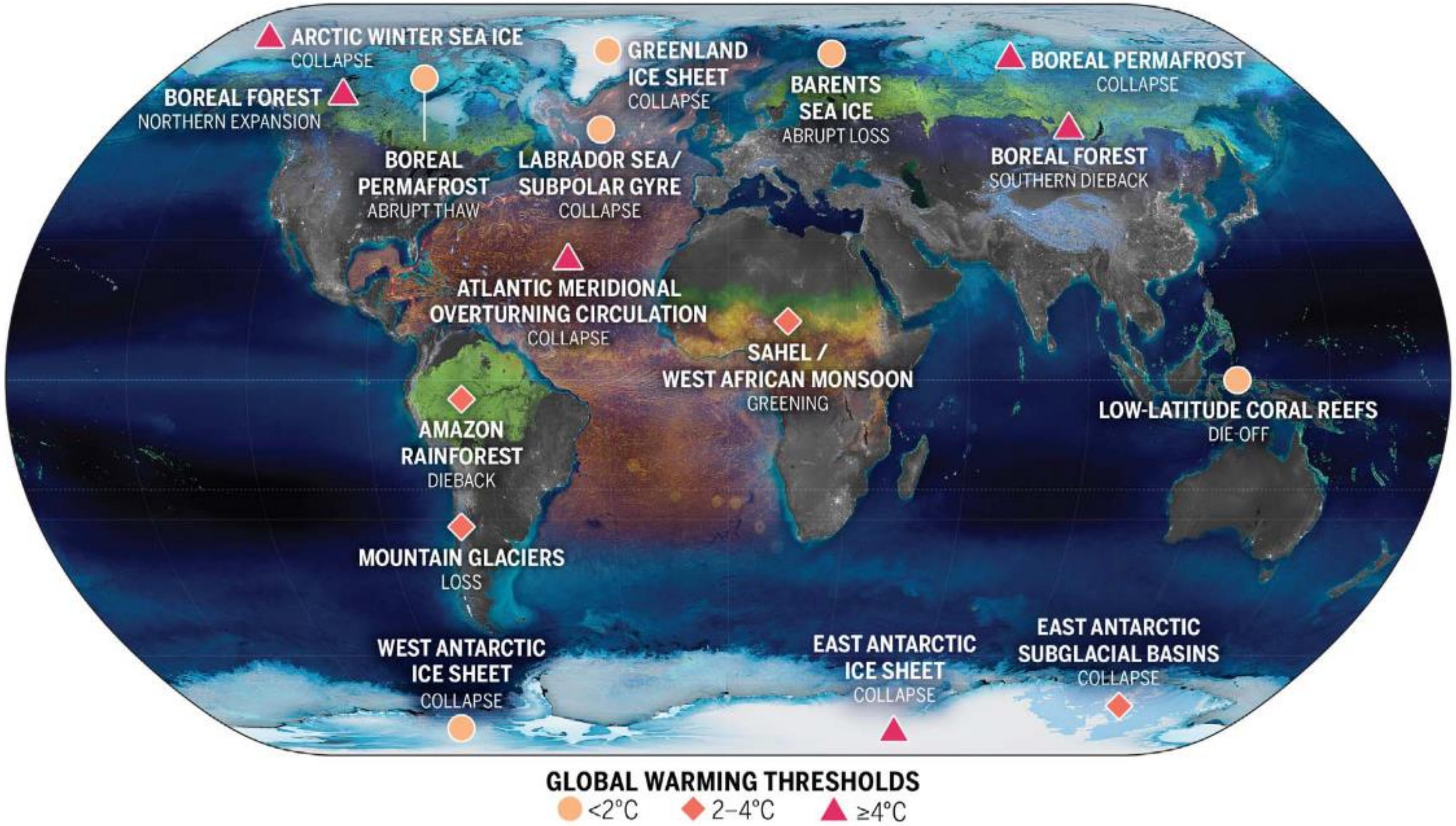
Tipping in Spatially Extended Systems

2022-12-06, One World Mathematics for Climate

Robbin Bastiaansen (r.bastiaansen@uu.nl)

Tipping Points

IPCC AR6 (2021) : "a critical threshold beyond which a system reorganizes, often abruptly and/or irreversibly"

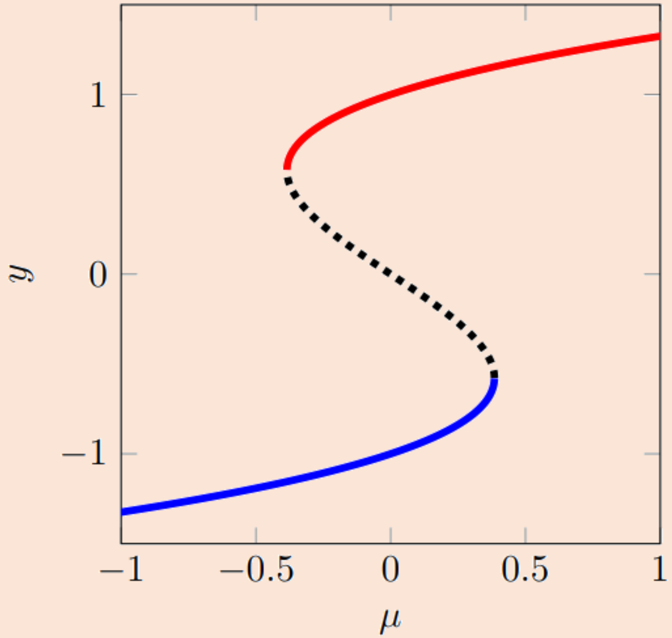


[source: McKay et al, 2022]

Mathematics

Tipping points ↔ Bifurcations

$$\frac{dy}{dt} = f(y, \mu)$$



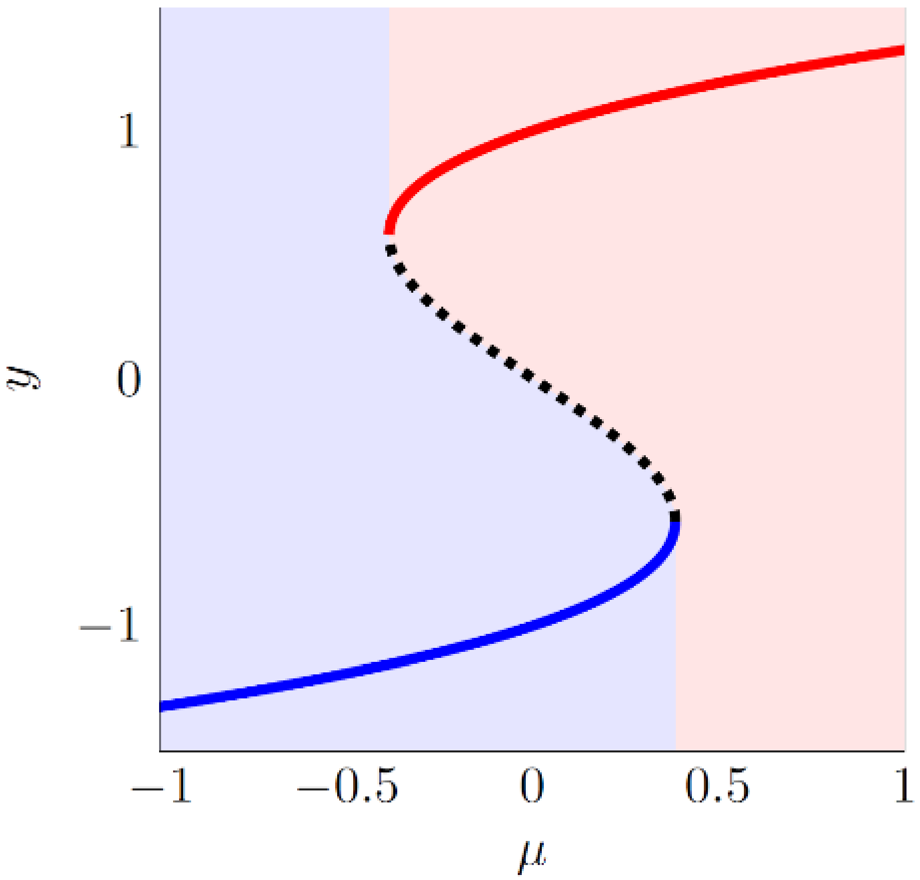
What about spatially extended systems?

Part 0:
Tipping in ODEs

Tipping in ODEs (1)

Canonical example:

$$\frac{dy}{dt} = y(1 - y^2) + \mu$$



Concrete example: Global Energy Balance Model

$$\frac{dT}{dt} = Q\big(1 \; - \alpha(T)\big) - \varepsilon\sigma_0 T^4 + \mu$$

Classic Literature

[Holling, 1973]

[Noy-Meier, 1975]

[May, 1977]

Tipping

[Ashwin et al, 2012]

Bifurcation-Tipping : Basin disappears

Noise-Tipping : Forced outside Basin

Rate-Tipping : *(more complicated)*

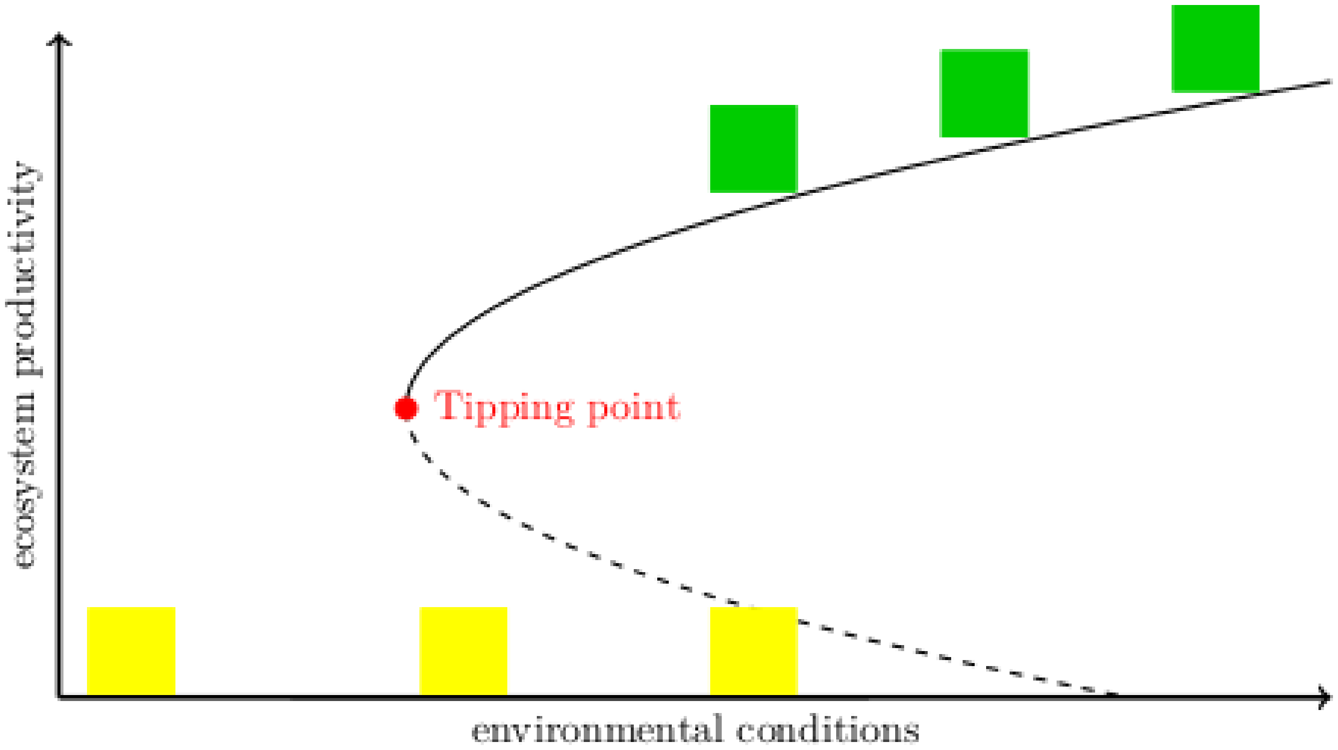
Tipping in ODEs (2)

Two components:

$$\begin{cases} \frac{du}{dt} = f(u,v) \\ \frac{dv}{dt} = g(u,v) \end{cases}$$

includes common models:

