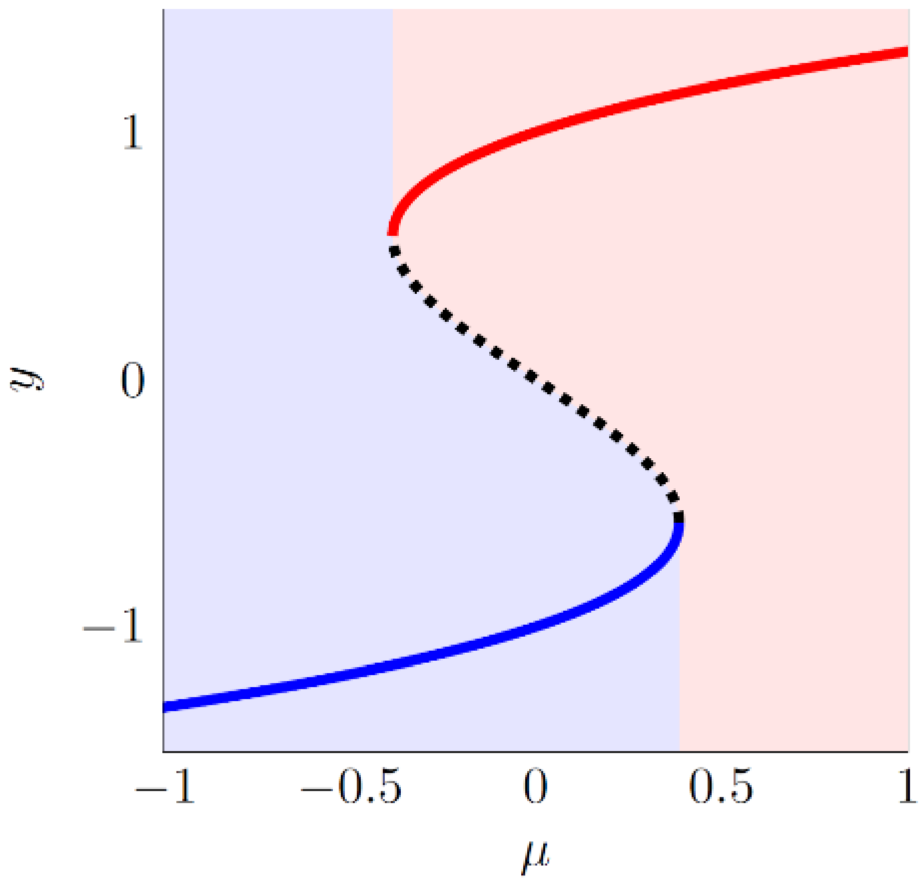
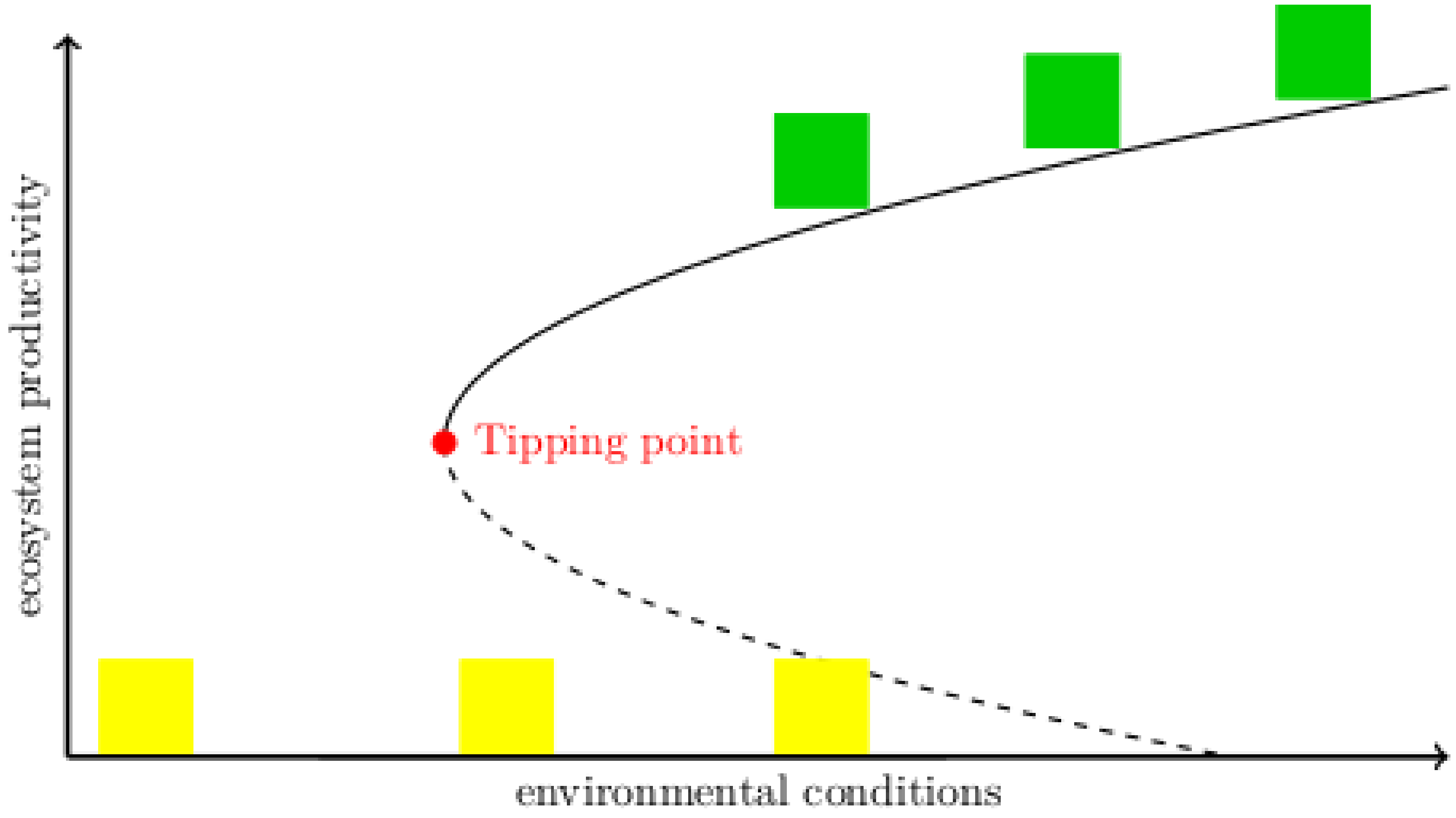
Tipping
in
Spatially Extended Systems

2023-04-19, Colloquium, MI, Potsdam University

Robbin Bastiaansen (r.bastiaansen@uu.nl)

Classic Theory of Tipping





Canonical example:

$$\frac{dy}{dt} = y(1 - y^2) + \mu$$

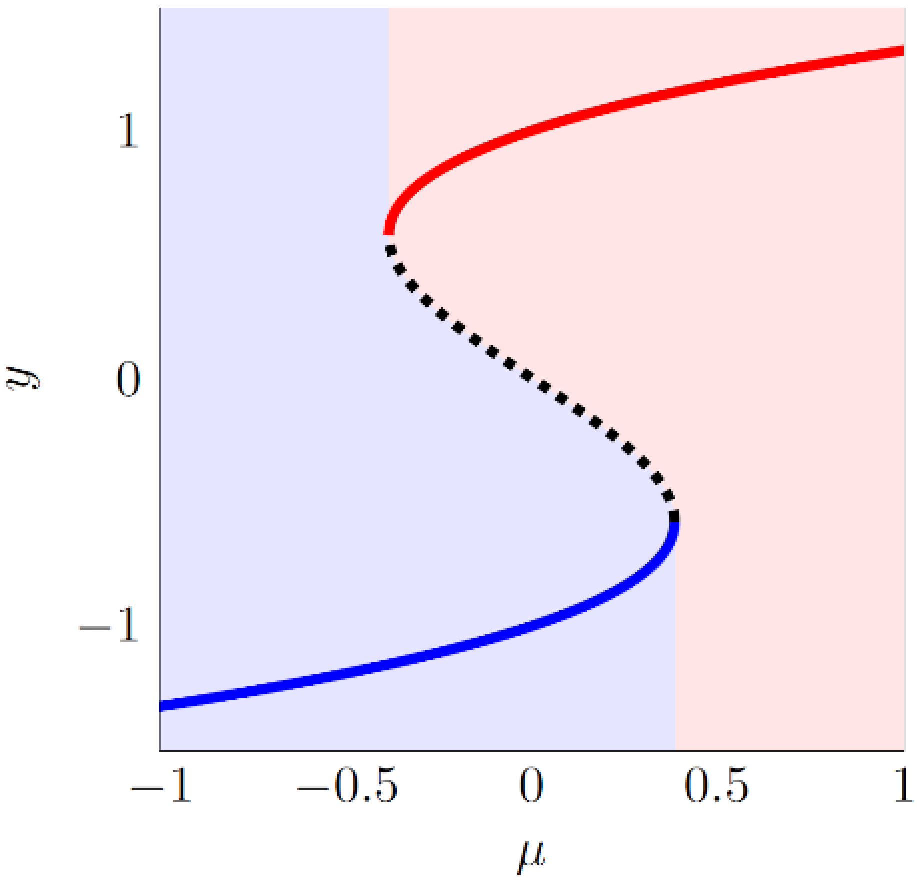
$$\frac{d\vec{y}}{dt} = f(\vec{y}; \mu)$$

$$\begin{cases} \frac{du}{dt} = f(u, v; \mu) \\ \frac{dv}{dt} = g(u, v; \mu) \end{cases}$$

Tipping in ODEs (1)

Canonical example:

$$\frac{dy}{dt} = y(1 - y^2) + \mu$$



Concrete example: Global Energy Balance Model

$$\frac{dT}{dt} = Q\big(1 - \alpha(T)\big) - \varepsilon\sigma_0 T^4 + \mu$$

Classic Literature

[Holling, 1973]

[Noy-Meier, 1975]

[May, 1977]

Tipping

[Ashwin et al, 2012]

Bifurcation-Tipping : Basin disappears

Noise-Tipping : Forced outside Basin

Rate-Tipping : *(more complicated)*

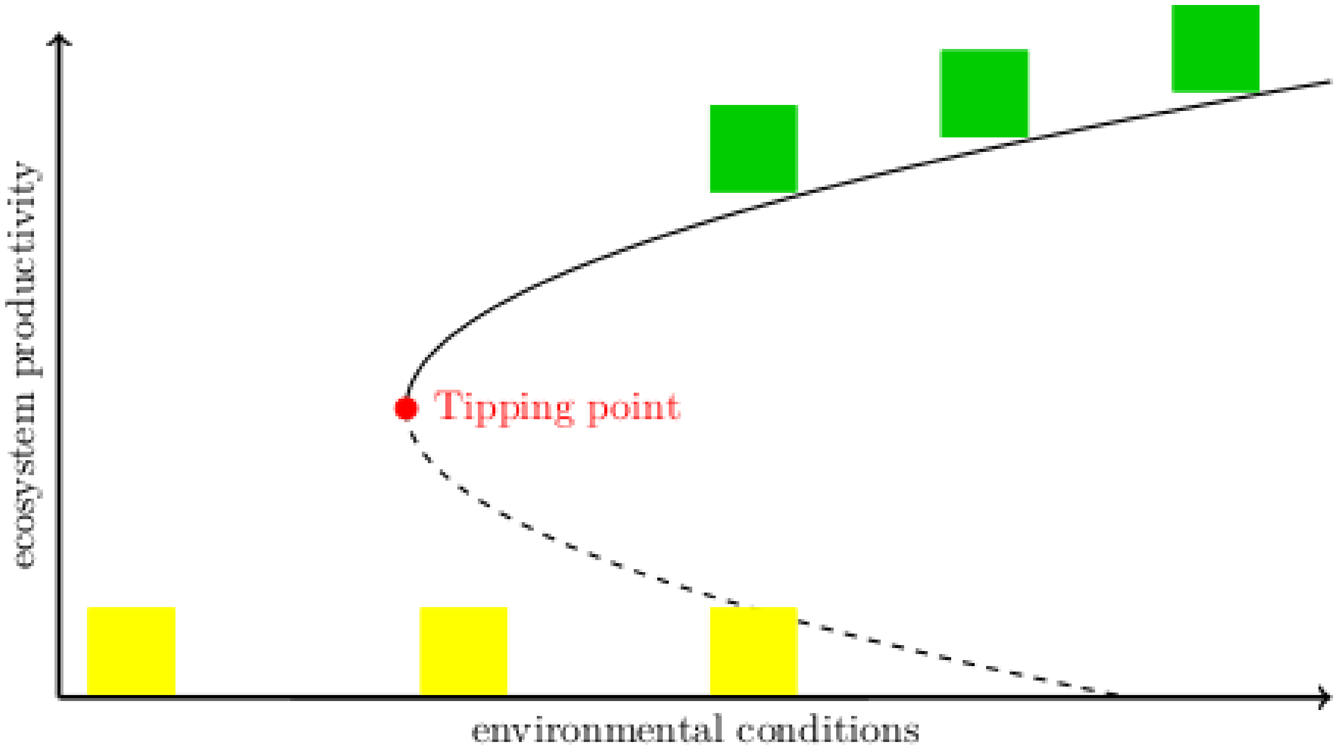
Tipping in ODEs (2)

Two components:

$$\begin{cases} \dfrac{du}{dt} = f(u,v) \\ \dfrac{dv}{dt} = g(u,v) \end{cases}$$

includes common models:

