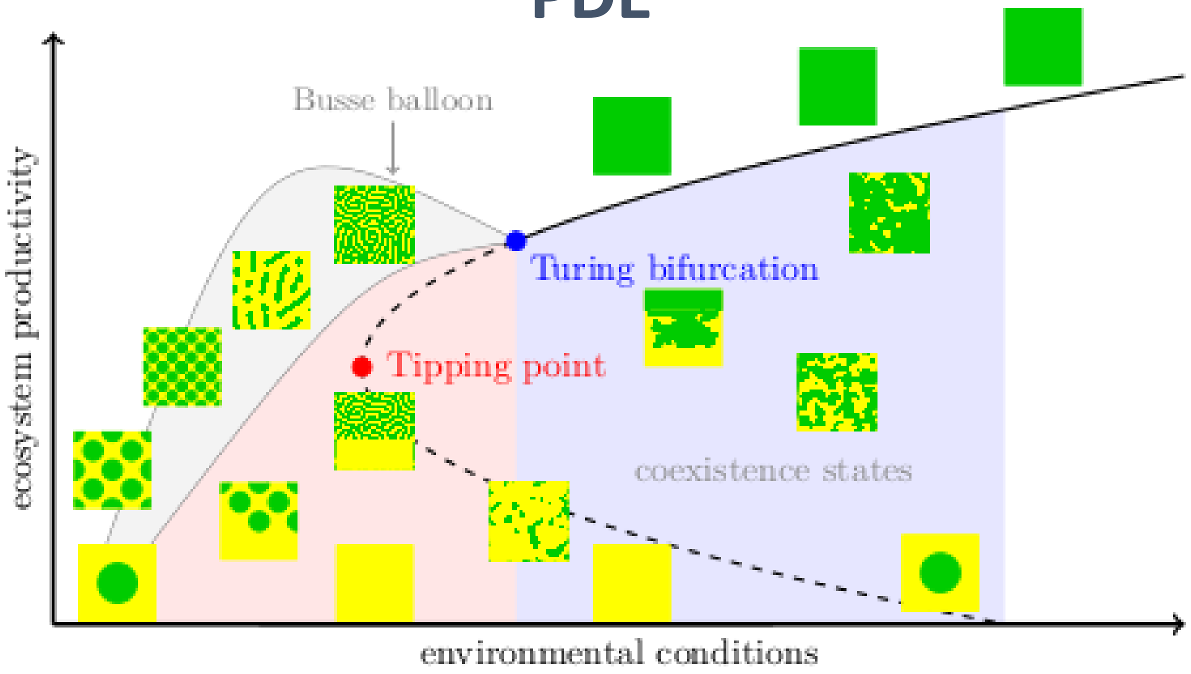
Crossing a Tipping Point:

→ Always full reorganization

Early Warning Signals

signal for <u>WHEN</u>

PDE



Crossing a bifurcation:

Now also possible:

→ Spatial reorganization (Turing patterns)

→ Fragmented tipping (coexistence states)

Early Warning Signals

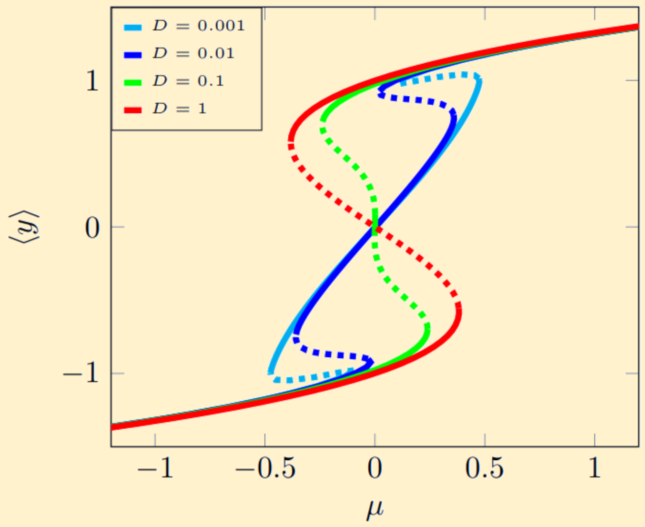
need to signal <u>WHEN</u> & <u>WHAT</u>

Do systems always behave like this? (a.k.a. the small print)

No.