- Predator-Prey
- Activator-Inhibitor

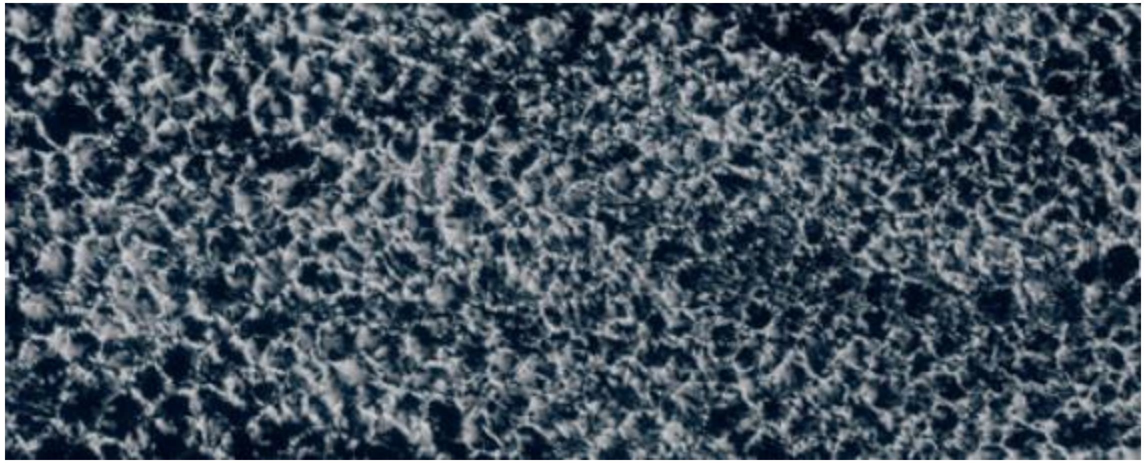


Examples of tipping in ODEs include:

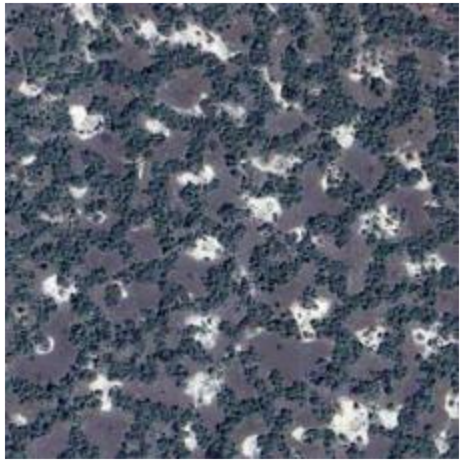
- Forest-Savanna bistability
- Deep ocean exchange
- Cloud formation
- Ice sheet melting
- Turbidity in shallow lakes

Examples of spatial patterning – regular patterns

mussel beds

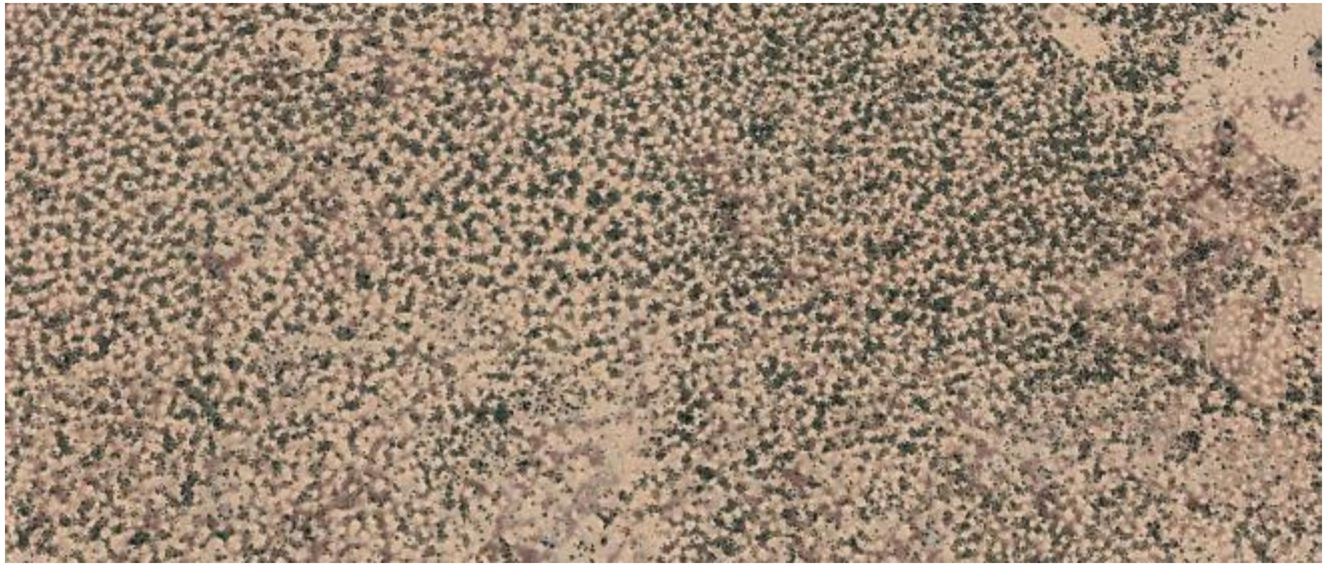


clouds



savannas

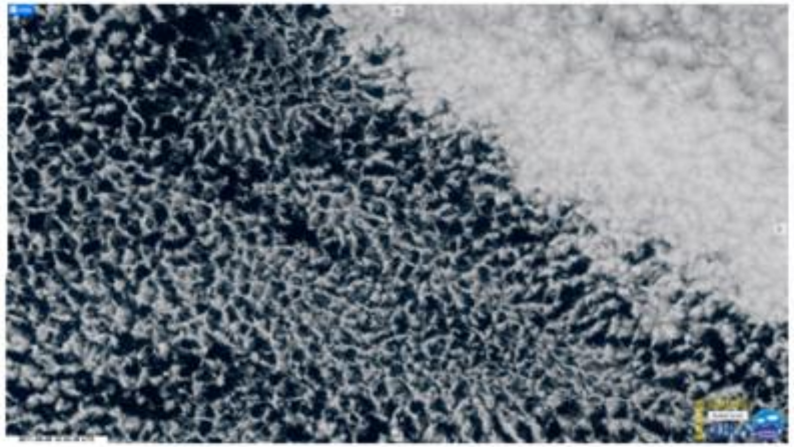
melt ponds



drylands

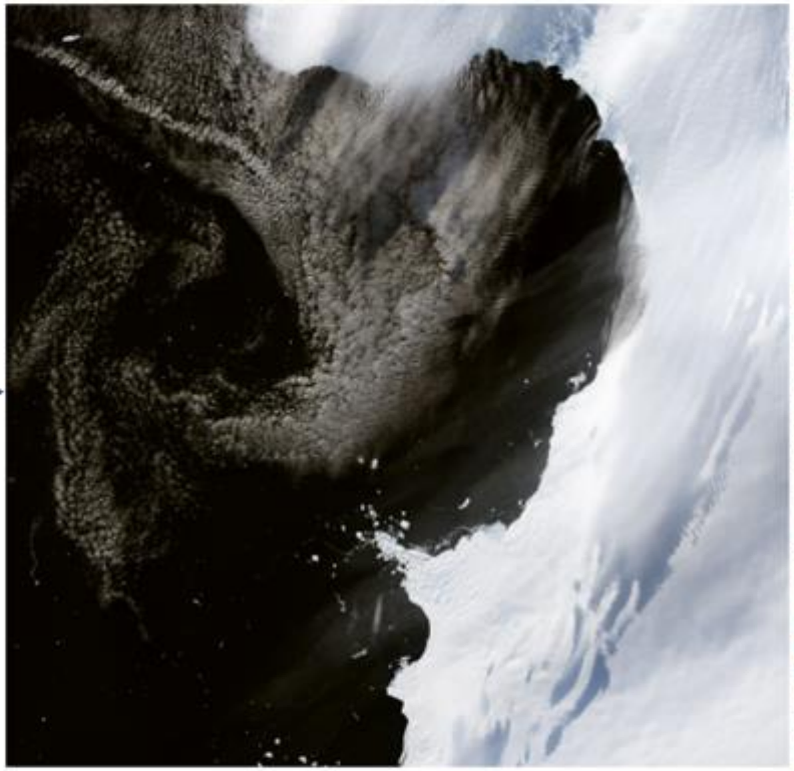
Examples of spatial patterning – spatial interfaces

tropical forest
& savanna
ecosystems
[Google Earth]



types of
stratocumulus
clouds
[RAMMB/CIRA SLIDER]

sea-ice & water
at Eltanin Bay
[NASA's Earth observatory]



algae bloom
in Lake St. Clair
[NASA's Earth observatory]

Part 1:
Turing Patterns

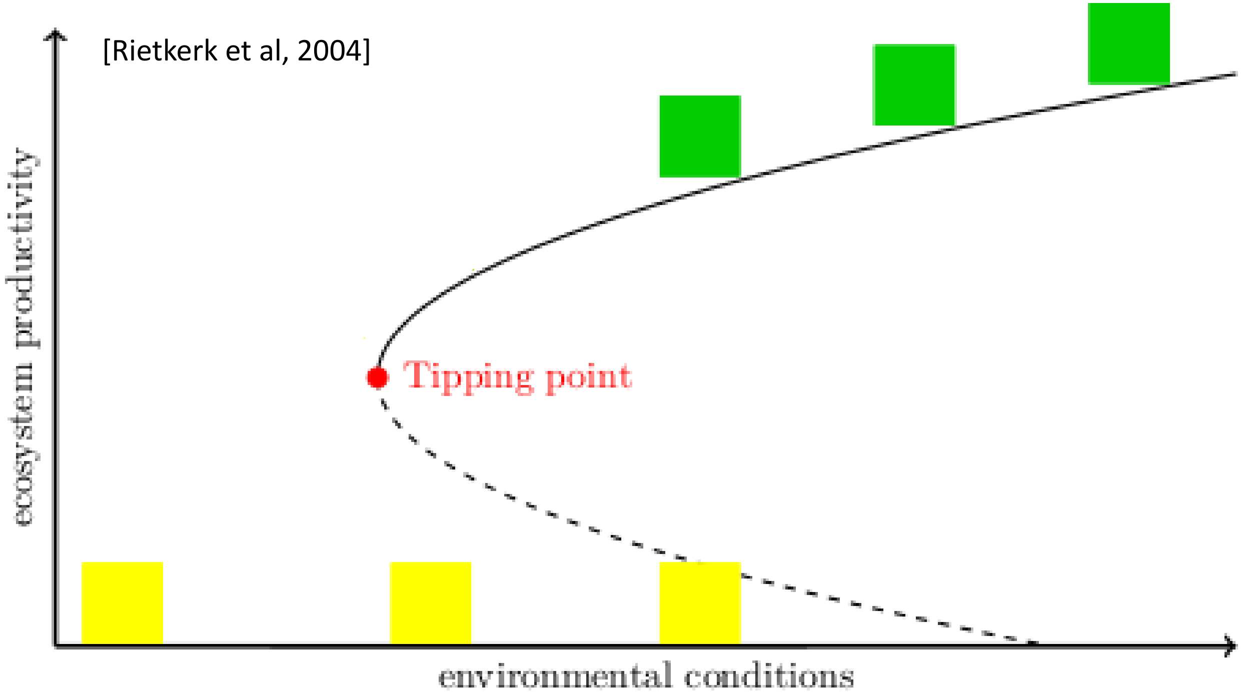
image courtesy: National Science Foundation

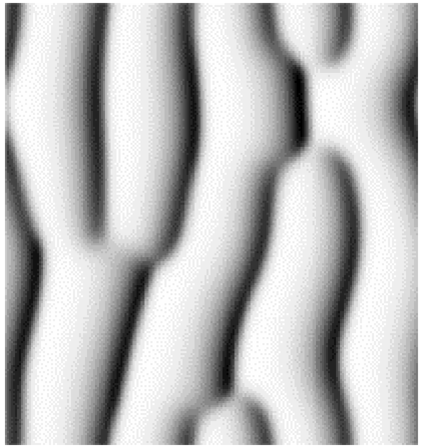
Patterns in models

Add spatial transport:

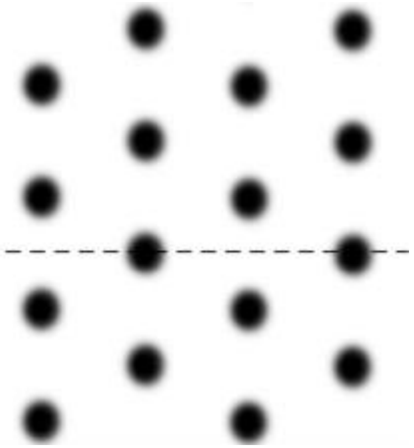
Reaction-Diffusion equations:

$$\begin{cases} \frac{du}{dt} = f(u,v) + D_u \, \Delta u \\ \frac{dv}{dt} = g(u,v) + D_v \, \Delta v \end{cases}$$