- Predetor-Prey
- Activator-Inhibitor

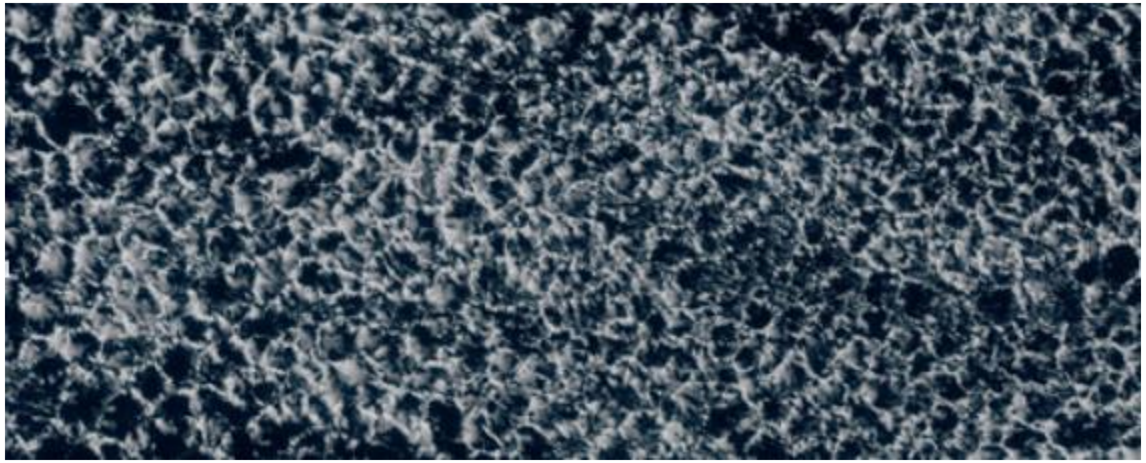


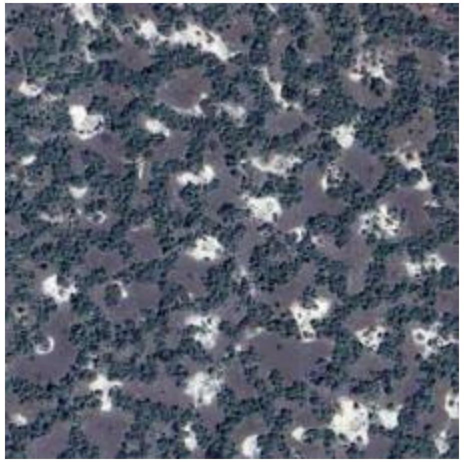
Examples of tipping in ODEs include:

- Forest-Savanna bistability
- Deep ocean exchange
- Cloud formation
- Ice sheet melting
- Turbidity in shallow lakes

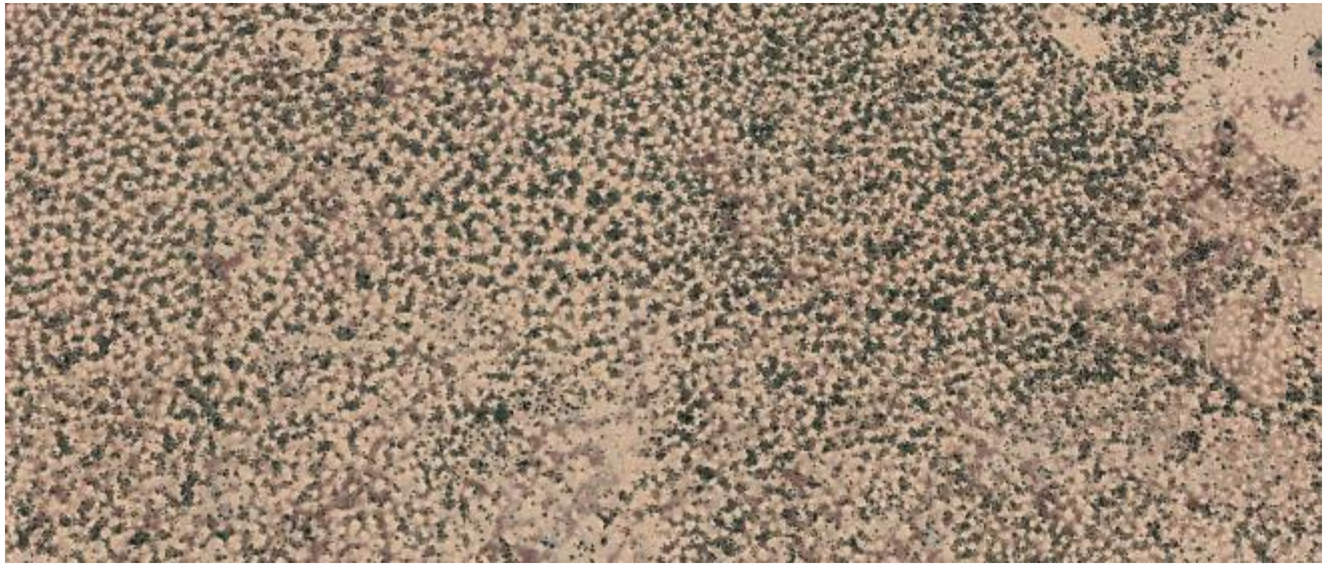
Examples of spatial patterning – regular patterns


mussel beds


clouds


savannas

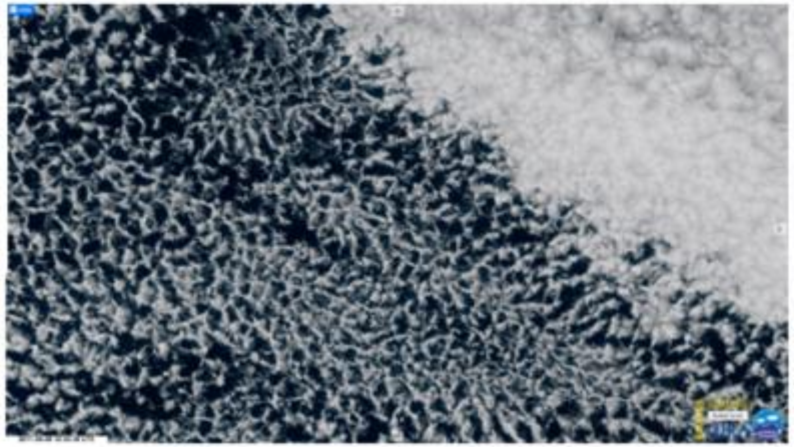

melt ponds


drylands

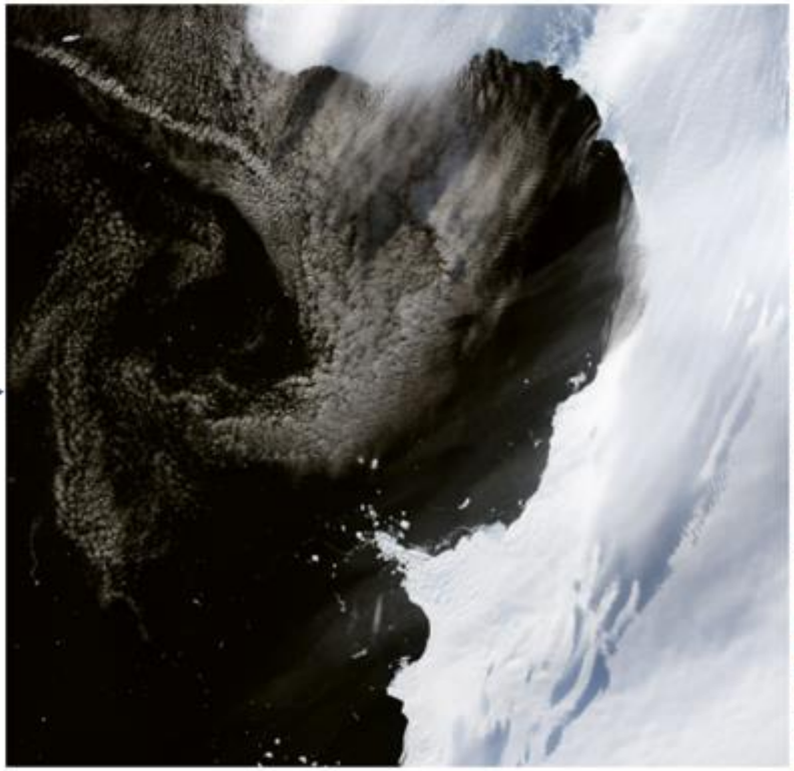
Examples of spatial patterning – spatial interfaces

tropical forest
& savanna
ecosystems
[Google Earth]

types of
stratocumulus
clouds
[RAMMB/CIRA SLIDER]



sea-ice & water
at Eltanin Bay
[NASA's Earth observatory]



algae bloom
in Lake St. Clair
[NASA's Earth observatory]

Part 1: Turing Patterns

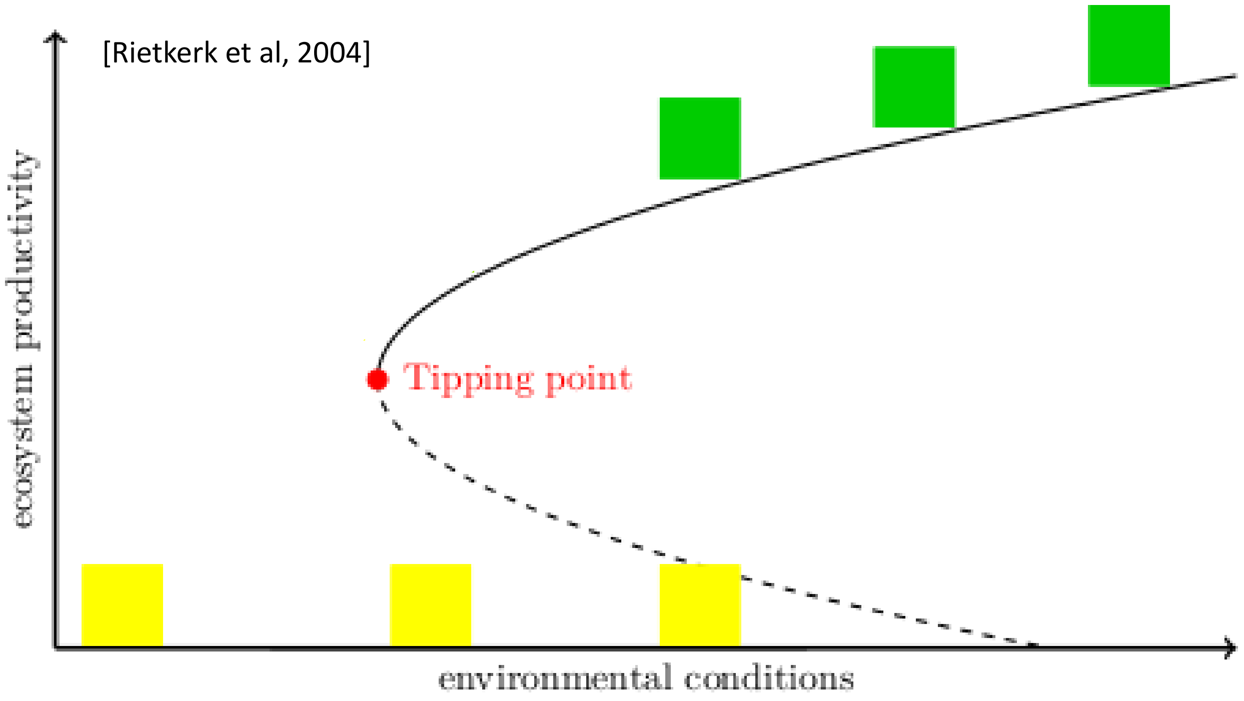
image courtesy: National Science Foundation

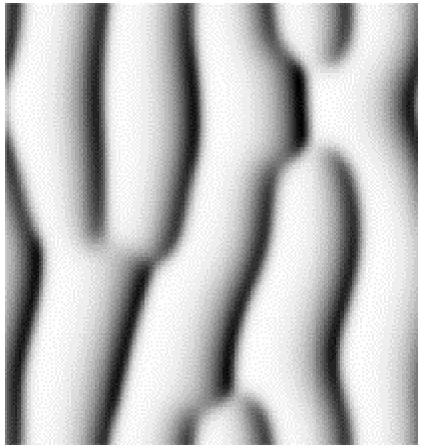
Patterns in models

Add spatial transport:

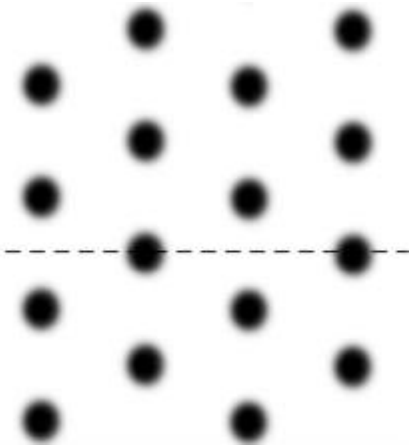
Reaction-Diffusion equations:

$$\begin{cases} \dfrac{du}{dt} = f(u,v) + D_u\,\Delta u \\ \dfrac{dv}{dt} = g(u,v) + D_v\,\Delta v \end{cases}$$



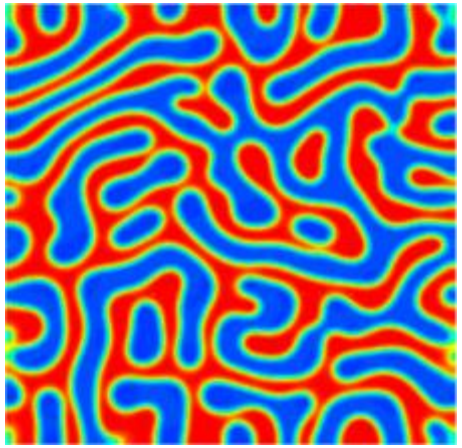


[Klausmeier, 1999]



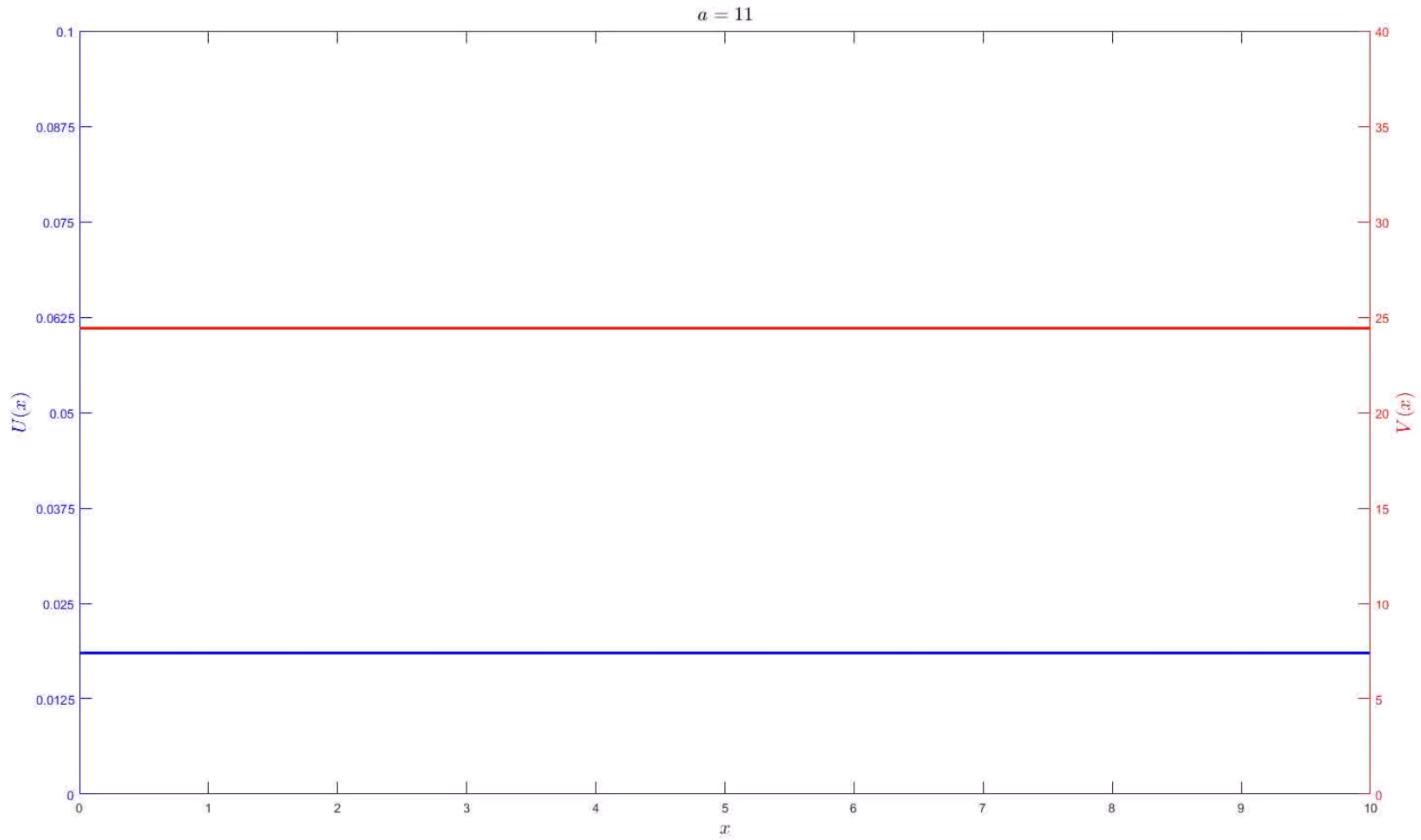
[Gilad et al, 2004]