Well-mixed systems

Spatially confined systems



→ Such systems (again) behave like ODEs ←

But even in other systems terms & conditions apply:
System-specific knowledge is required!

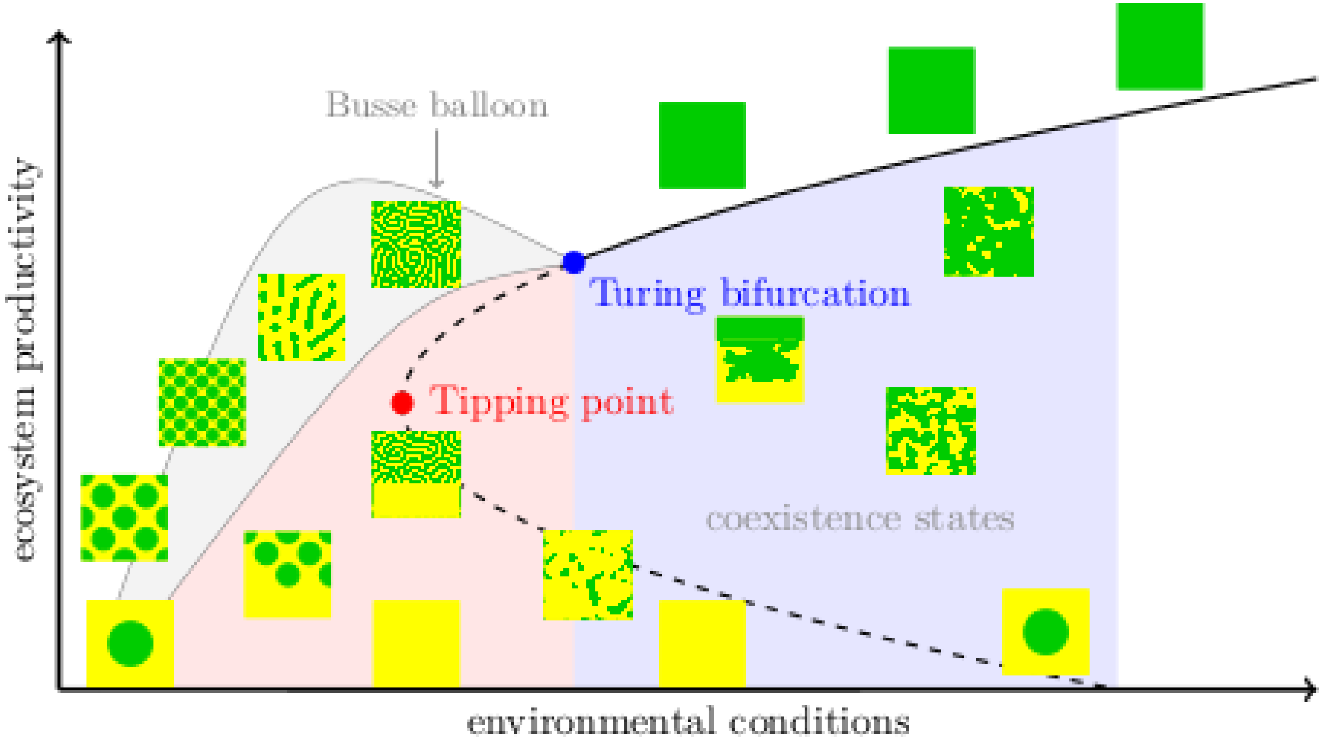
Summary

Spatial patterns:

- Turing Patterns
- Coexistence States

Tipping can be more subtle:

- Spatial reorganization
- Fragmented Tipping



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Rietkerk, M., Bastiaansen, R., Banerjee, S., van de Koppel, J., Baudena, M., & Doelman, A. (2021). Evasion of tipping in complex systems through spatial pattern formation. Science, 374(6564), eabj0359.

Bastiaansen, R., Dijkstra, H. A., & von der Heydt, A. S. (2021). Fragmented Tipping in a spatially heterogeneous world. *Environmental Research Letters, 17, 045006*