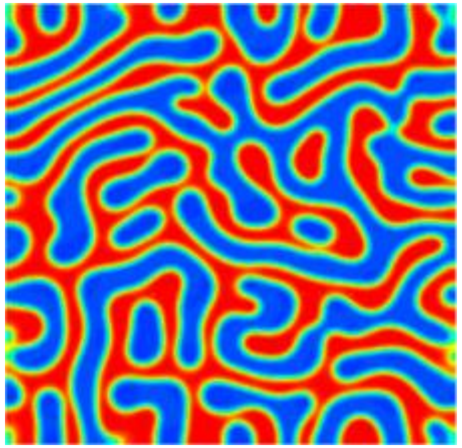


[Klausmeier, 1999]



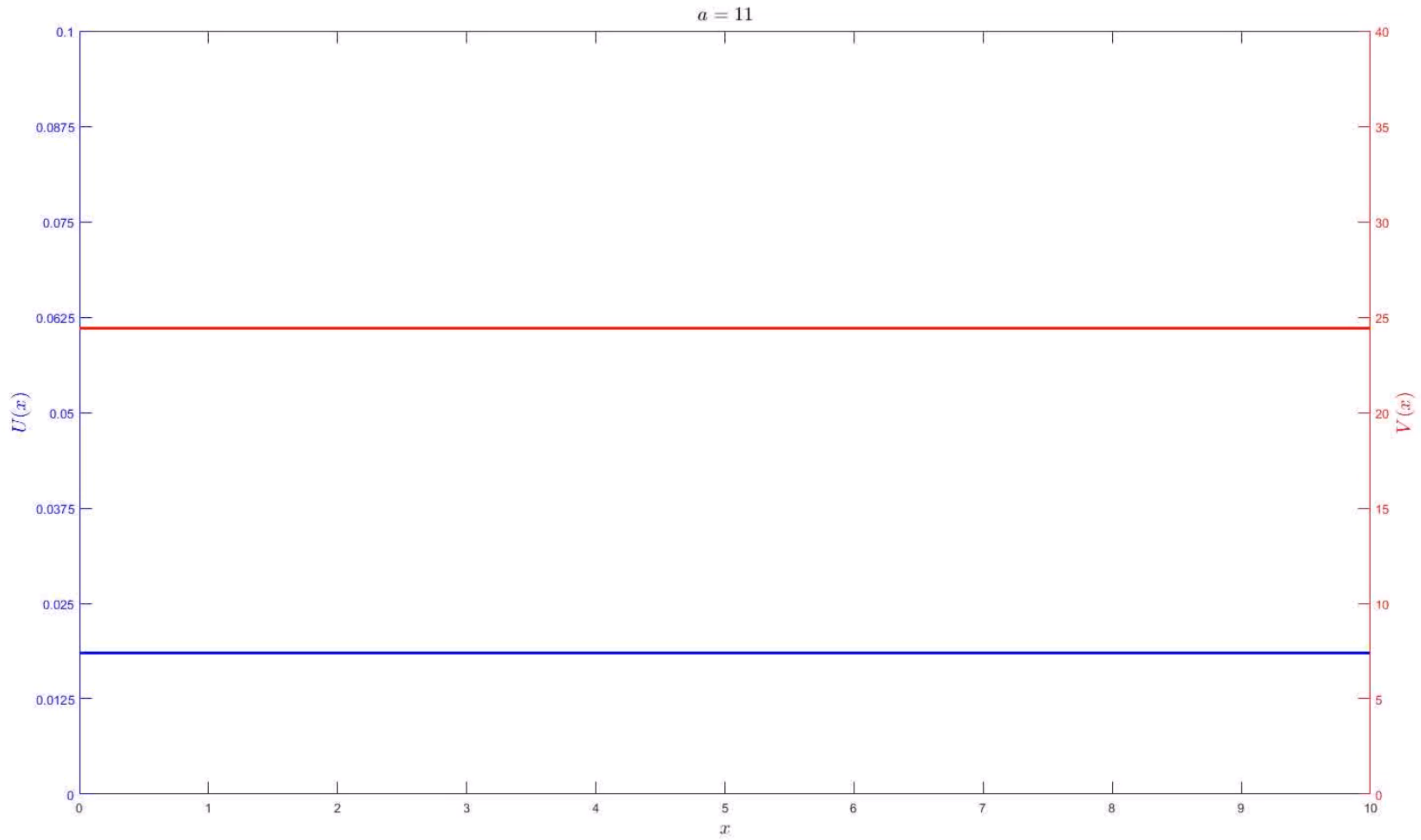
[Gilad et al, 2004]

[Rietkerk et al, 2002]



[Liu et al, 2013]

Behaviour of PDEs



Turing patterns

$$\begin{cases} \frac{du}{dt} = f(u,v) + D_u \Delta u \\ \frac{dv}{dt} = g(u,v) + D_v \Delta v \end{cases}$$

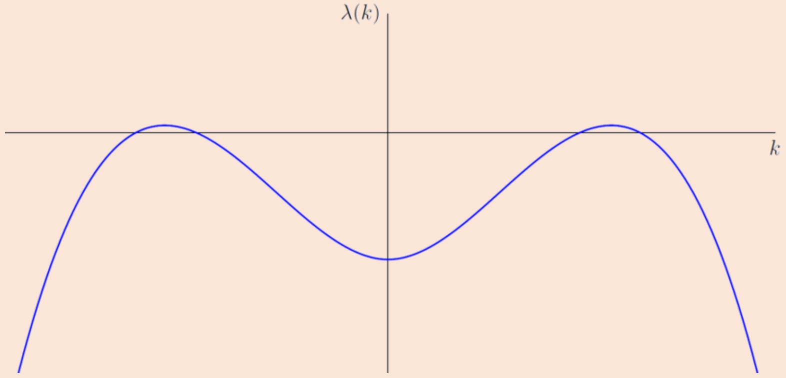
Turing bifurcation

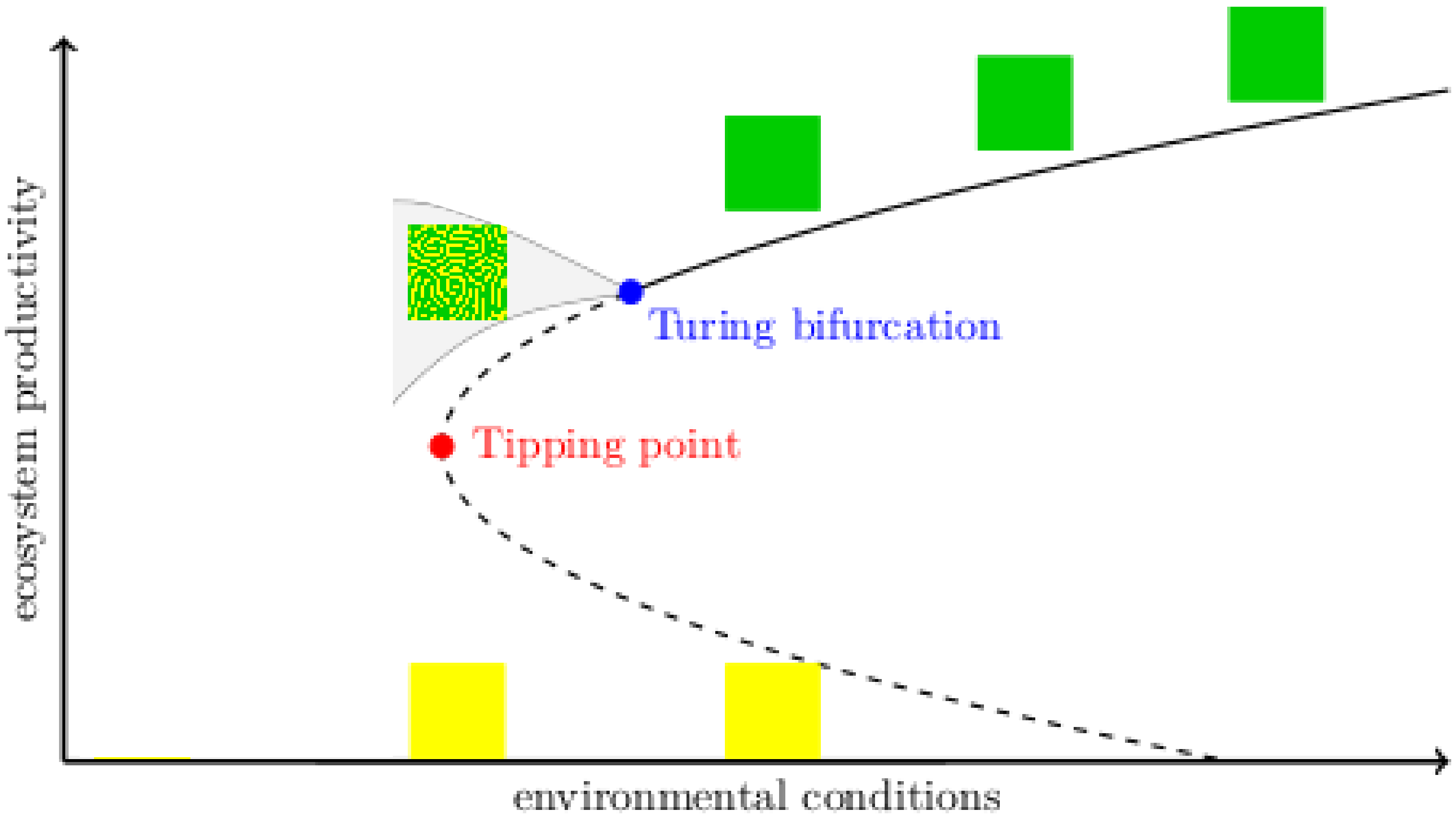
Instability to non-uniform perturbations

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u_* \\ v_* \end{pmatrix} + e^{\lambda t} e^{ikx} \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix}$$

→ Dispersion relation

$$\lambda(k) = \cdots$$





Weakly non-linear analysis

Ginzburg-Landau equation / Amplitude Equation

& Eckhaus/Benjamin-Feir-Newel criterion

[Eckhaus, 1965; Benjamin & Feir, 1967; Newell, 1974]

Busse balloon

$$\begin{cases} \frac{du}{dt} = f(u,v) + D_u \Delta u \\ \frac{dv}{dt} = g(u,v) + D_v \Delta v \end{cases}$$

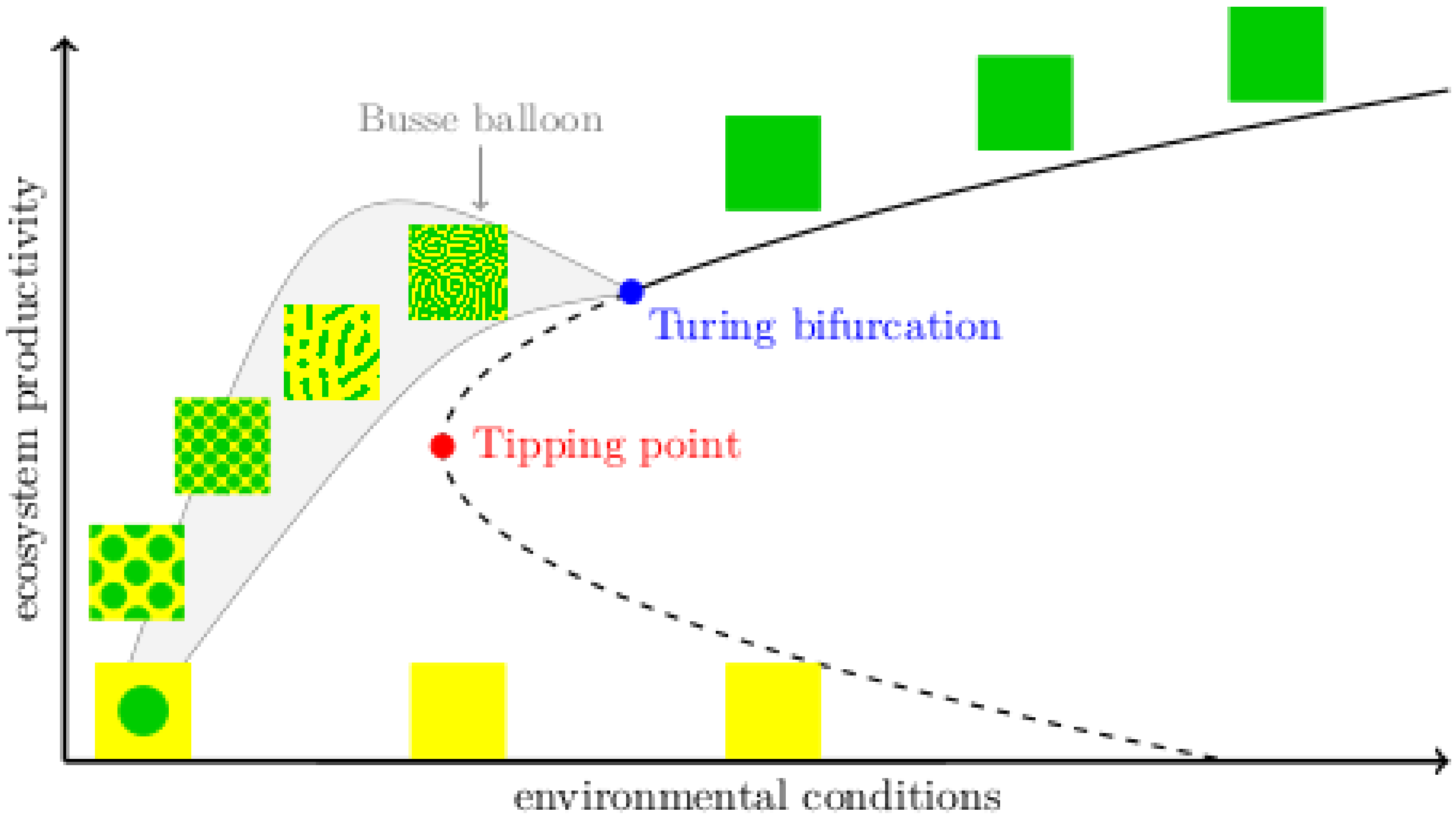
Busse balloon

A model-dependent shape in *(parameter, observable)* space that indicates all stable patterned solutions to the PDE.