[Rietkerk et al, 2002]



[Liu et al, 2013]

Behaviour of PDEs



Turing patterns

$$\begin{cases} \frac{du}{dt} = f(u,v) + D_u \Delta u \\ \frac{dv}{dt} = g(u,v) + D_v \Delta v \end{cases}$$

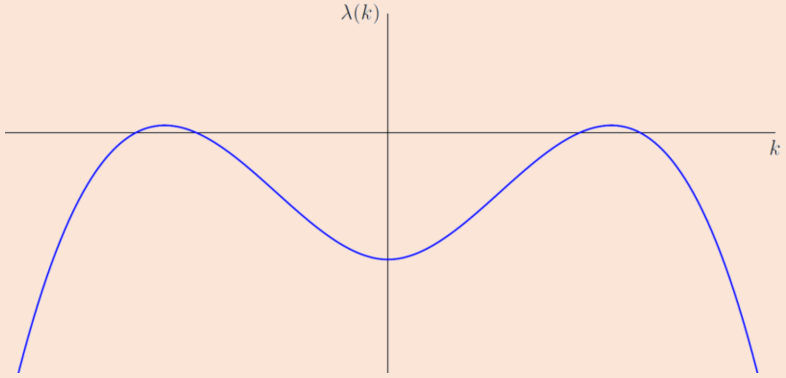
Turing bifurcation

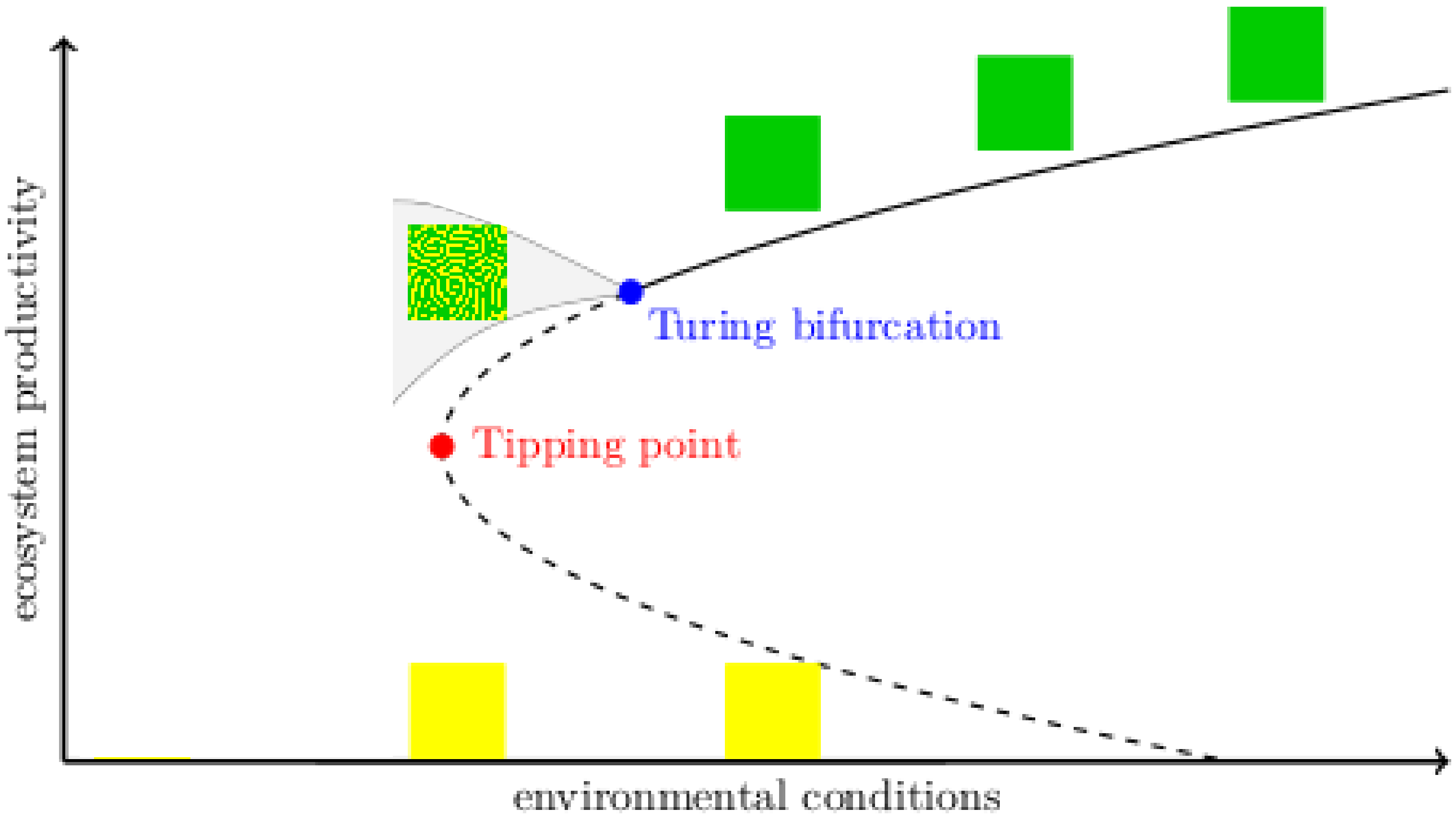
Instability to non-uniform perturbations

$$\binom{u}{v} = \binom{u_*}{v_*} + e^{\lambda t} e^{ikx} \binom{\bar{u}}{\bar{v}}$$

→ Dispersion relation

$$\lambda(k) = \cdots$$





Weakly non-linear analysis

Ginzburg-Landau equation / Amplitude Equation

& Eckhaus/Benjamin-Feir-Newel criterion

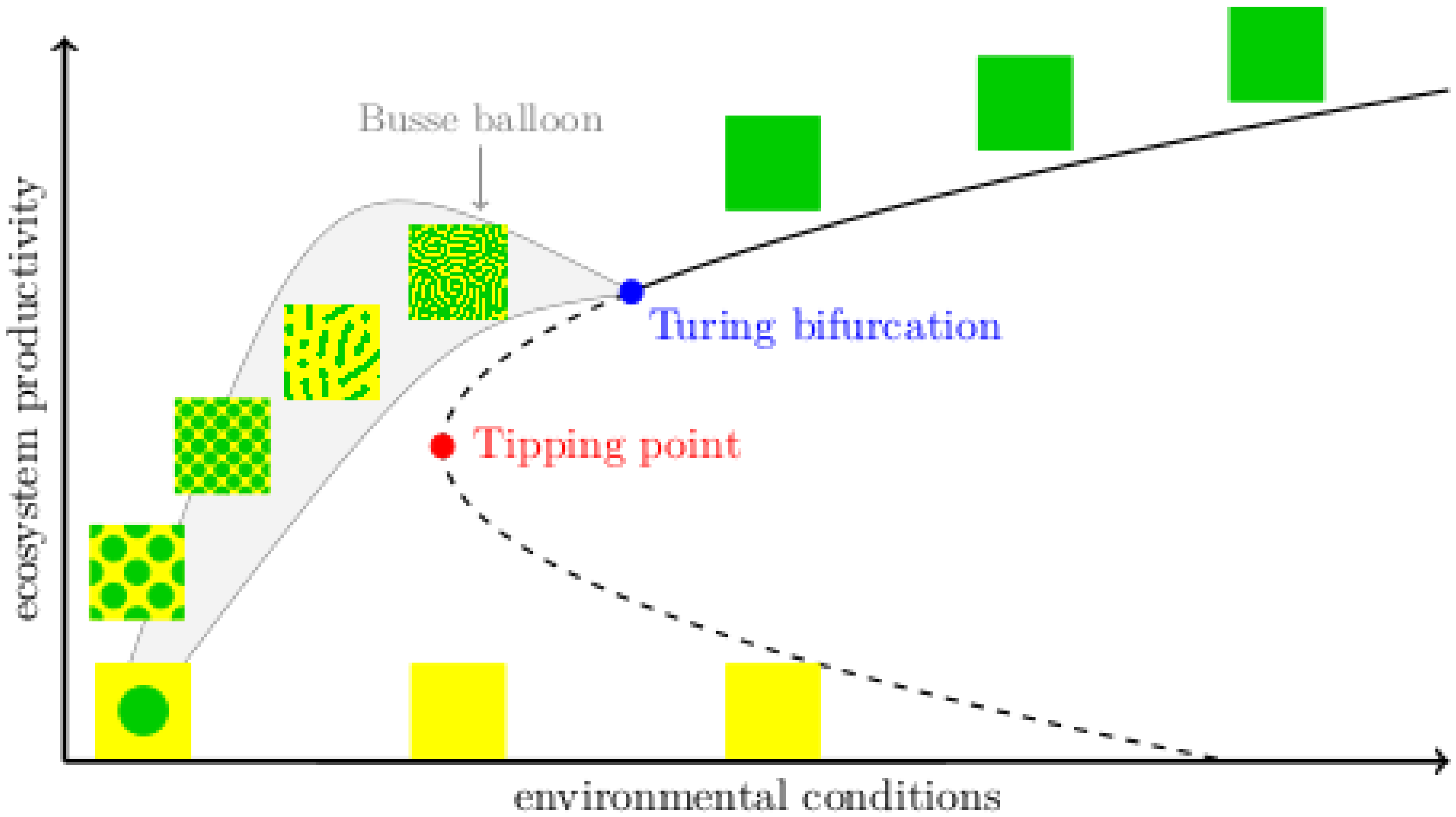
[Eckhaus, 1965; Benjamin & Feir, 1967; Newell, 1974]

Busse balloon

$$\begin{cases} \frac{du}{dt} = f(u,v) + D_u \Delta u \\ \frac{dv}{dt} = g(u,v) + D_v \Delta v \end{cases}$$

Busse balloon

A model-dependent shape in *(parameter, observable)* space that indicates all stable patterned solutions to the PDE.



Construction Busse balloon

Via numerical continuation

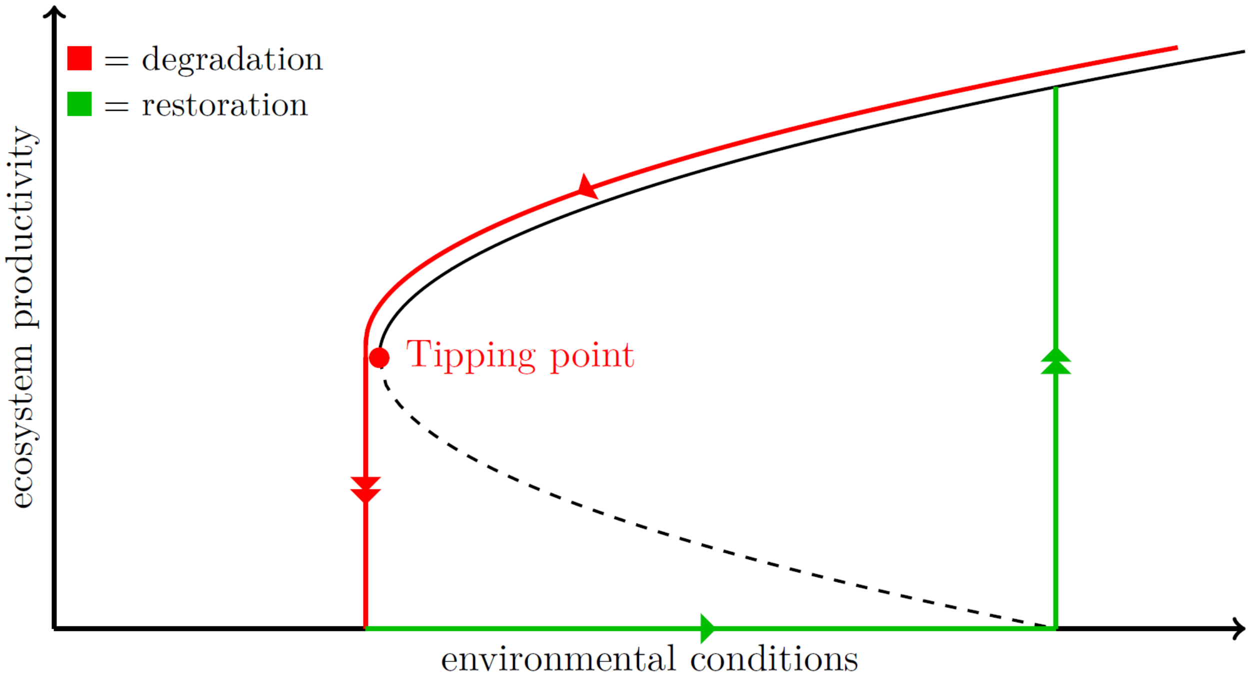
few general results on the shape of Busse balloon