Construction Busse balloon

Via numerical continuation

few general results on the shape of Busse balloon

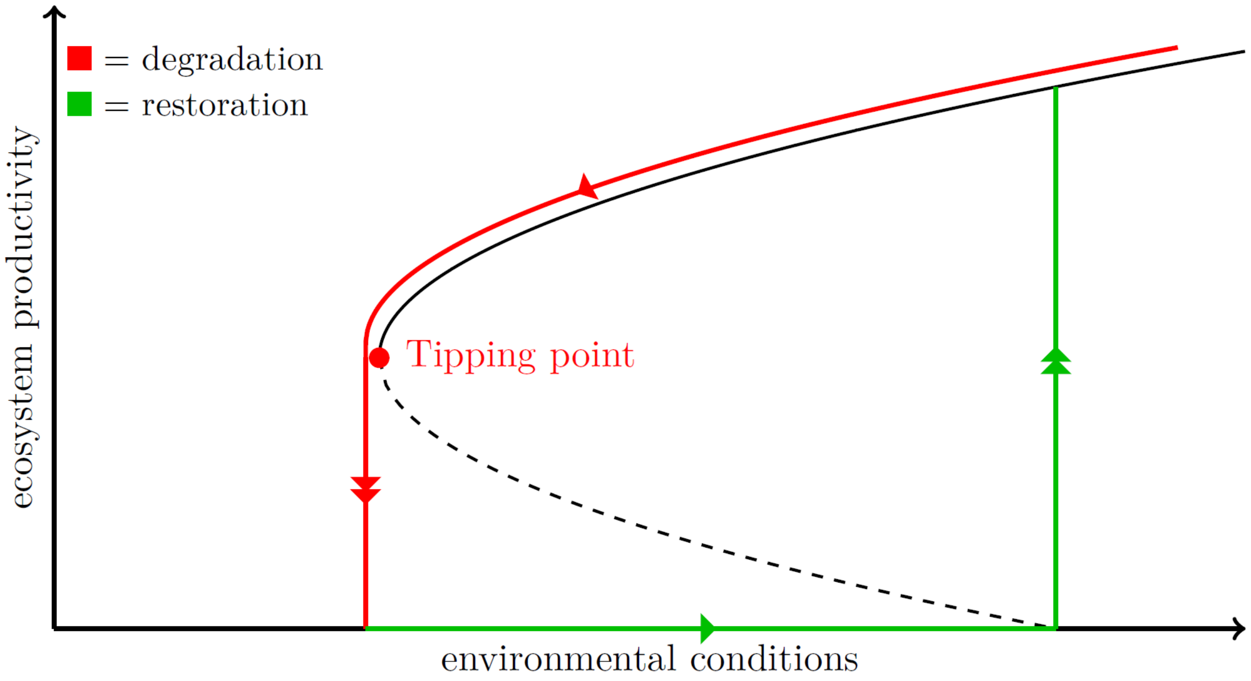


Busse balloon

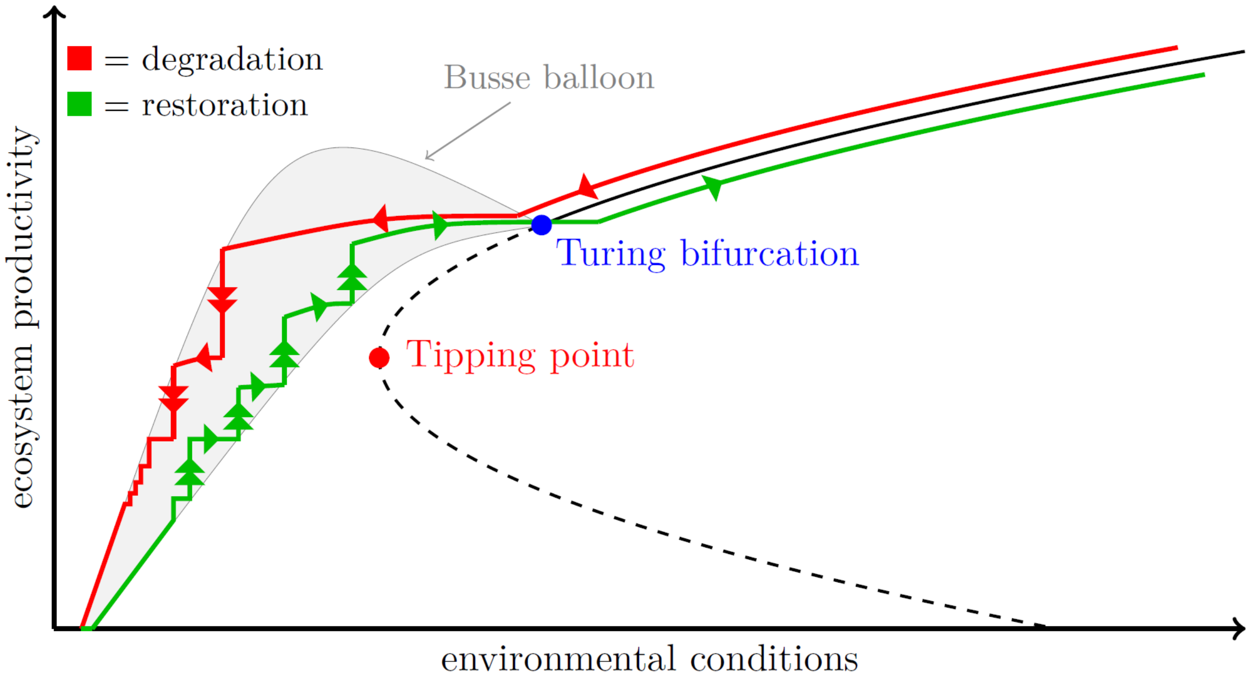
Idea originates from thermal convection

[Busse, 1978]

Tipping of (Turing) patterns

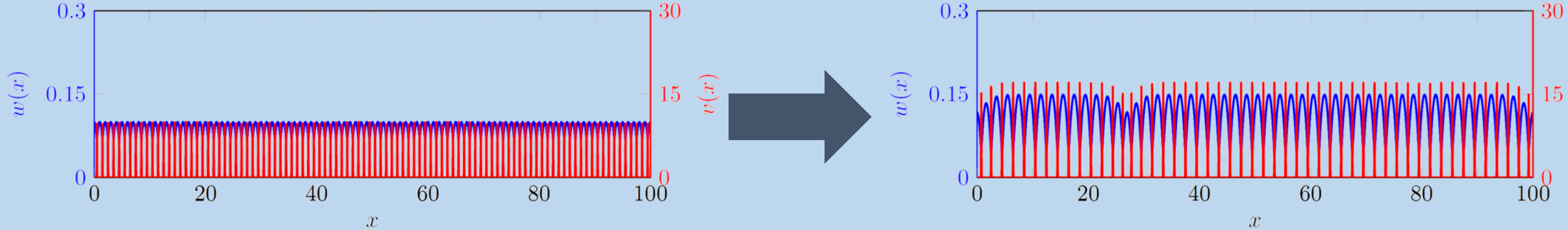


Classic tipping



Tipping of patterns

Degradation of patterns

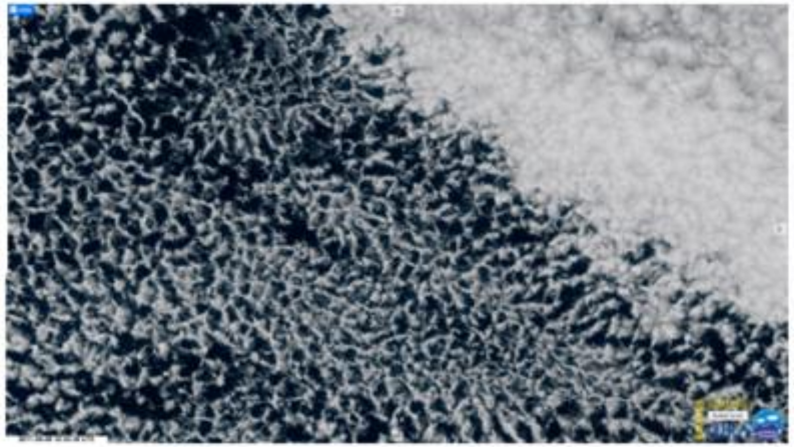


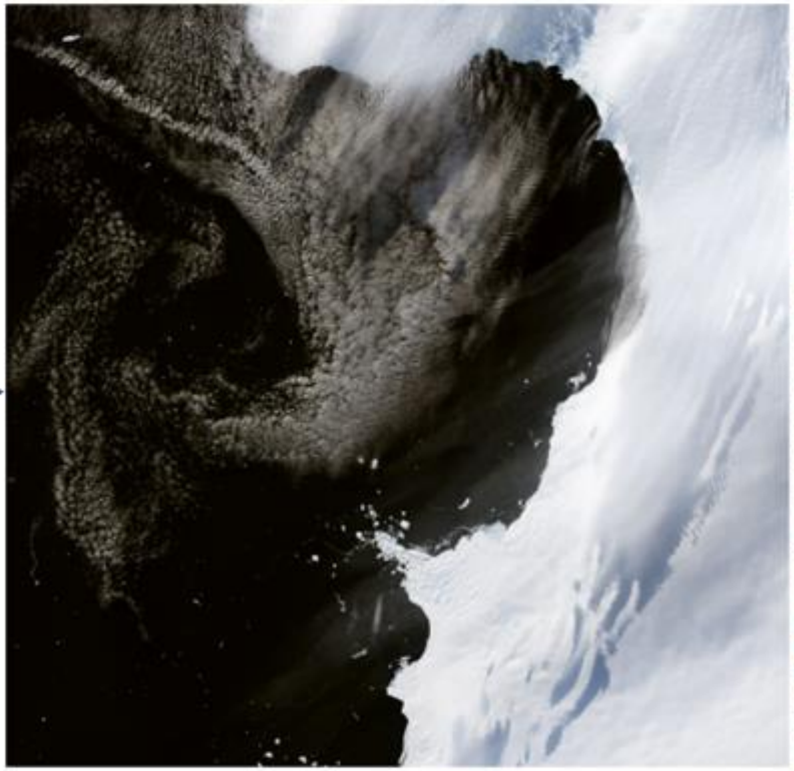

Part 2:
Coexistence States
and spatial heterogeneities

Examples of spatial patterning – spatial interfaces

tropical forest
& savanna
ecosystems
[Google Earth]

types of
stratocumulus
clouds
[RAMMB/CIRA SLIDER]

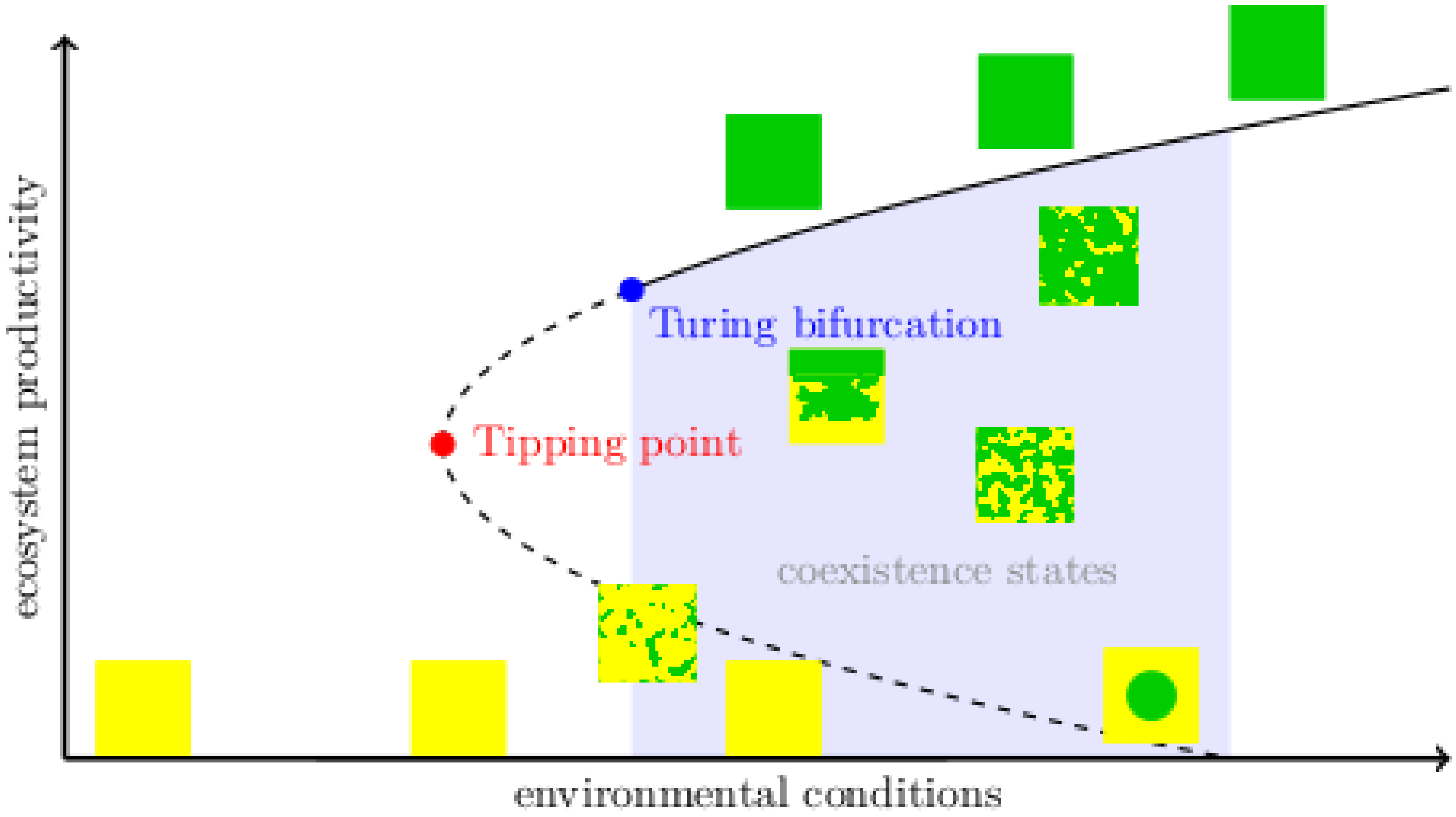




sea-ice & water
at Eltanin Bay
[NASA's Earth observatory]

algae bloom
in Lake St. Clair
[NASA's Earth observatory]

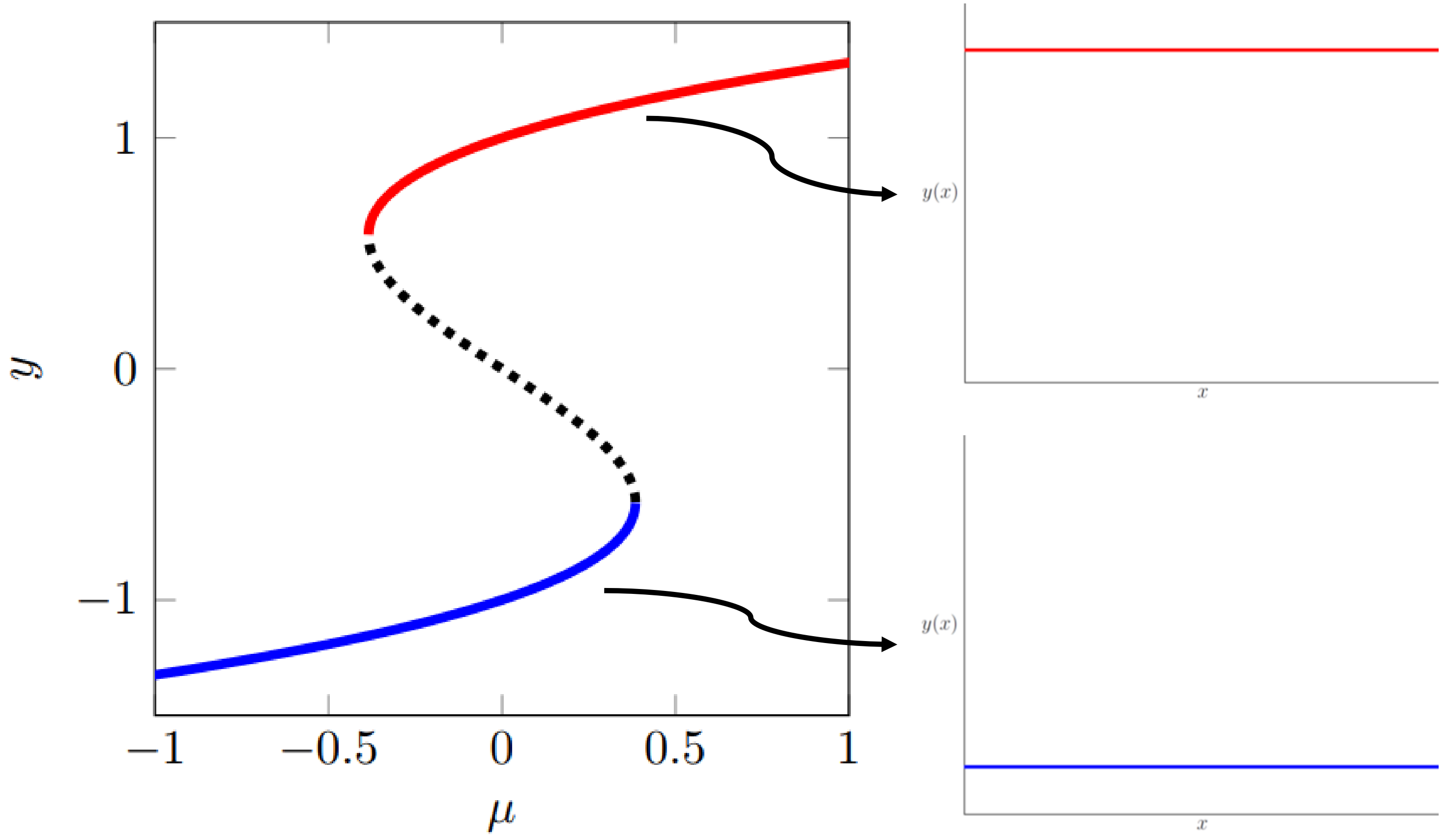
Coexistence states in bifurcation diagram

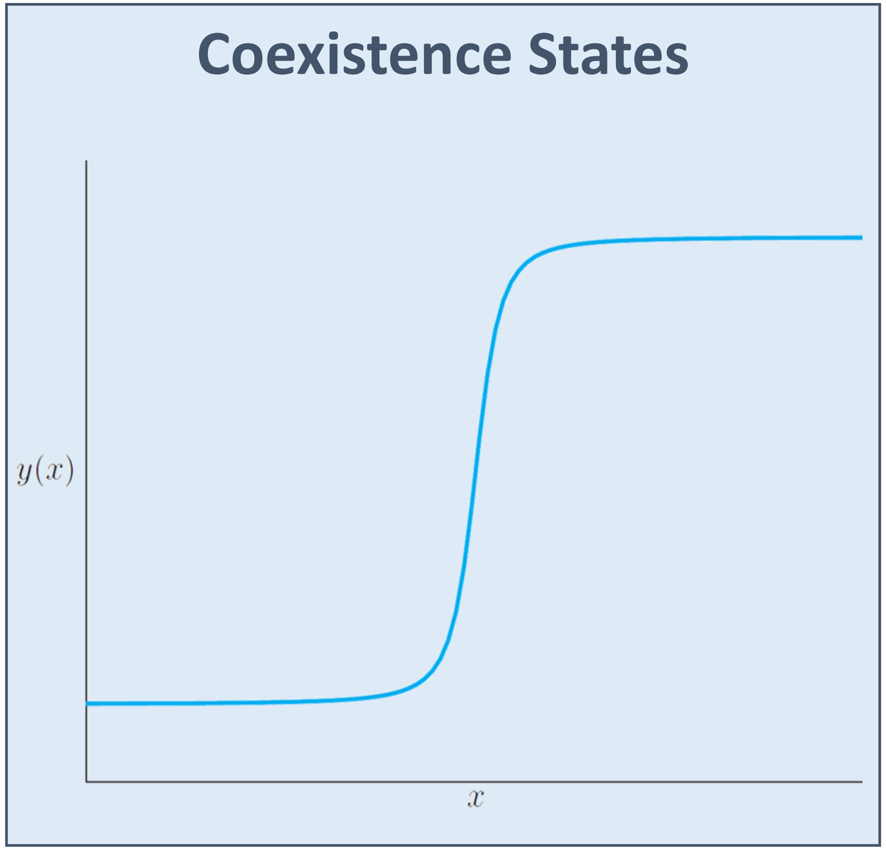


Coexistence states

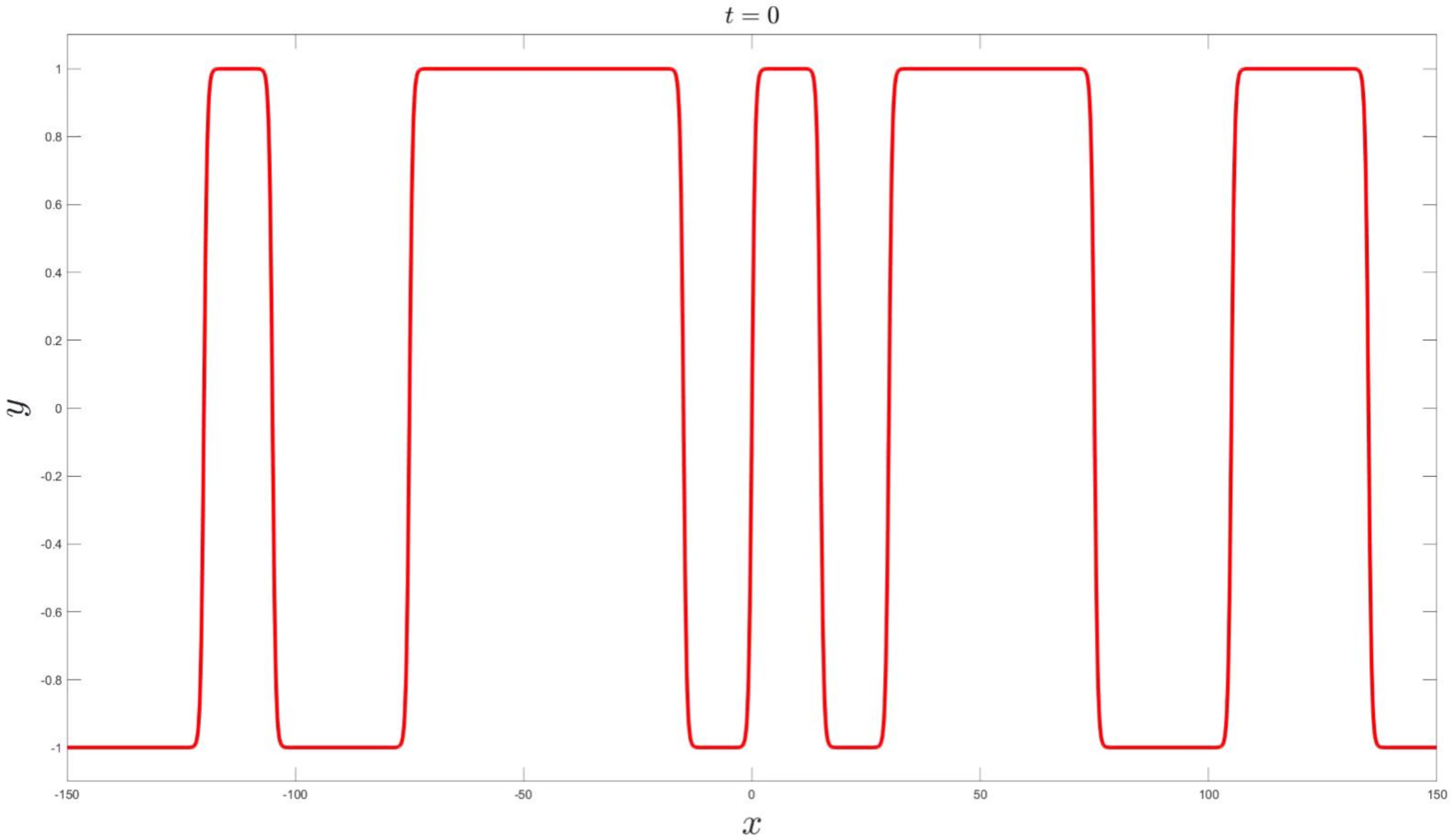
Bistable (Allen-Cahn/Nagumo) equation:

$$\frac{\partial y}{\partial t} = y(1 - y^2) + \mu + D\,\frac{\partial^2 y}{\partial x^2}$$