Busse balloon

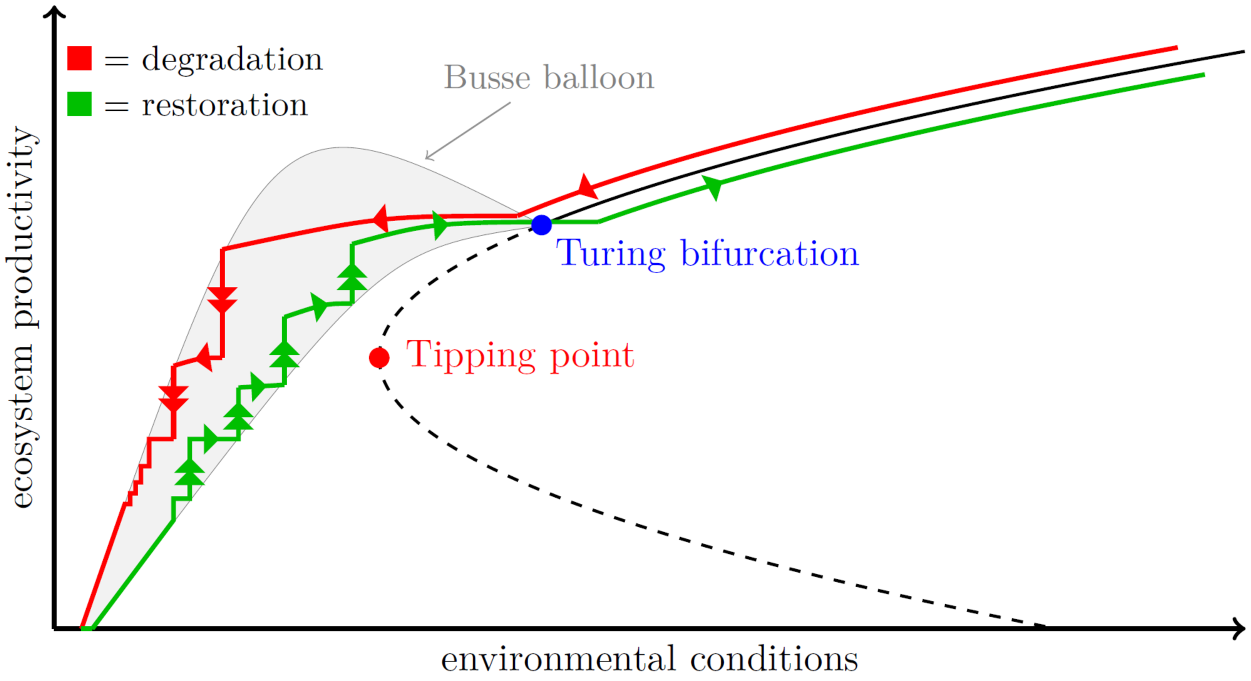
Idea originates from thermal convection

[Busse, 1978]

Tipping of (Turing) patterns

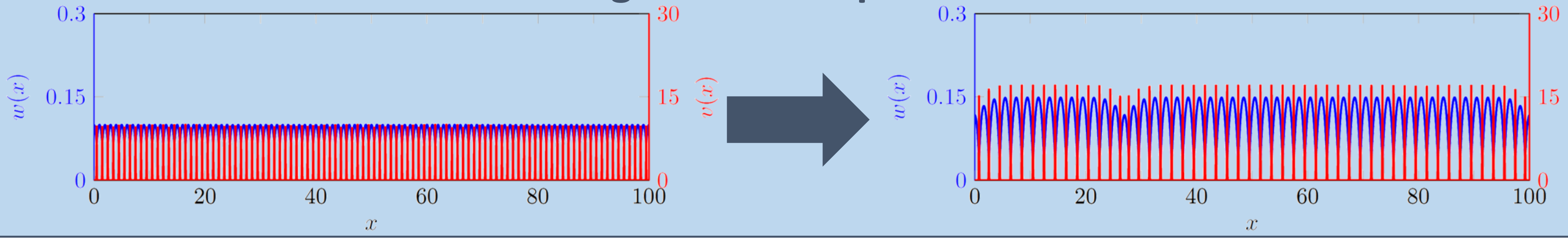


Classic tipping



Tipping of patterns

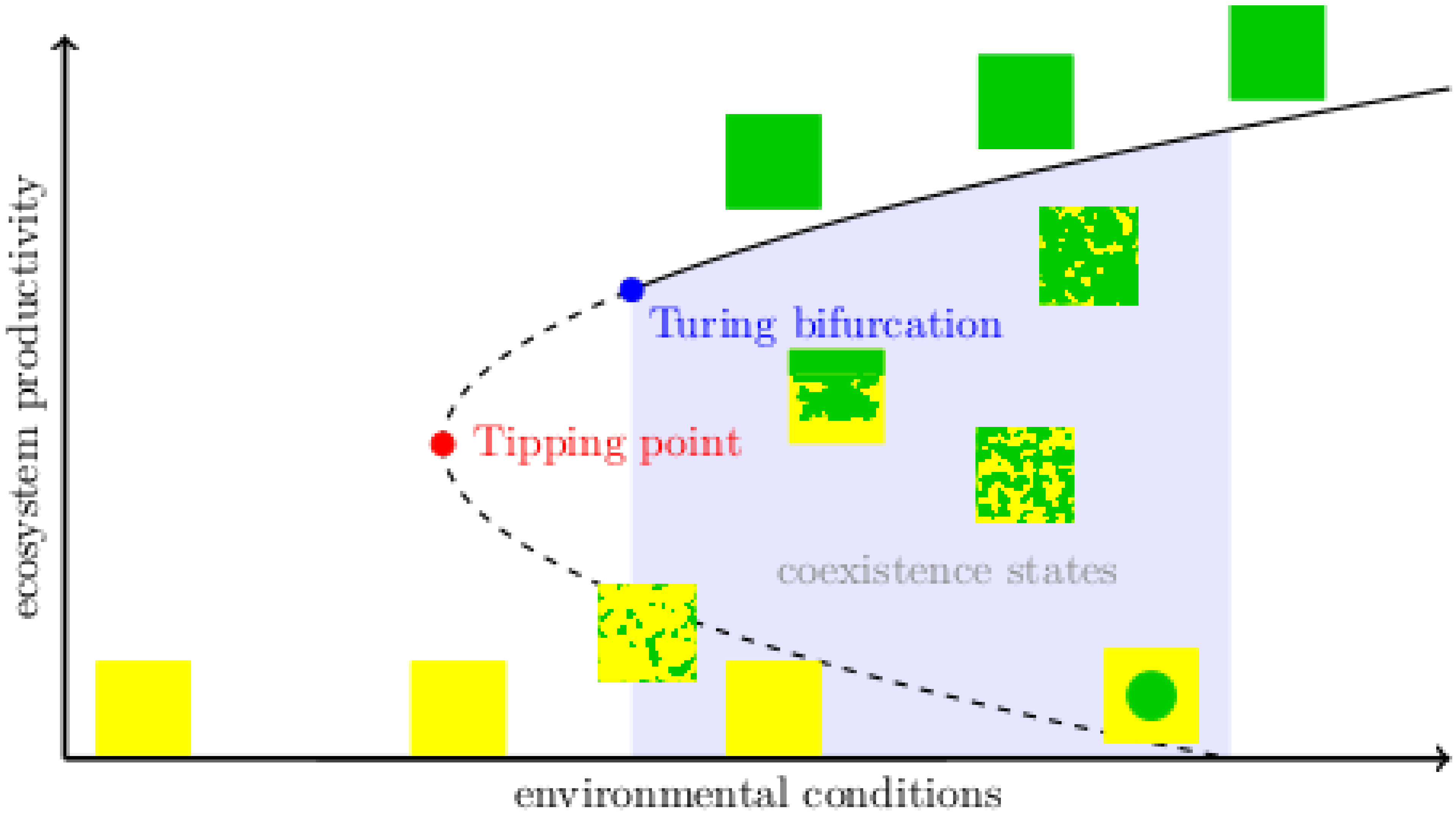
Degradation of patterns



Part 2: Coexistence States

and spatial heterogeneities

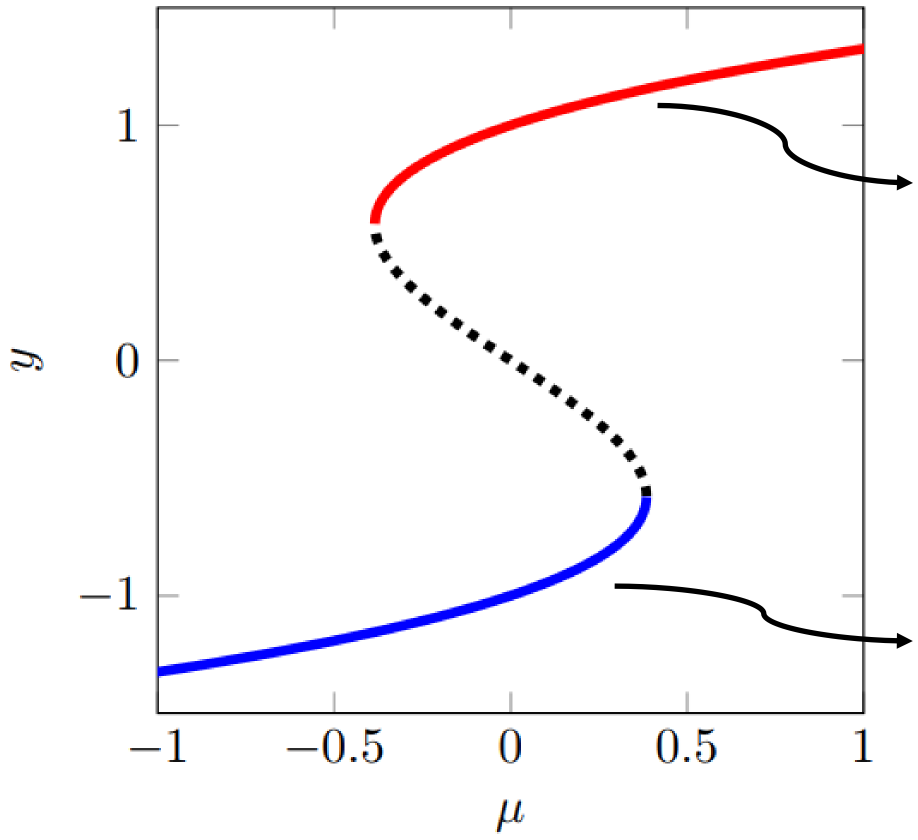
Coexistence states in bifurcation diagram

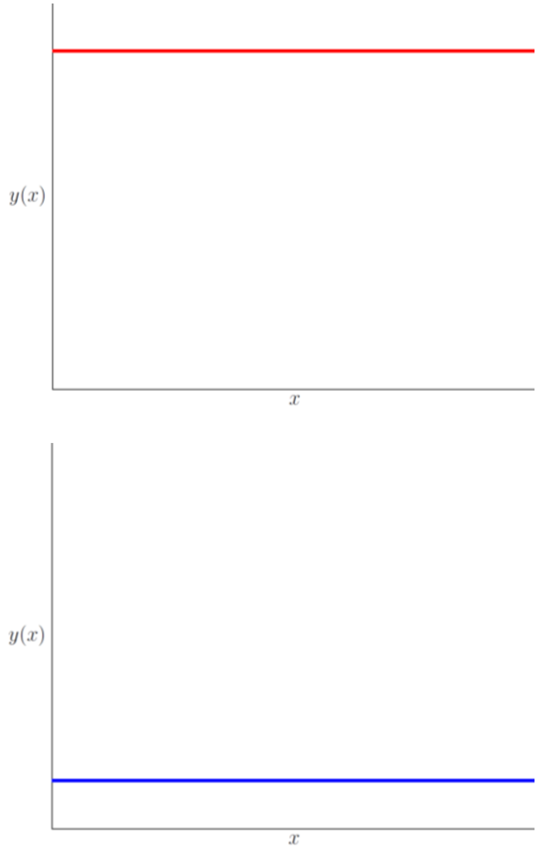


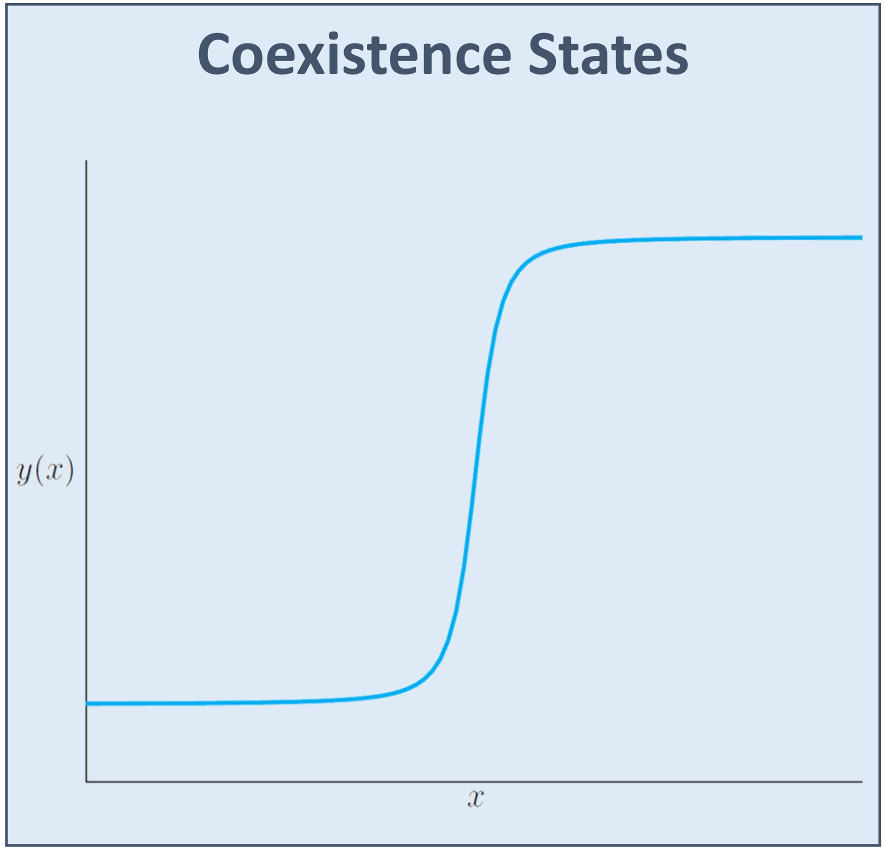
Coexistence states

Bistable (Allen-Cahn/Nagumo) equation:

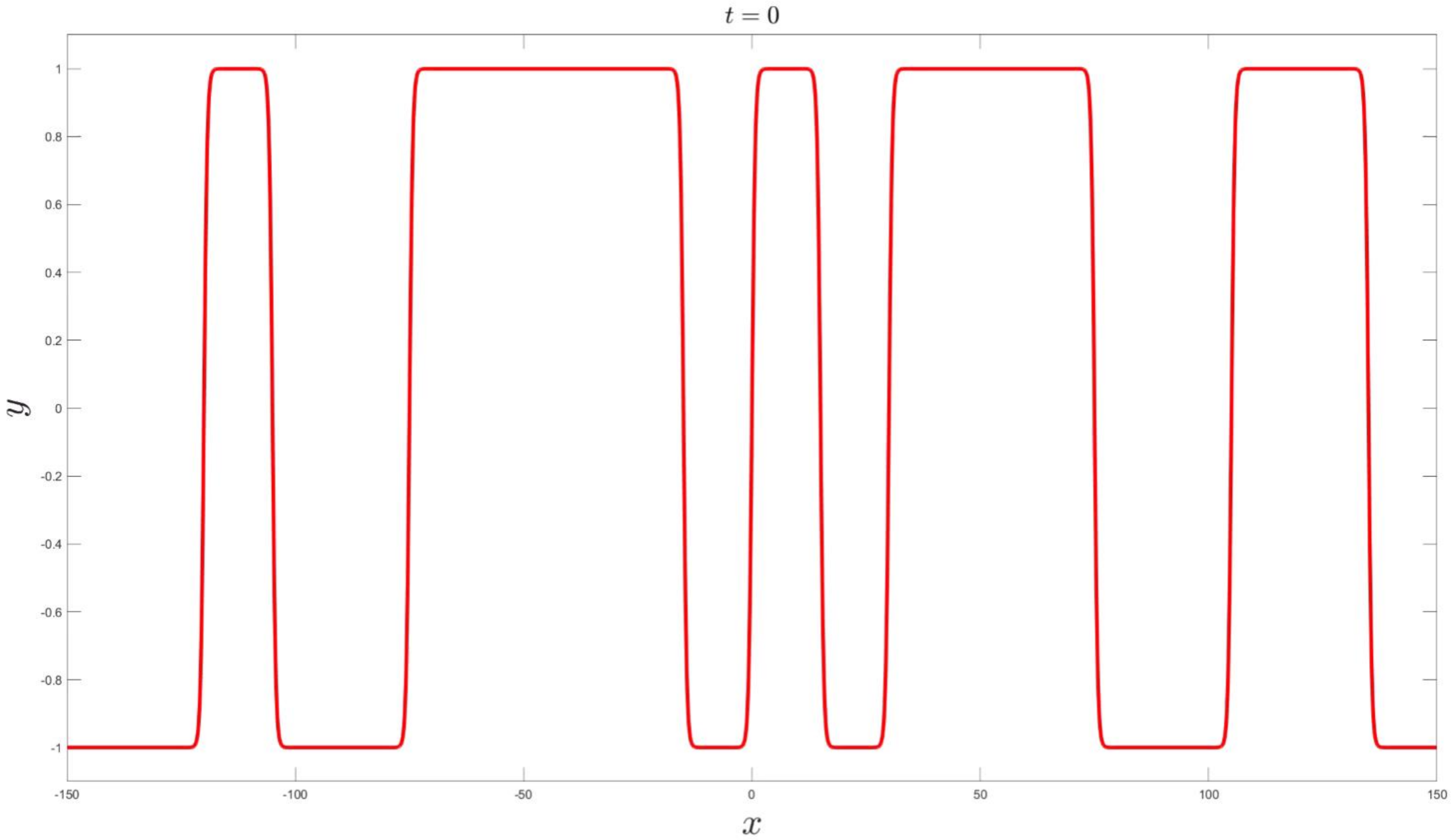
$$\frac{\partial y}{\partial t} = y(1 - y^2) + \mu + D\,\frac{\partial^2 y}{\partial x^2}$$







Dynamics of $\frac{\partial y}{\partial t} = D\ \frac{\partial^2 y}{\partial x^2} + y(1 - y^2) + \mu$

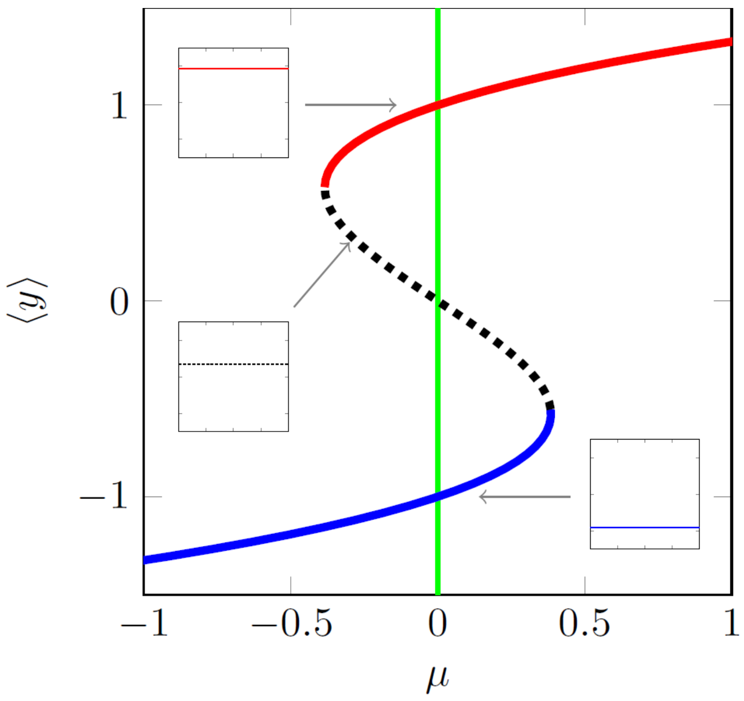


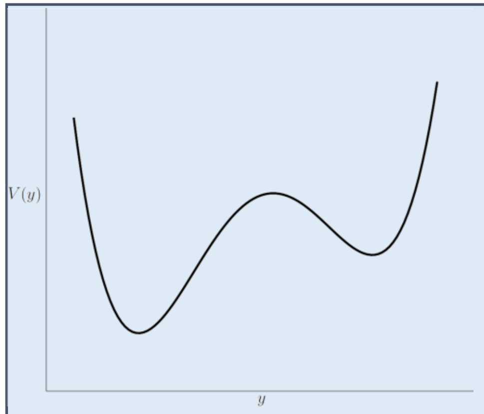
Front Dynamics

$$\frac{\partial y}{\partial t} = D\frac{\partial^2 y}{\partial x^2} + f(y;\mu)$$

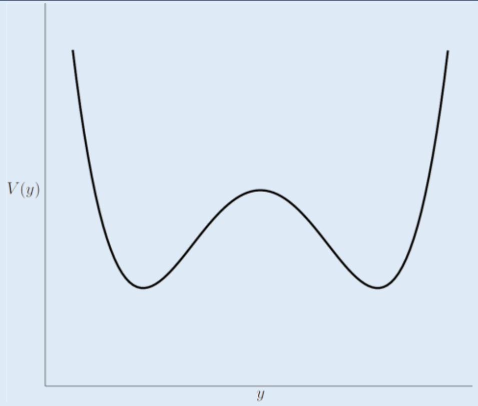
Potential function $V(y;\mu)$:

$$\frac{\partial V}{\partial y}(y;\mu) = -f(y;\mu)$$

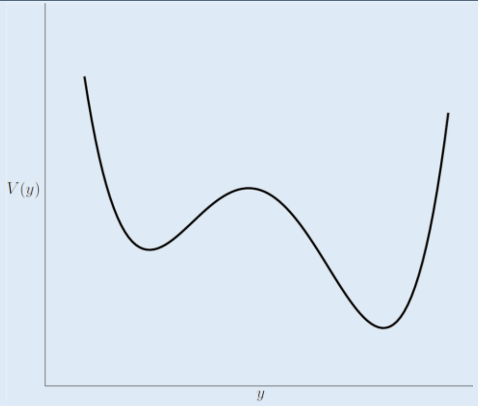




moves right

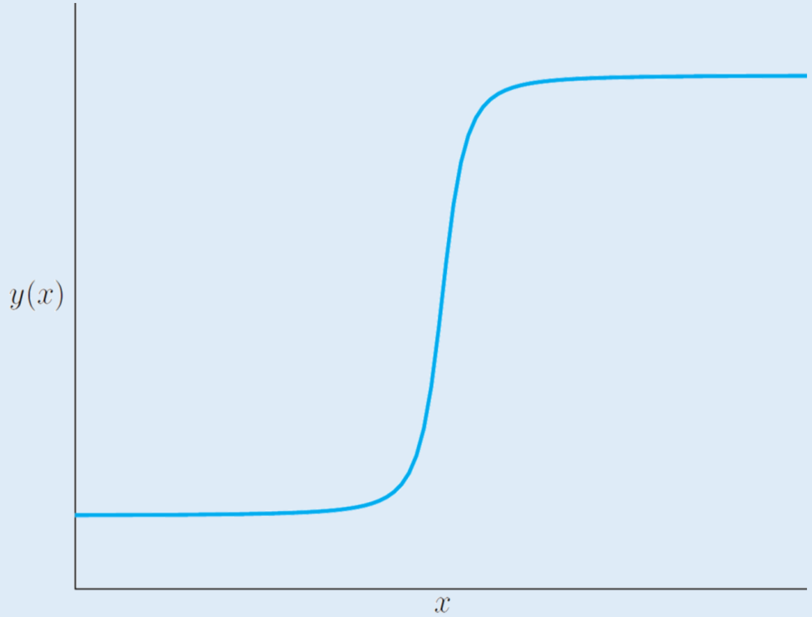


stationary



moves left

Maxwell Point $\mu_{maxwell}$



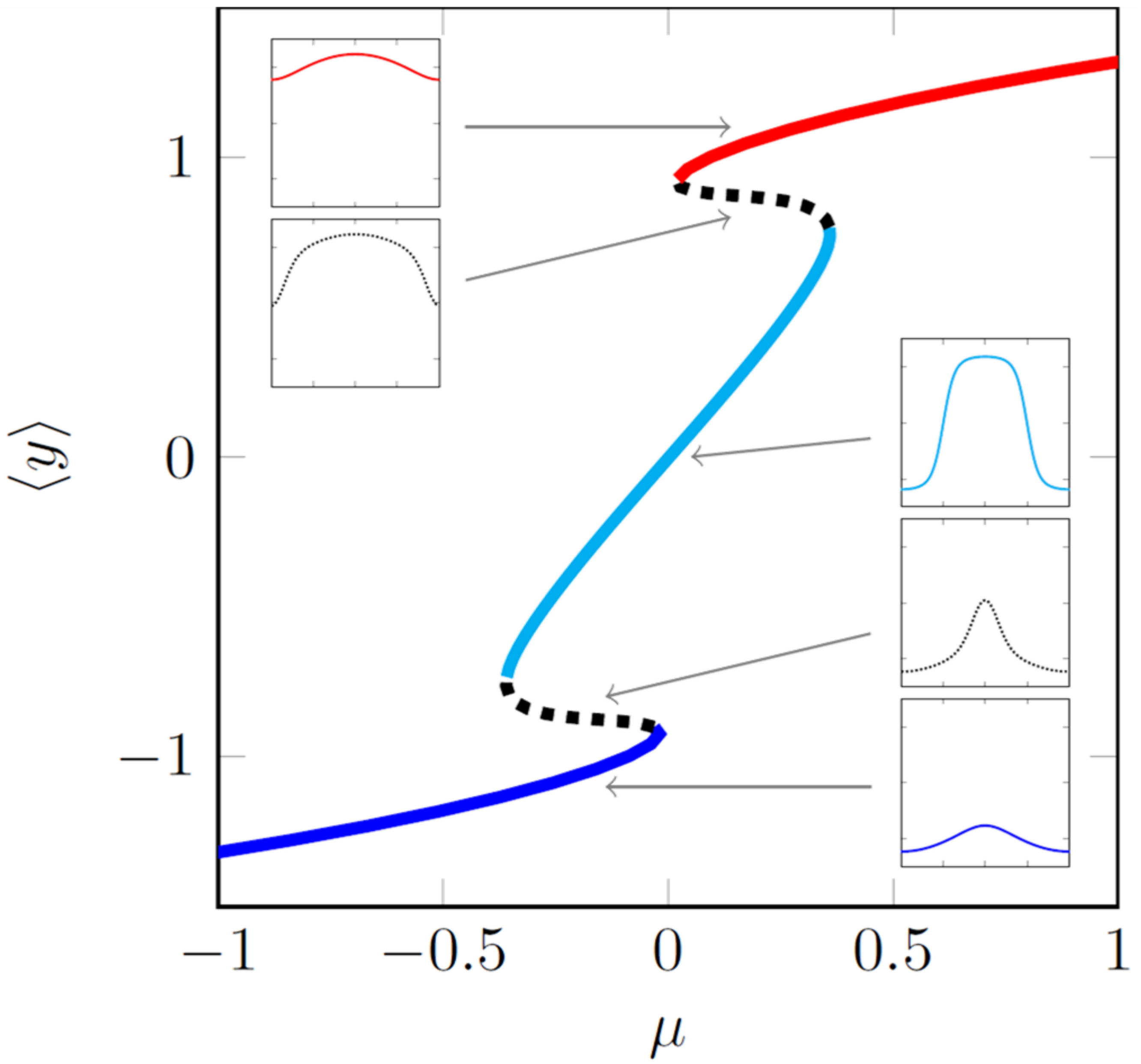
Adding Spatial Heterogeneity

$$\frac{\partial y}{\partial t} = D\frac{\partial^2 y}{\partial x^2} + f(y, x; \mu)$$