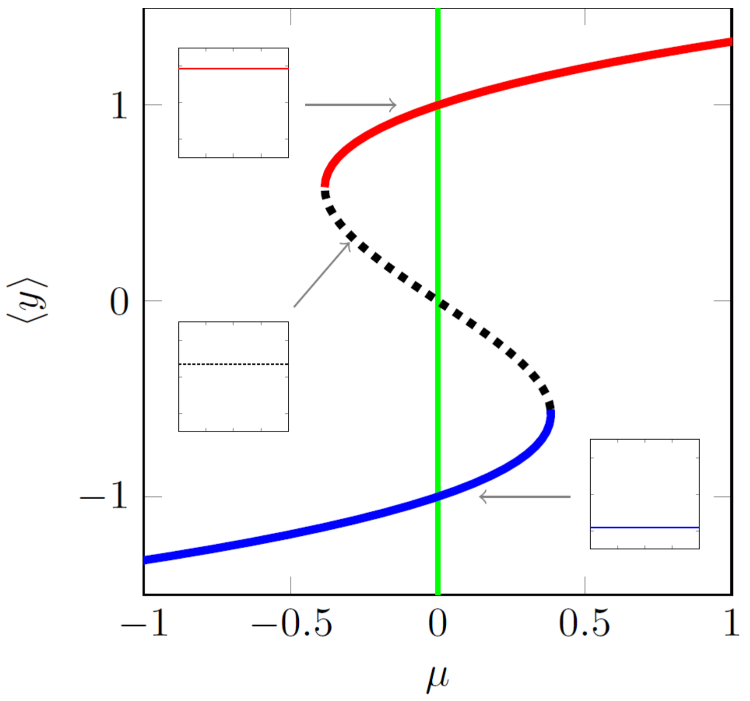




Dynamics of $\frac{\partial y}{\partial t} = D\ \frac{\partial^2 y}{\partial x^2} + y(1 - y^2) + \mu$



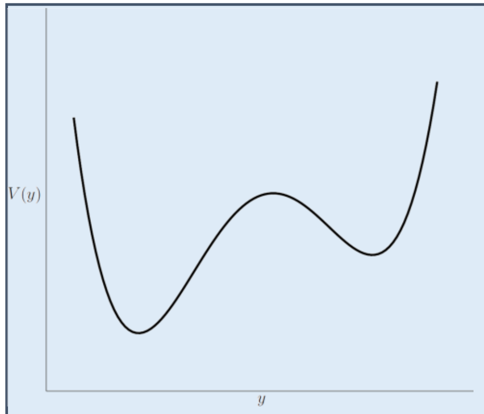
Front Dynamics



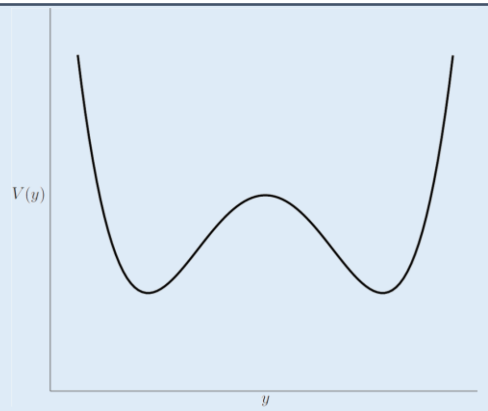
$$\frac{\partial y}{\partial t} = D\,\frac{\partial^2 y}{\partial x^2} + f(y;\mu)$$

Potential function $V(y;\mu)$:

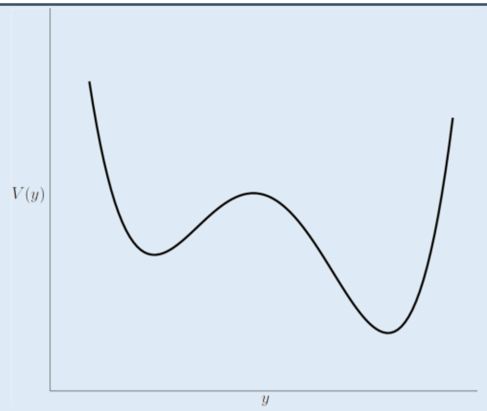
$$\frac{\partial V}{\partial y}(y;\mu) = -f(y;\mu)$$



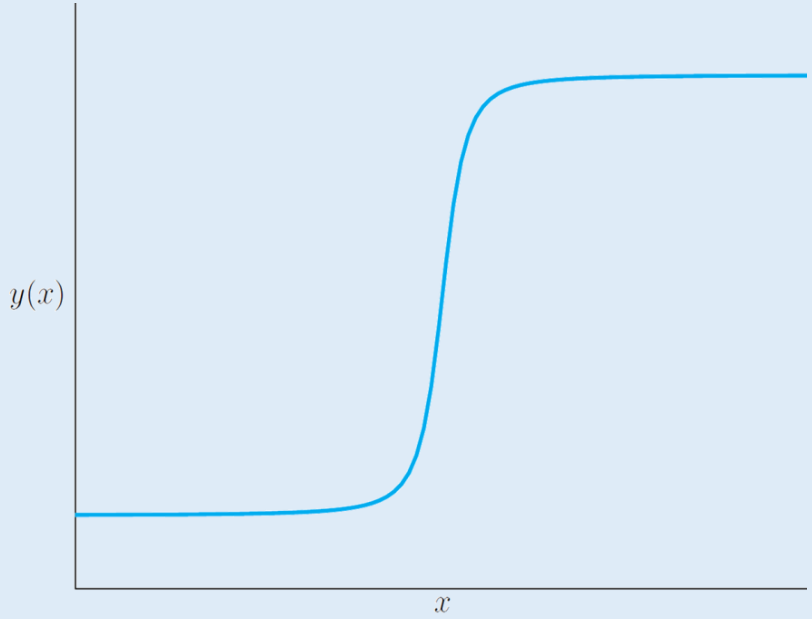
moves right



stationary



moves left



Maxwell Point $\mu_{maxwell}$

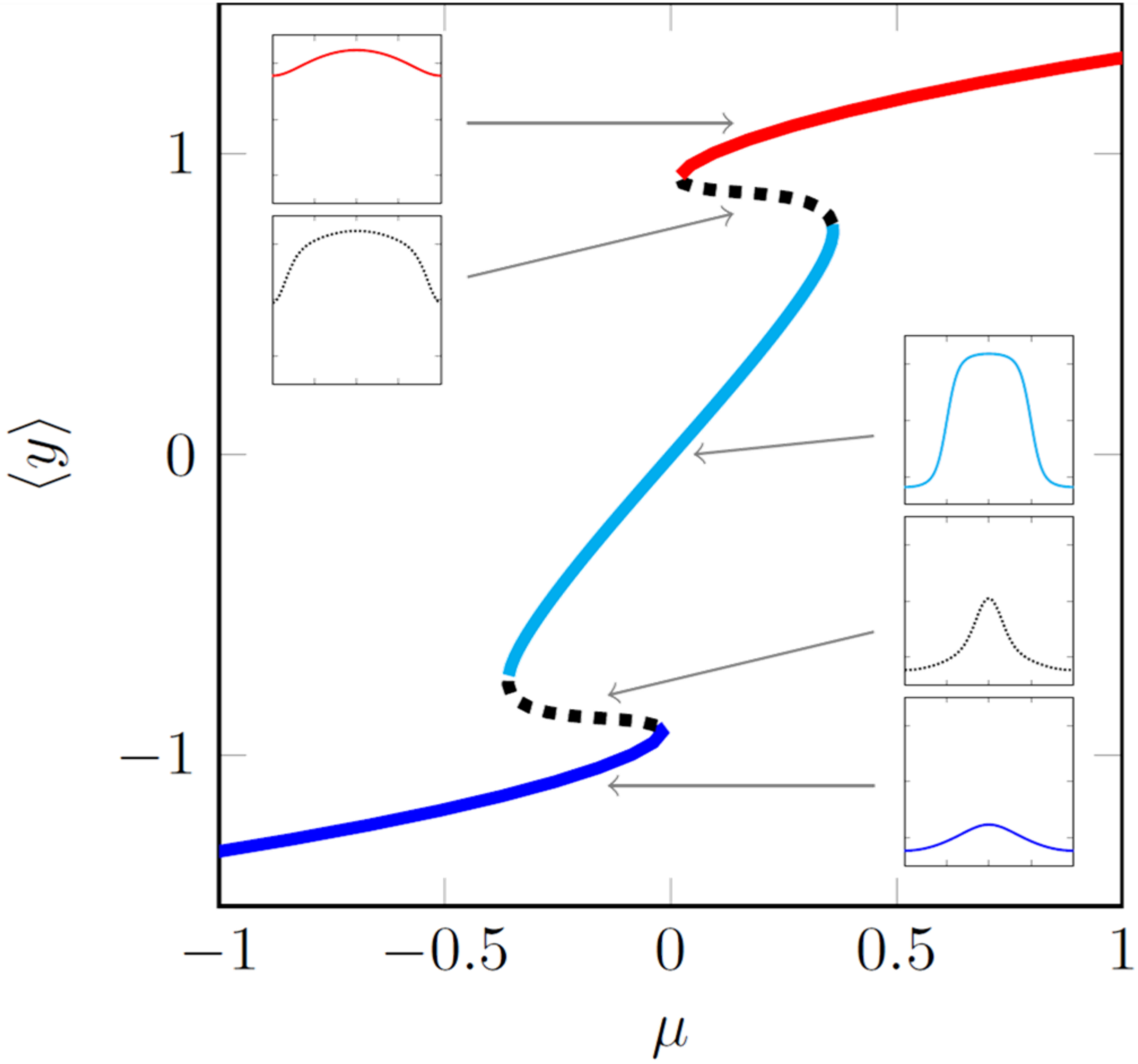
Adding Spatial Heterogeneity

$$\frac{\partial y}{\partial t} = D\frac{\partial^2 y}{\partial x^2} + f(y, x; \mu)$$

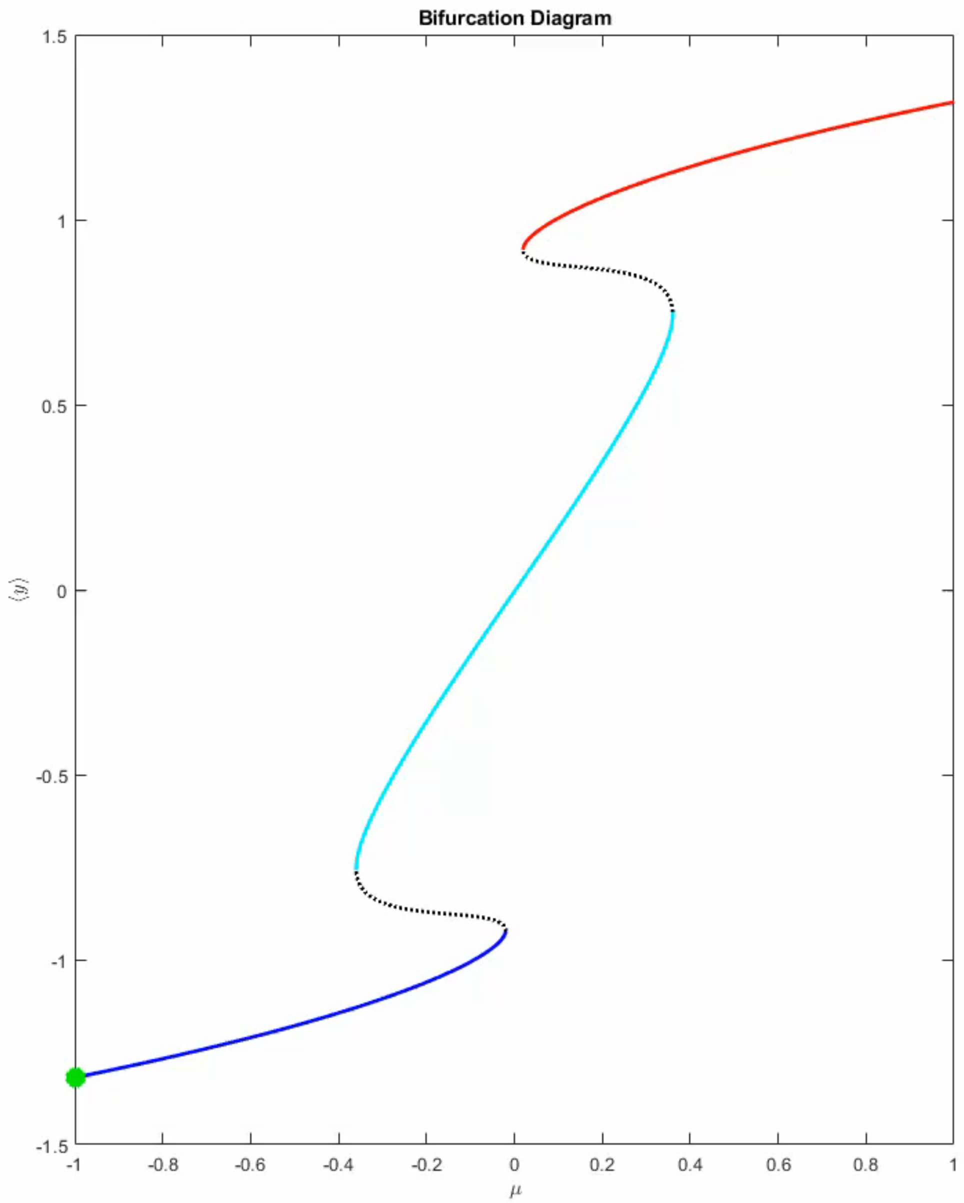
Now, the **local** difference in potentials determines the front movement