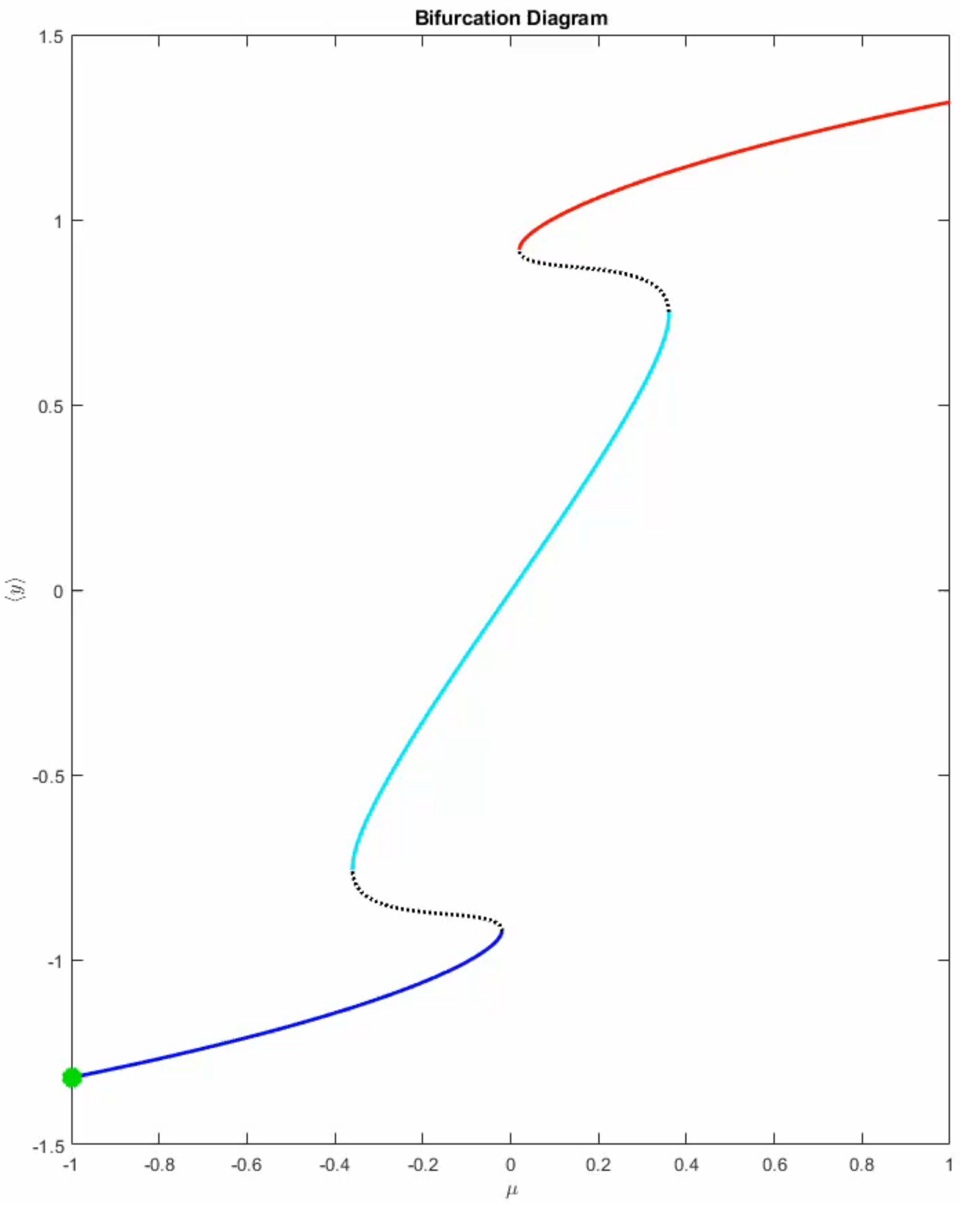
Now, the **local** difference in potentials determines the front movement

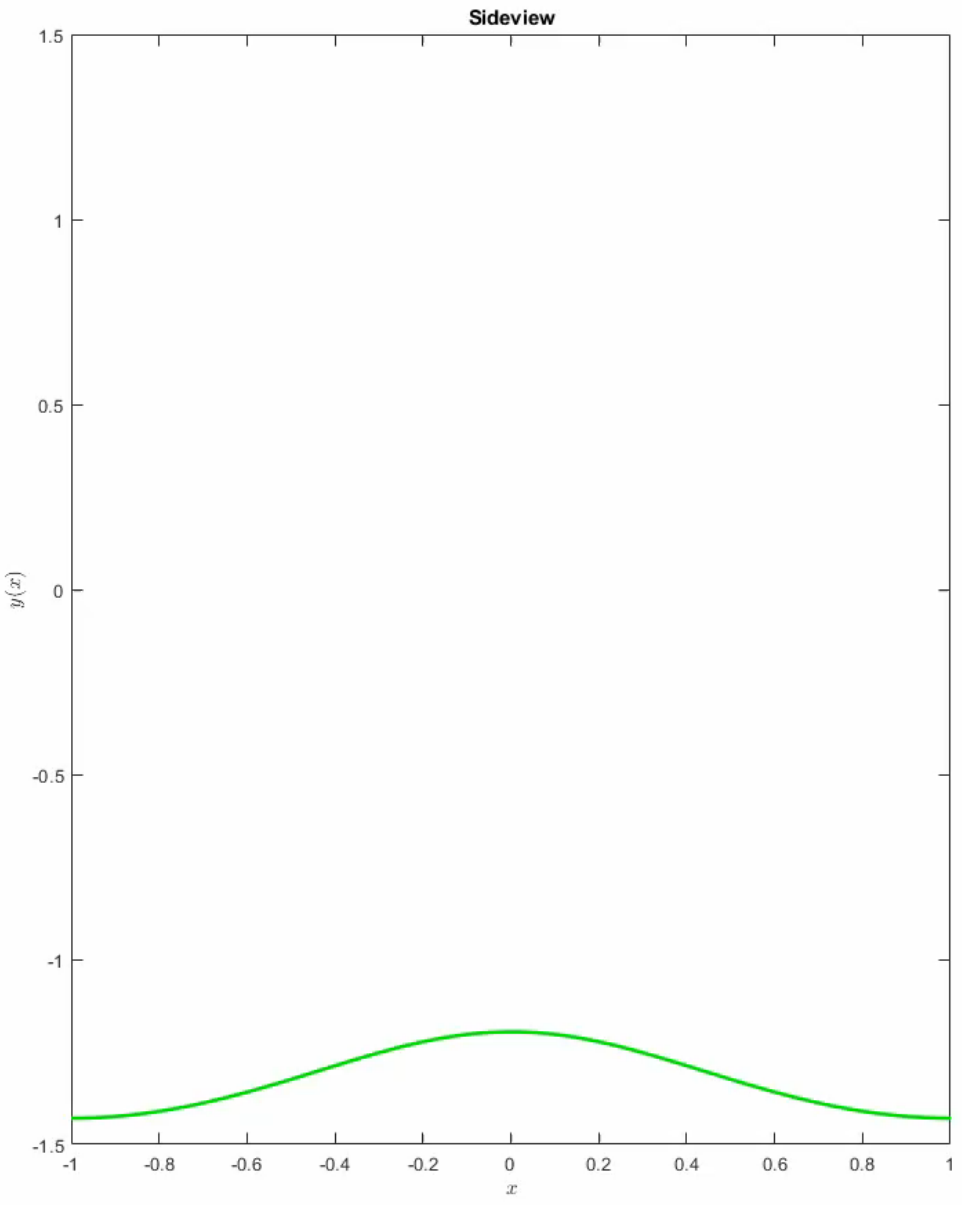
New behaviour:

- Multi-fronts can be stationary
- Maxwell point is smeared out



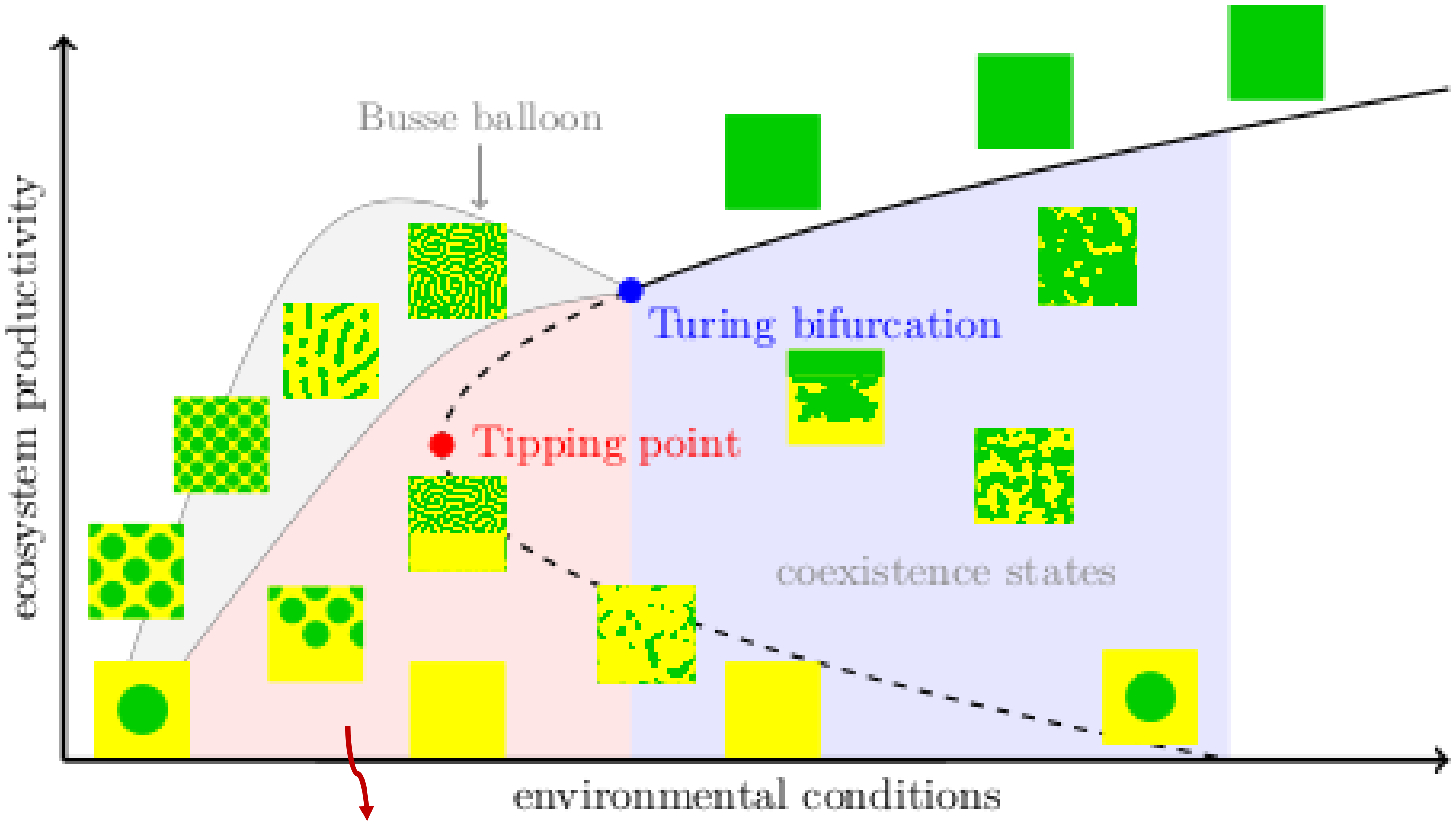
Fragmented Tipping



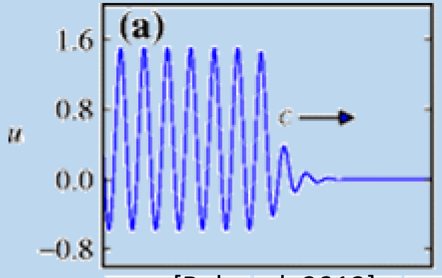


Part 3:
Tipping in Spatially Extended Systems?

“Bifurcation Diagram” for spatially extended systems



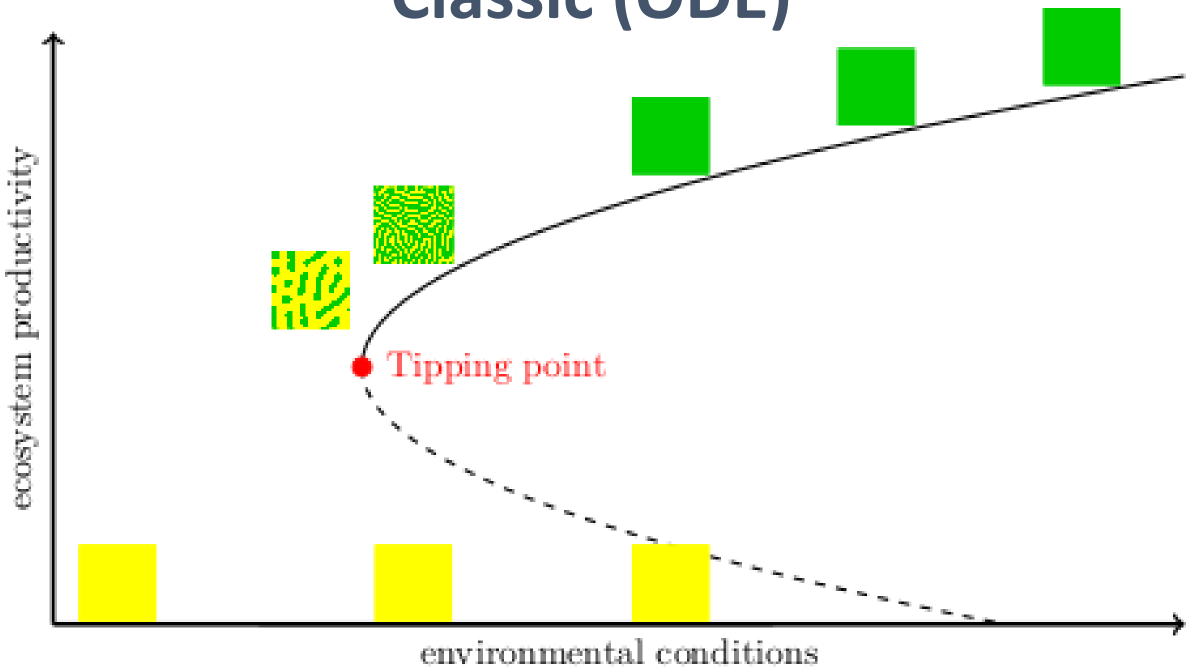
Coexistence states between patterned and uniform states also exist



[Bel et al, 2012]

What if the system tips?