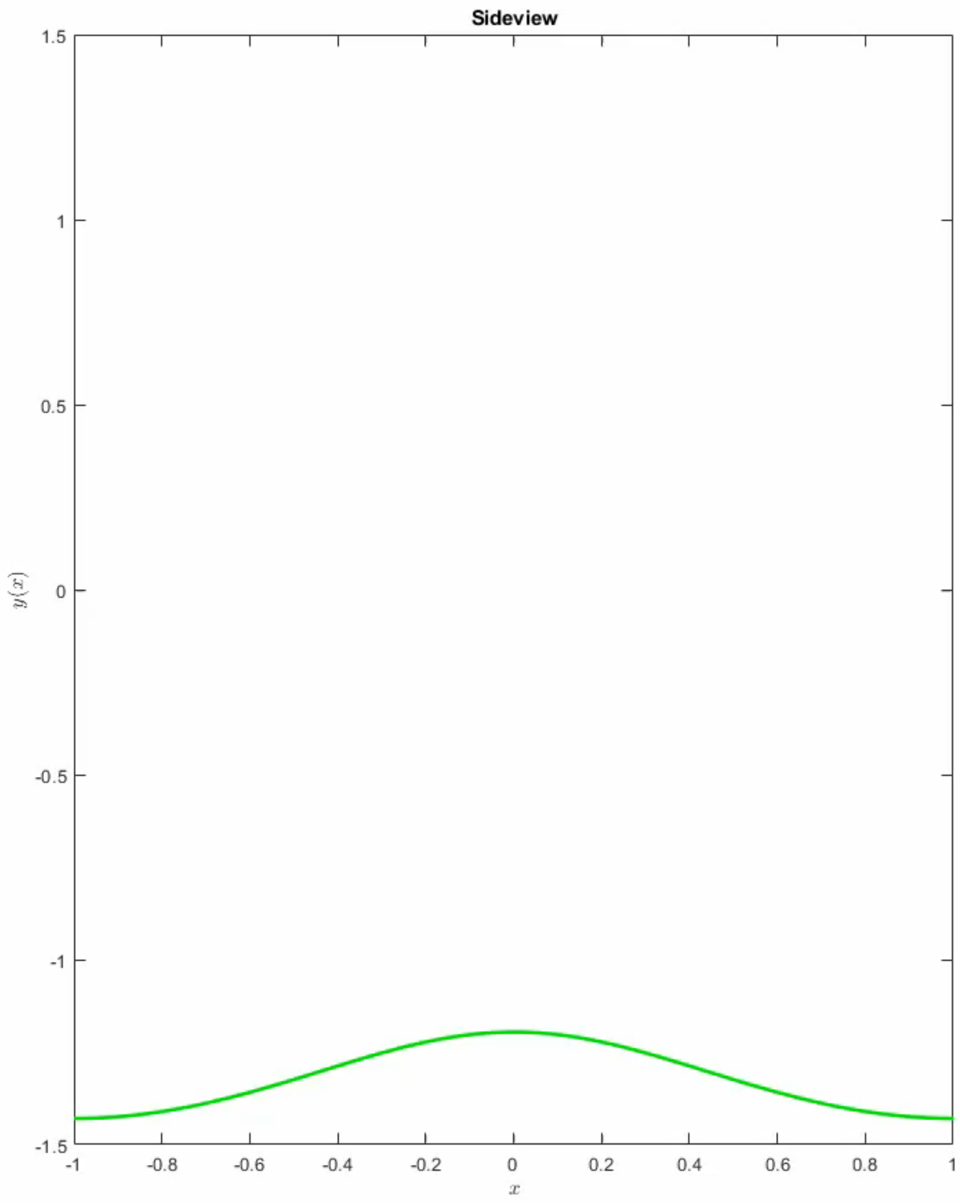
New behaviour:

- Multi-fronts can be stationary
- Maxwell point is smeared out

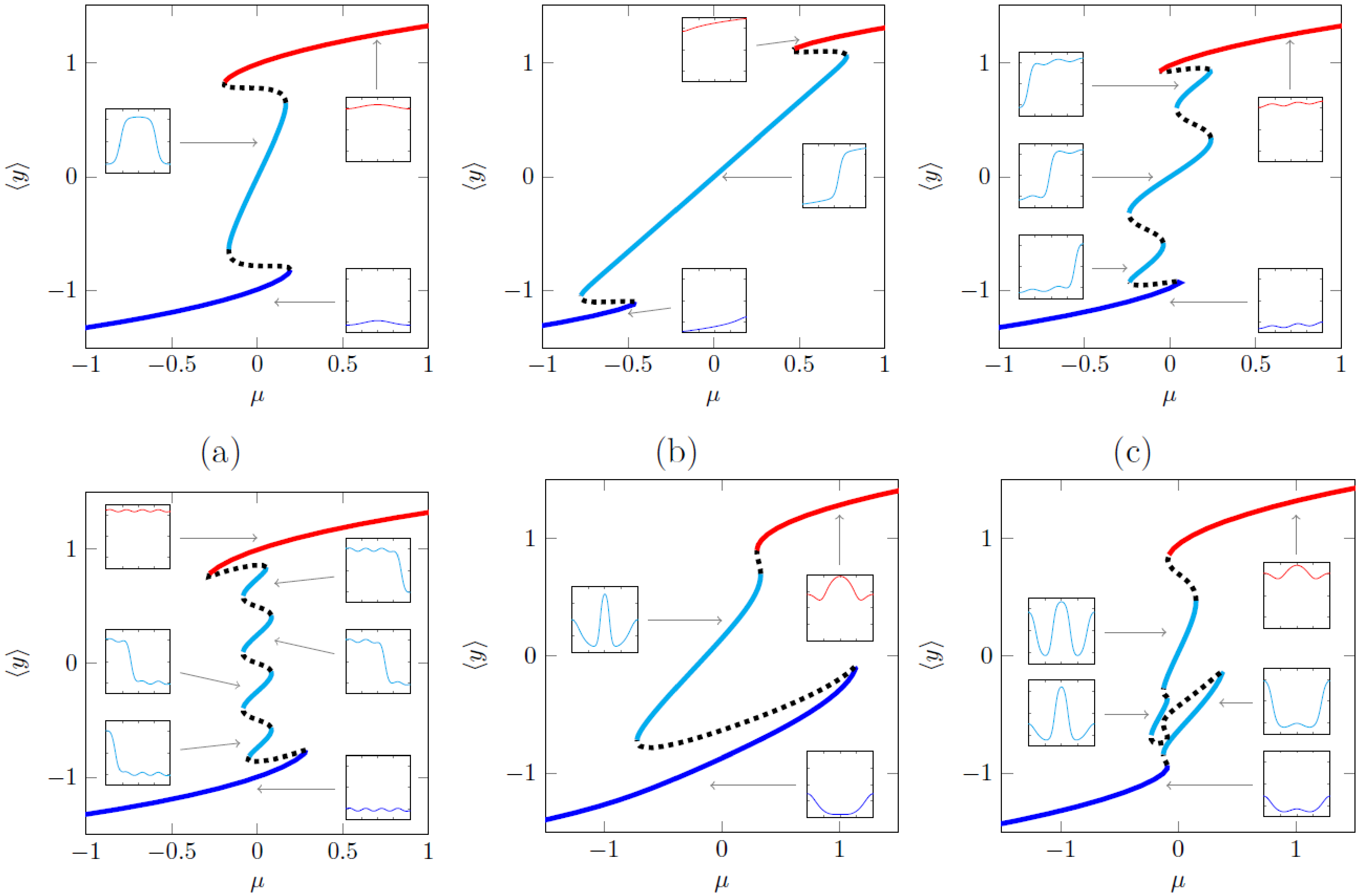


Fragmented Tipping



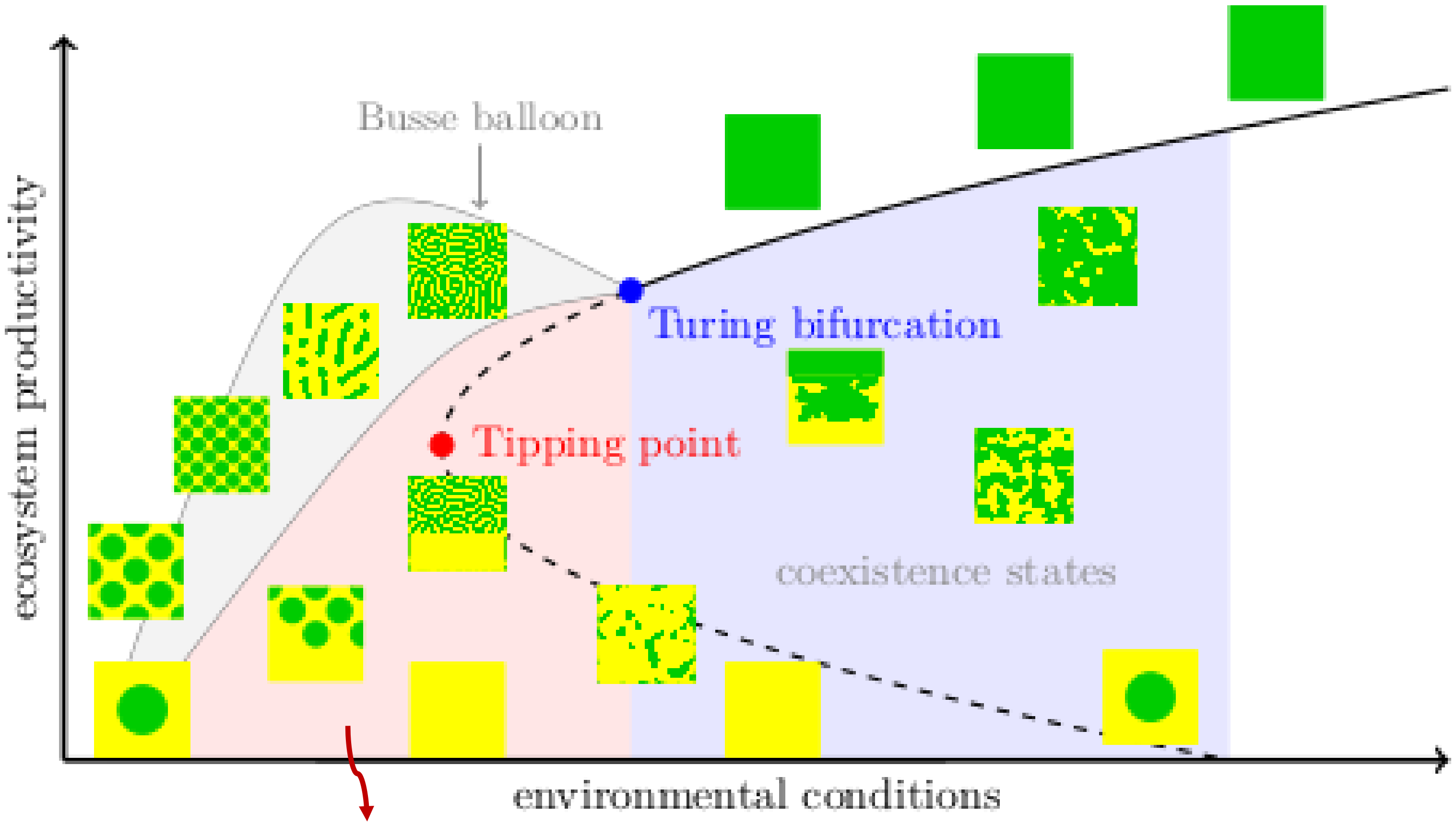


Other Spatial Heterogeneities

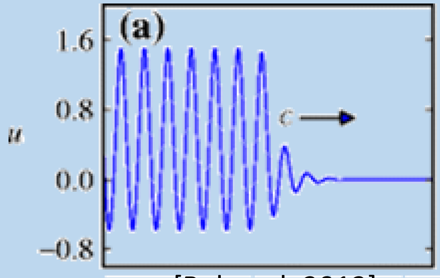



Part 3:
Tipping in Spatially
Extended Systems?