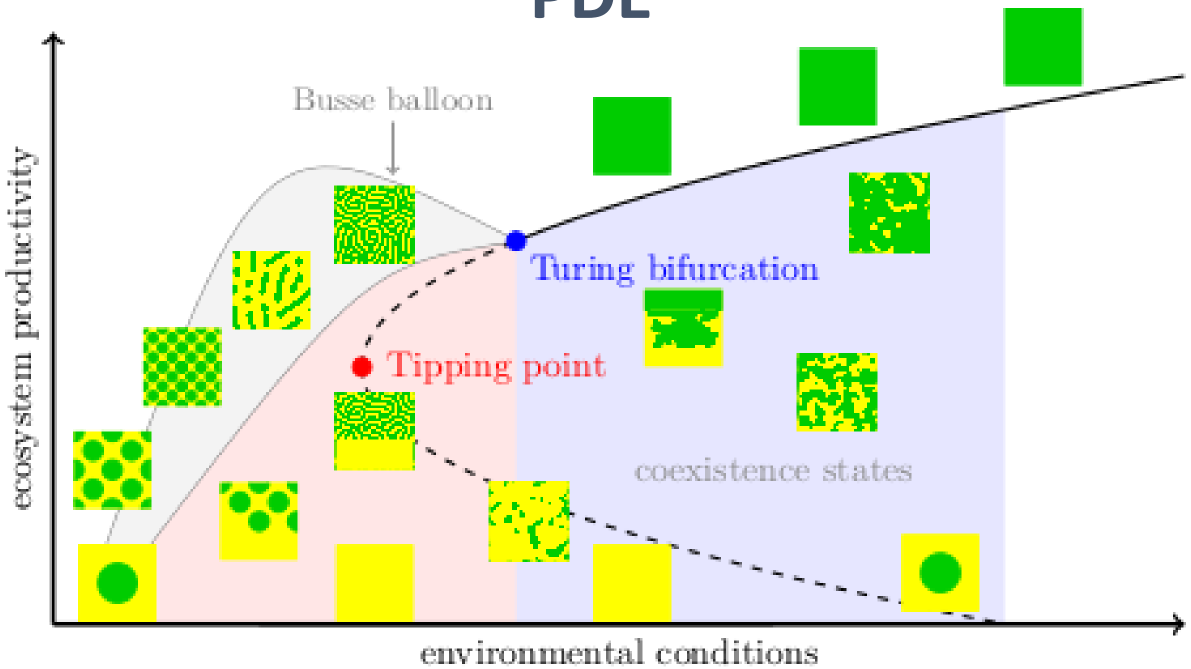
Classic (ODE)



Crossing a Tipping Point:
→ Always full reorganization

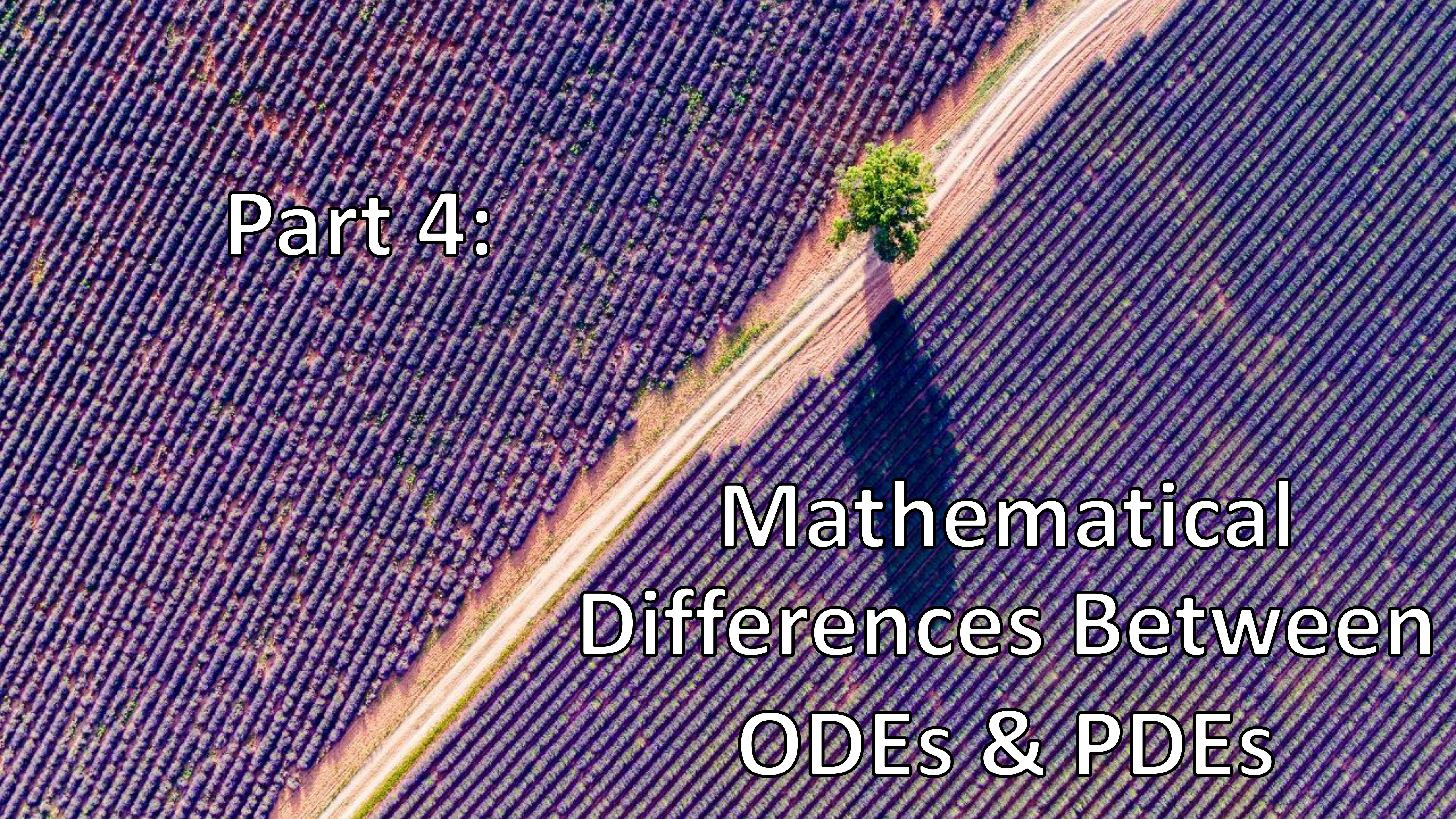
Early Warning Signals
signal for <u>WHEN</u>

PDE



Crossing a bifurcation:
Now also possible:
→ Spatial reorganization (Turing patterns)
→ Fragmented tipping (coexistence states)

Early Warning Signals
need to signal <u>WHEN</u> & <u>WHAT</u>


Part 4:
Mathematical
Differences Between
ODEs & PDEs

Differences between ODEs and PDEs

	ODE $y_t = f(y;\mu)$	**PDE** $y_t = y_{xx} + f(y;\mu)$
Stationary States	$0 = f(y^*;\mu)$	$0 = y^*_{xx} + f(y^*;\mu)$
Linear Stability	$\lambda\,\bar{y} = f_y(y^*;\mu)\,\bar{y}$	$\lambda\,\bar{y} = \bar{y}_{xx} + f_y(y^*(x);\mu)\,\bar{y}$

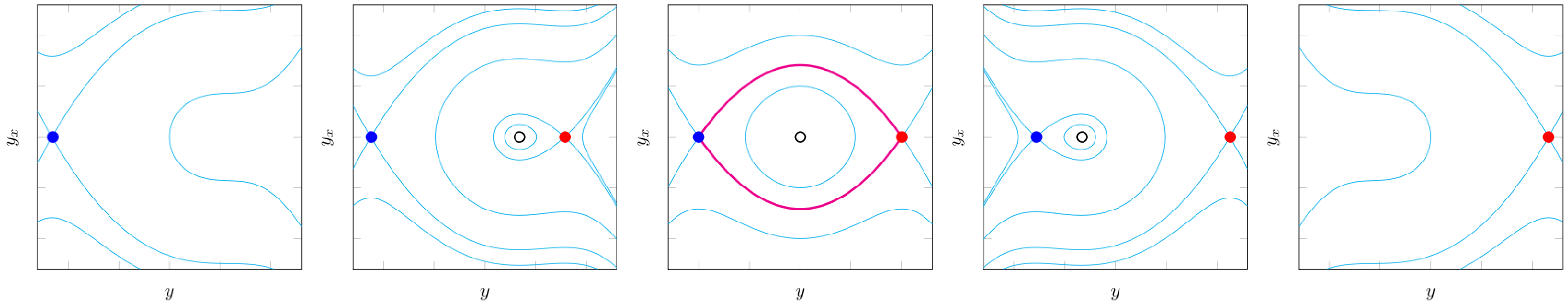
Stationary States

$$y_t = y_{xx} + f(y;\mu)$$

Stationary states

$$0 = y_{xx} + f(y;\mu)$$

$$\begin{cases} y_x = p \\ p_x = -f(y;\mu) \end{cases}$$

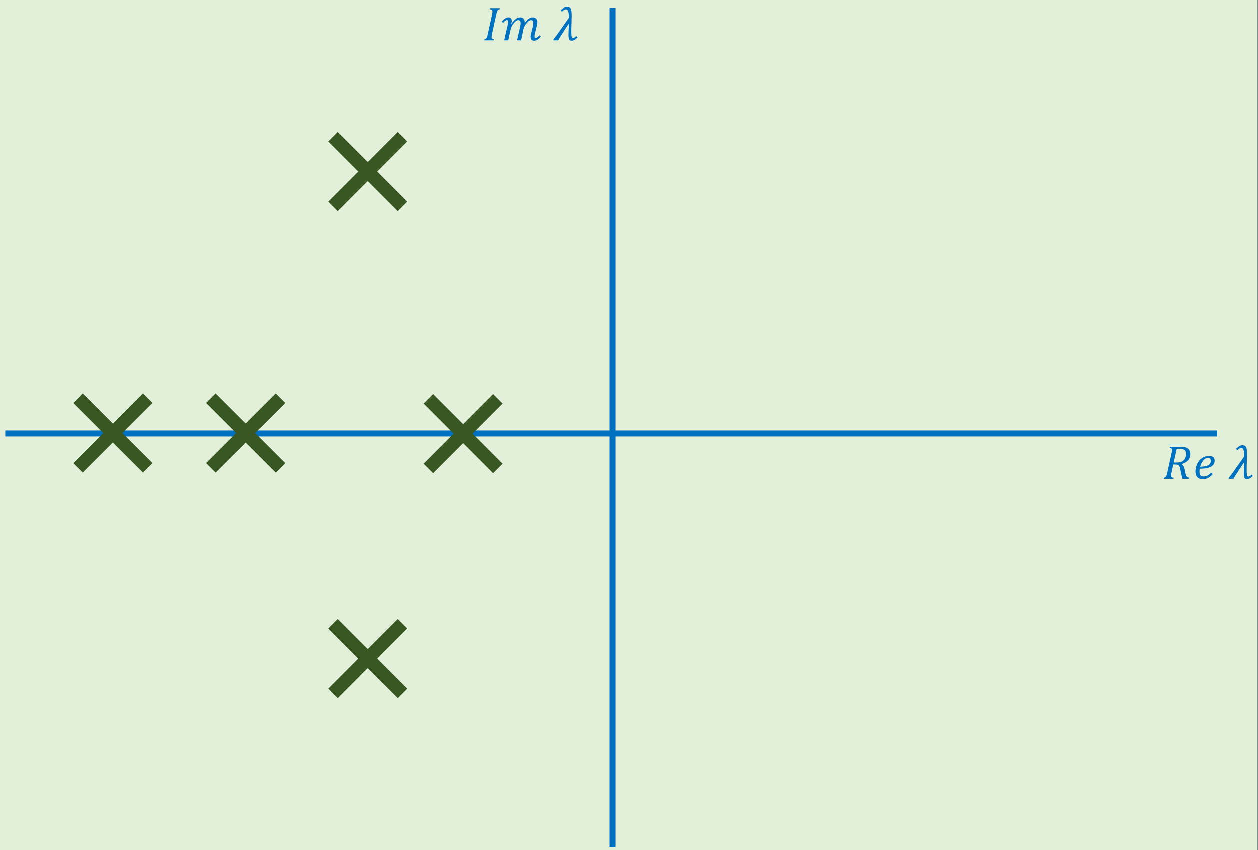


(a) $\mu < \mu_B$ (b) $\mu_B < \mu < \mu_M$ (c) $\mu = \mu_M$ (d) $\mu_M < \mu < \mu_A$ (e) $\mu > \mu_A$