“Bifurcation Diagram” for spatially extended systems



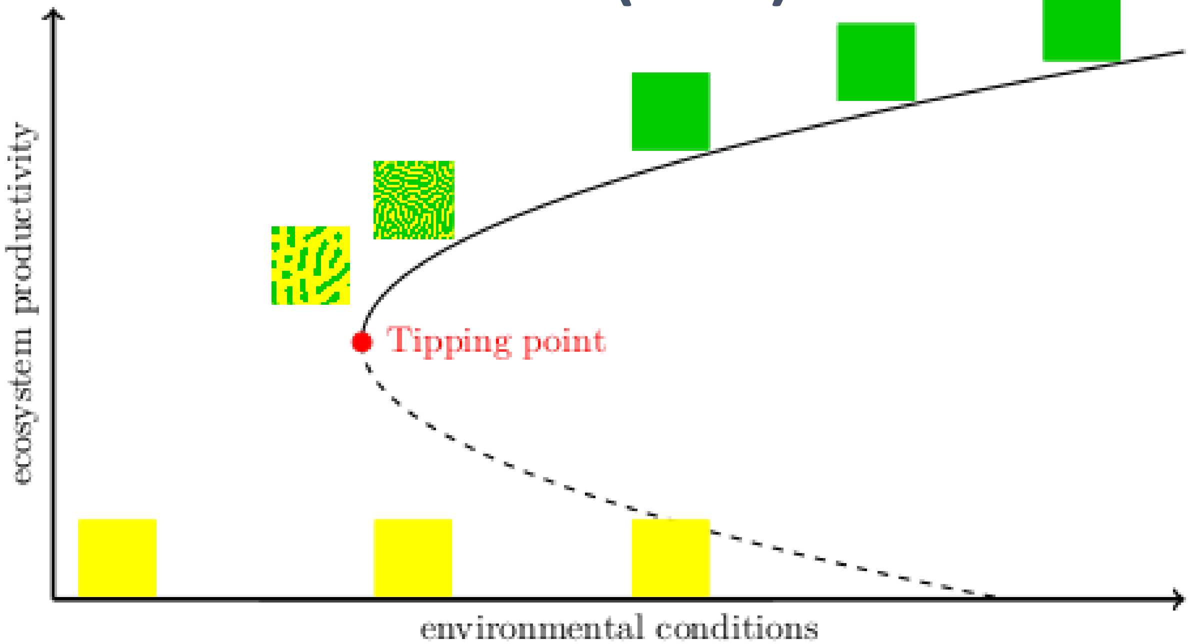
Coexistence states between patterned and uniform states also exist



[Bel et al, 2012]

What if the system tips?

Classic (ODE)



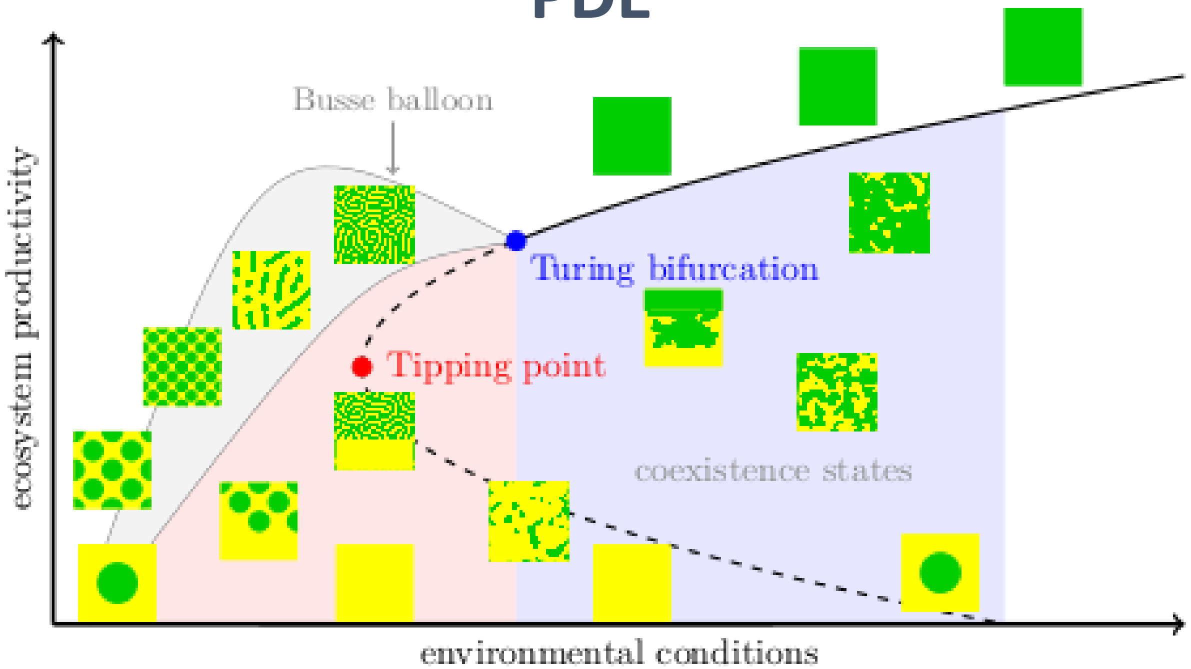
Crossing a Tipping Point:

→ Always full reorganization

Early Warning Signals

signal for <u>WHEN</u>

PDE



Crossing a bifurcation:

Now also possible:

→ Spatial reorganization (Turing patterns)

→ Fragmented tipping (coexistence states)

Early Warning Signals

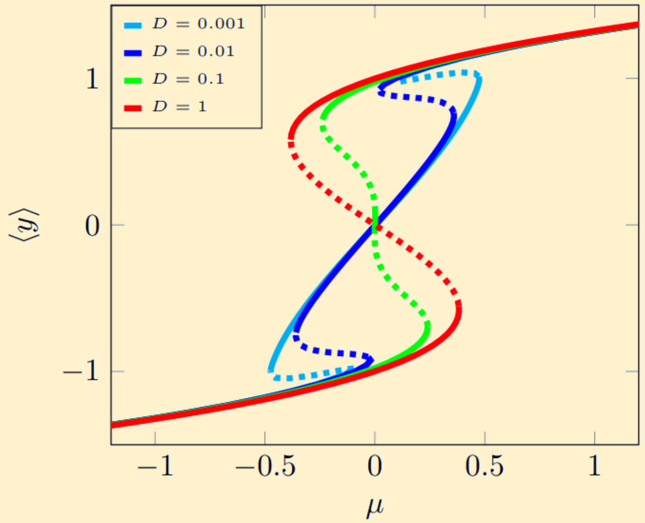
need to signal <u>WHEN</u> & <u>WHAT</u>

Do systems always behave like this? (a.k.a. the small print)

No.

Well-mixed systems

Spatially confined systems



→ Such systems (again) behave like ODEs ←

But even in other systems terms & conditions apply:
System-specific knowledge is required!

Summary

Spatial Patterns:

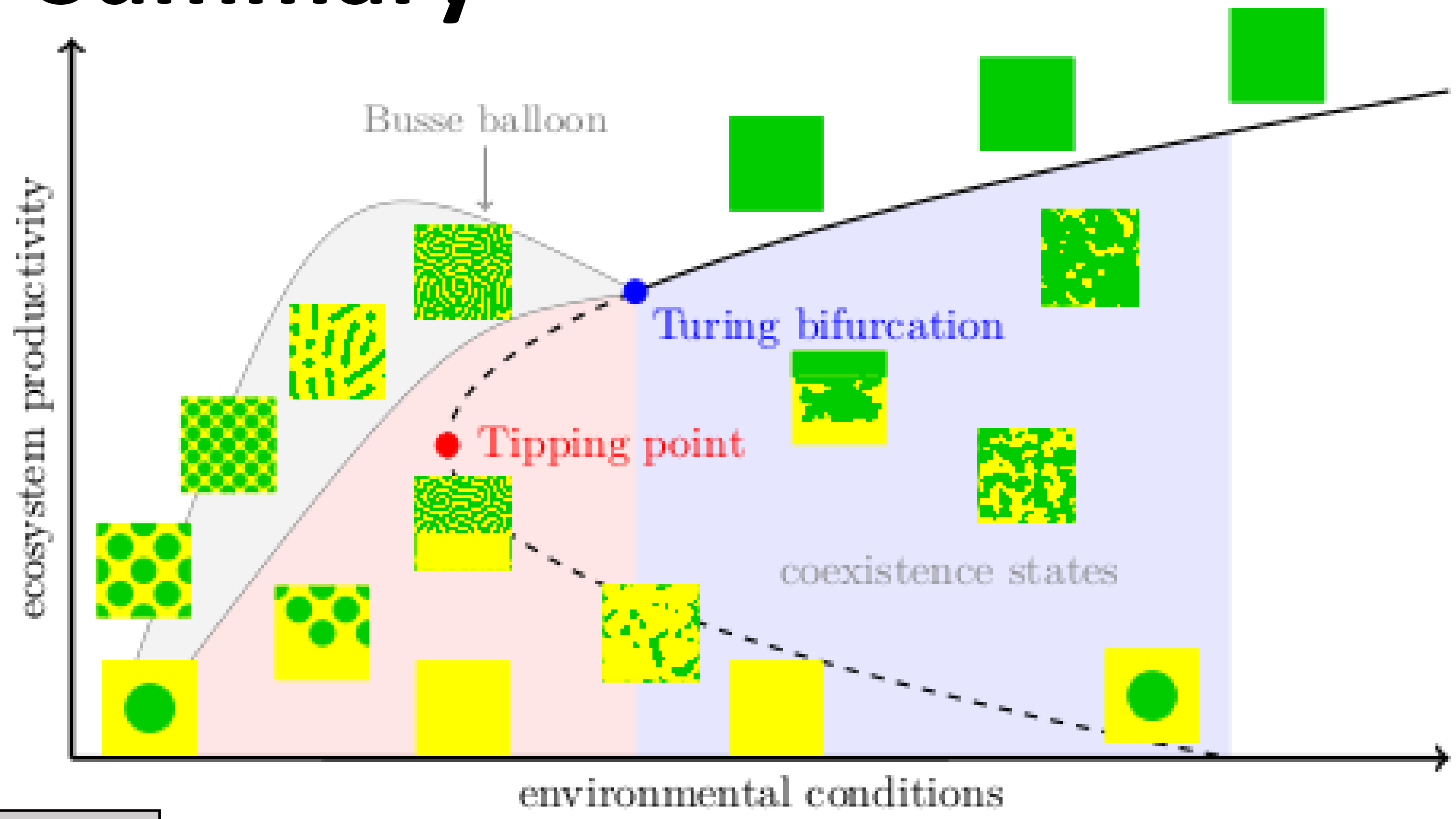
- Turing Patterns
- Coexistence States

Tipping can be more subtle:

- Spatial reorganization
- Fragmented Tipping

Dynamics of Patterns is:

- Slow Pattern Adaptation
- Fast Pattern Degradation



THANKS TO:

Swarnendu Banerjee	Mara Baudena	Alexandre Bouvet
Martina Chirilus-Bruckner	Vincent Deblauwe	Arjen Doelman
Henk Dijkstra	Maarten Eppinga	Anna von der Heydt
Olfa Jaïbi	Johan van de Koppel	Stéphane Mermoz
Max Rietkerk	Eric Siero	Koen Siteur

Rietkerk, M., Bastiaansen, R., Banerjee, S., van de Koppel, J., Baudena, M., & Doelman, A. (2021). Evasion of tipping in complex systems through spatial pattern formation. Science, 374(6564), eabj0359.

Bastiaansen, R., Dijkstra, H. A., & von der Heydt, A. S. (2021). Fragmented Tipping in a spatially heterogeneous world. *Environmental Research Letters, 17, 045006*