Stability of Stationary States

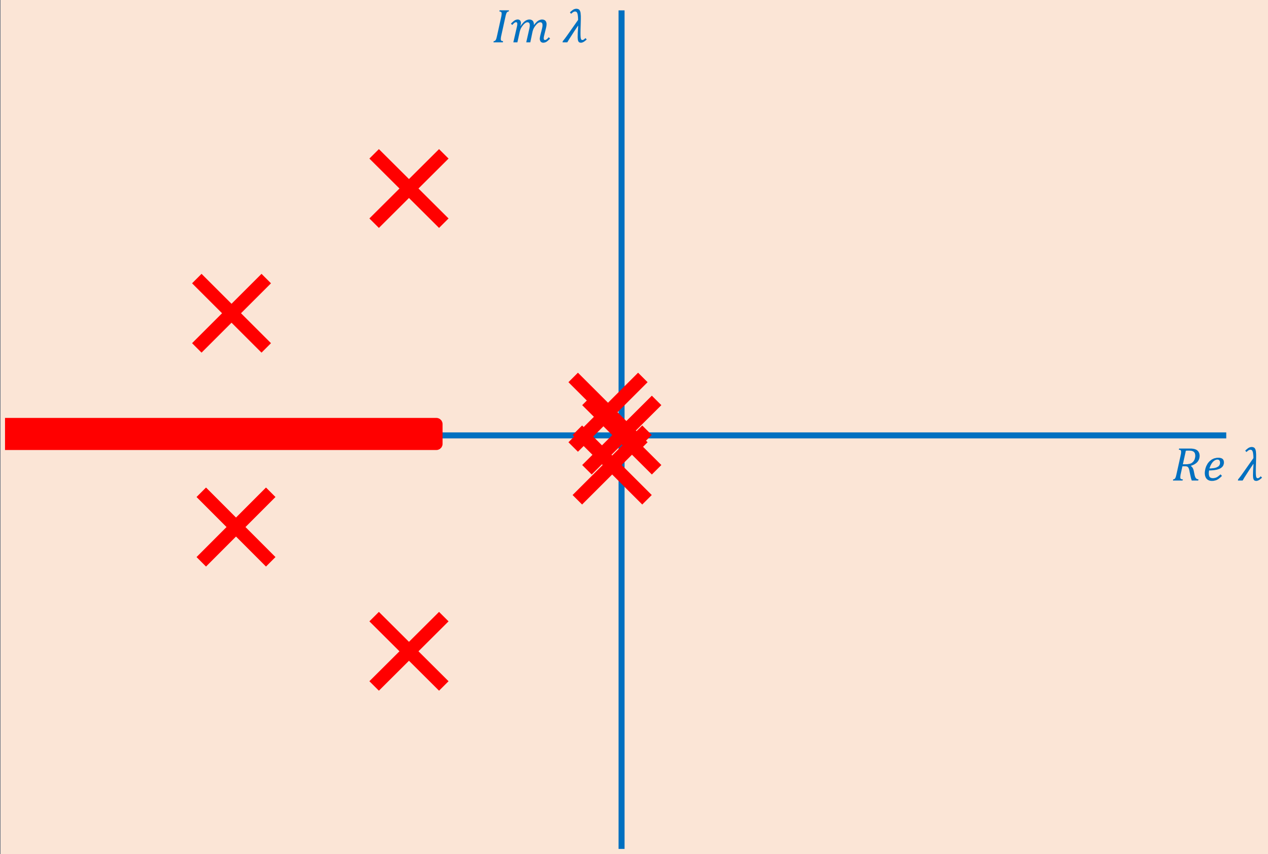
ODE

$$\lambda\, \bar{y} = f_y(y^*; \mu)\, \bar{y}$$



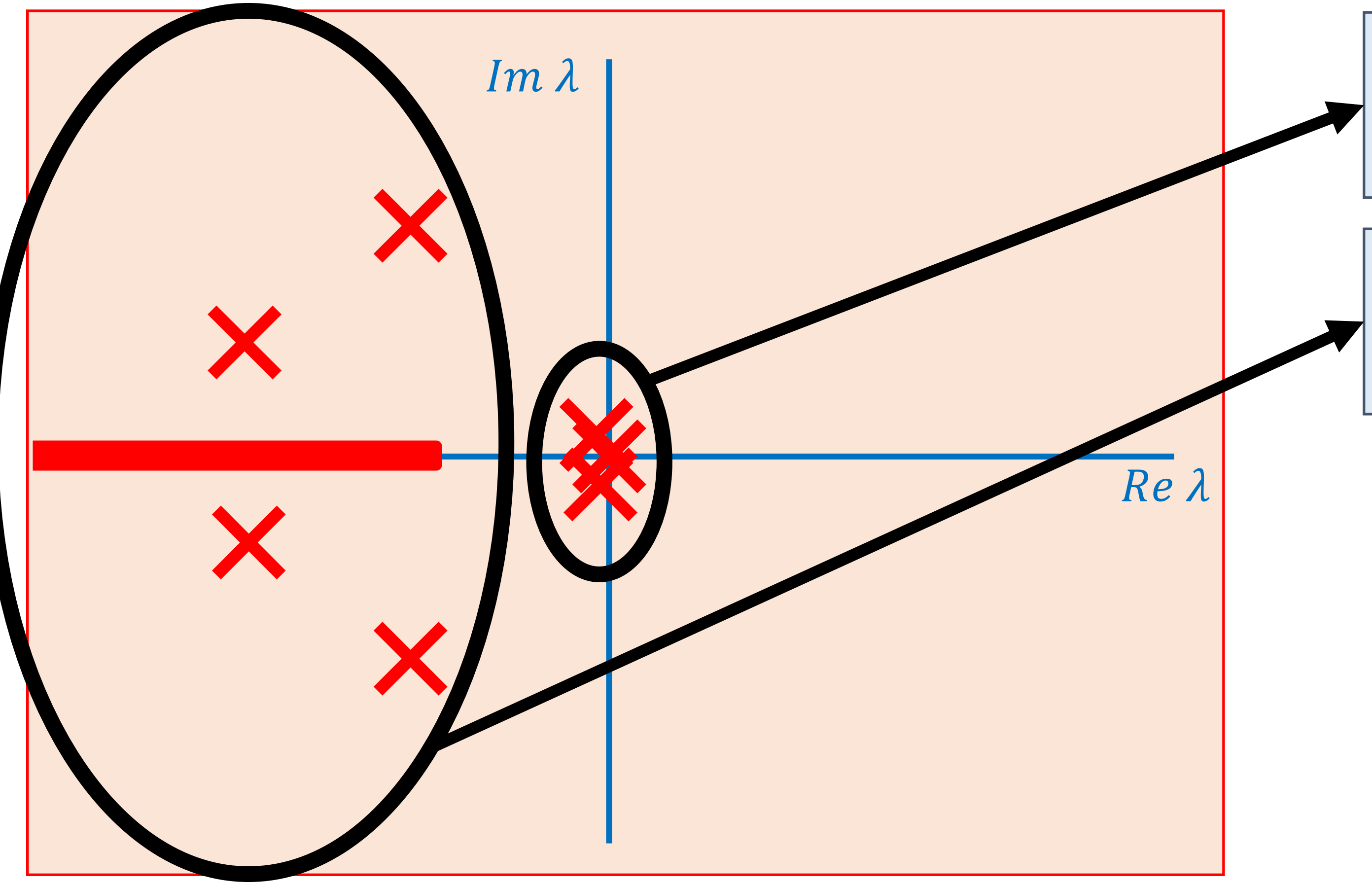
PDE

$$\lambda\, \bar{y} = \bar{y}_{xx} + f_y(y^*(x); \mu)\, \bar{y}$$



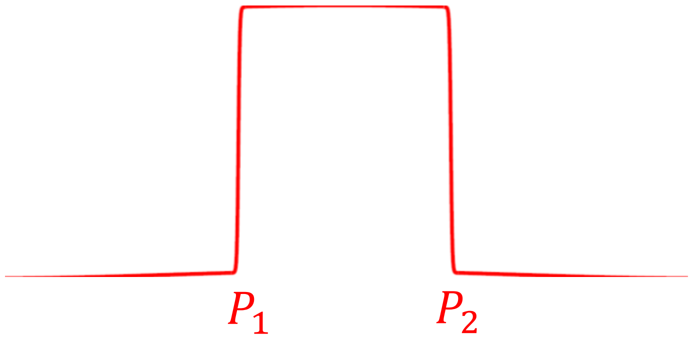

Part 5:
Dynamics &
Bifurcations of
Patterned States

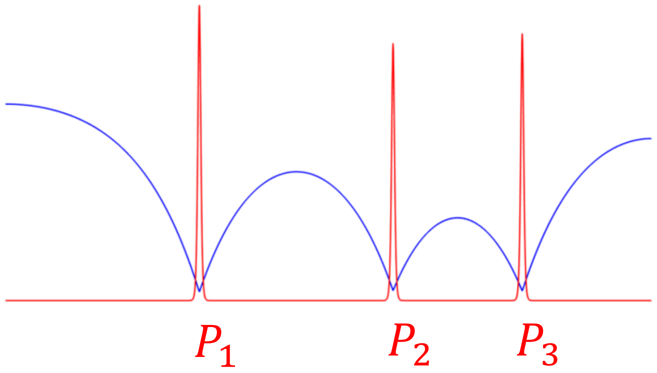
Dynamics of Patterned States

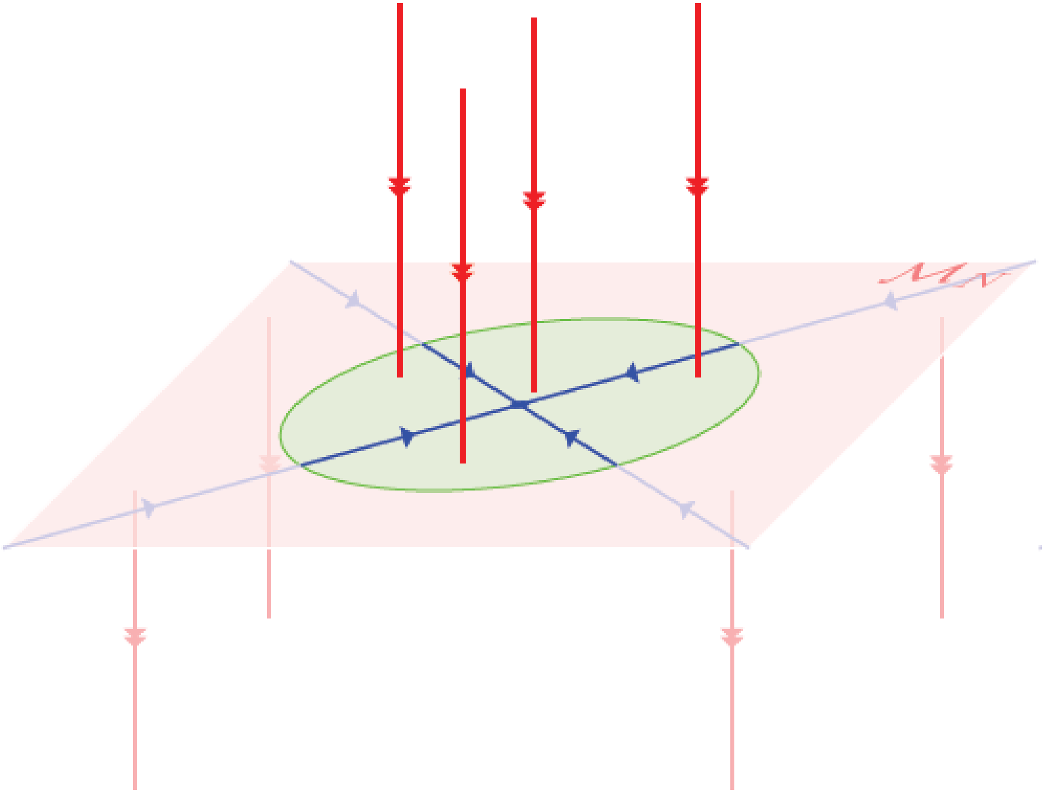


1. SLOW Pattern Adaptation

2. FAST Pattern Degradation





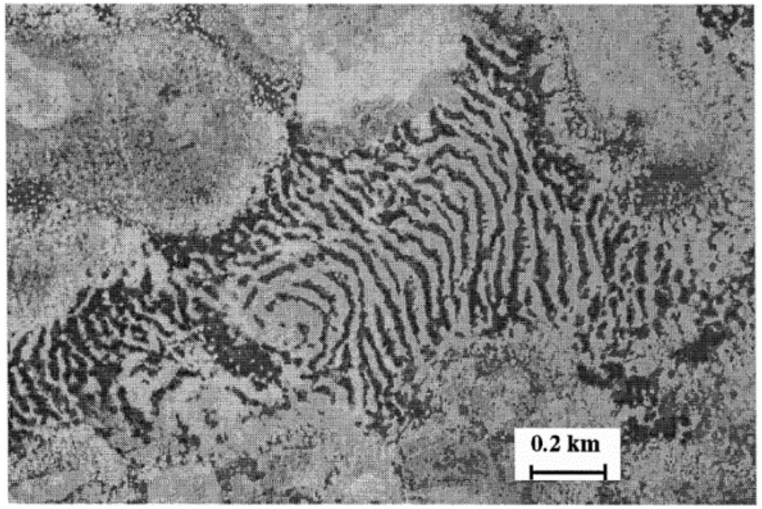


1. SLOW pattern adaptation

Somaliland, 1948 [Macfadyen, 1950]

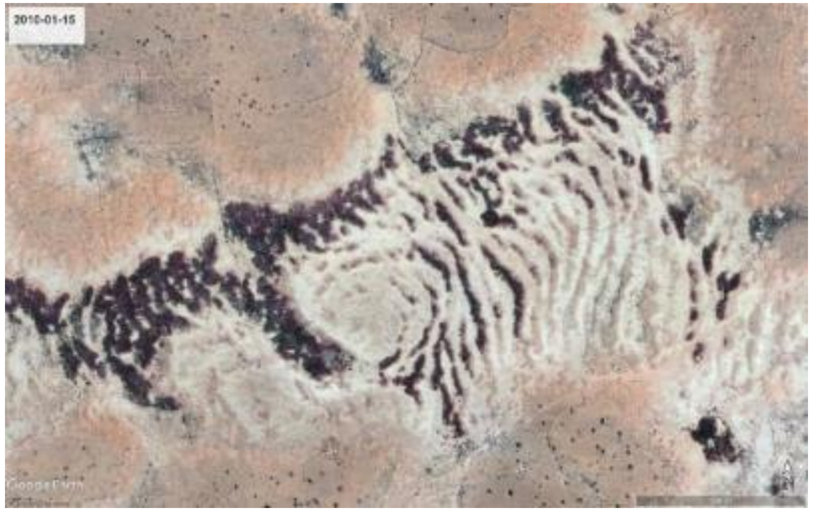
Somaliland, 2008

2. FAST Pattern Degradation

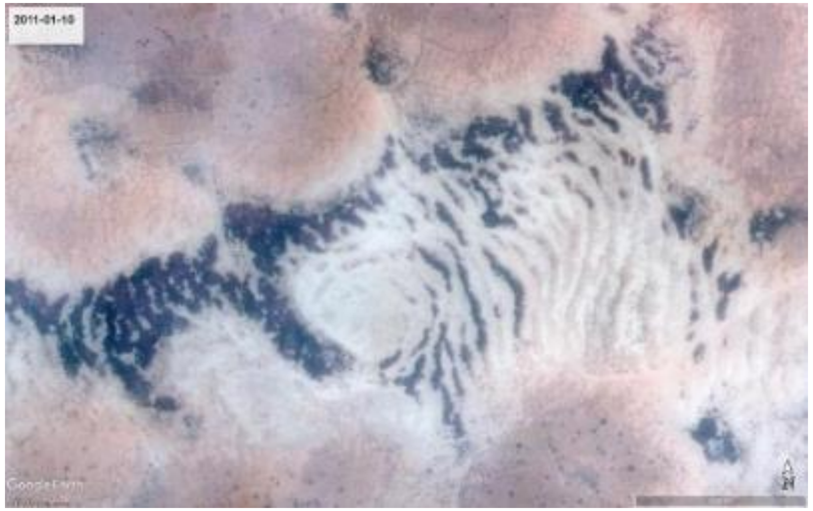


Niger, 1950 [Valentin, 1999]

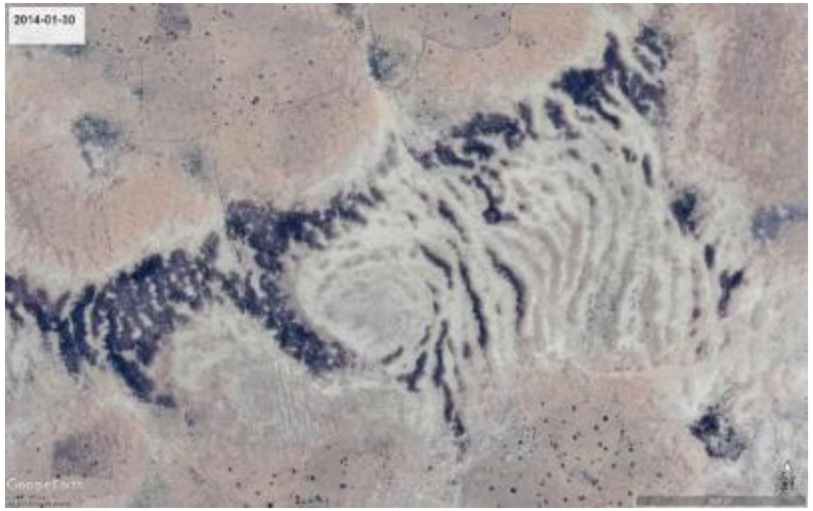
Niger, 2008



Niger, 2010



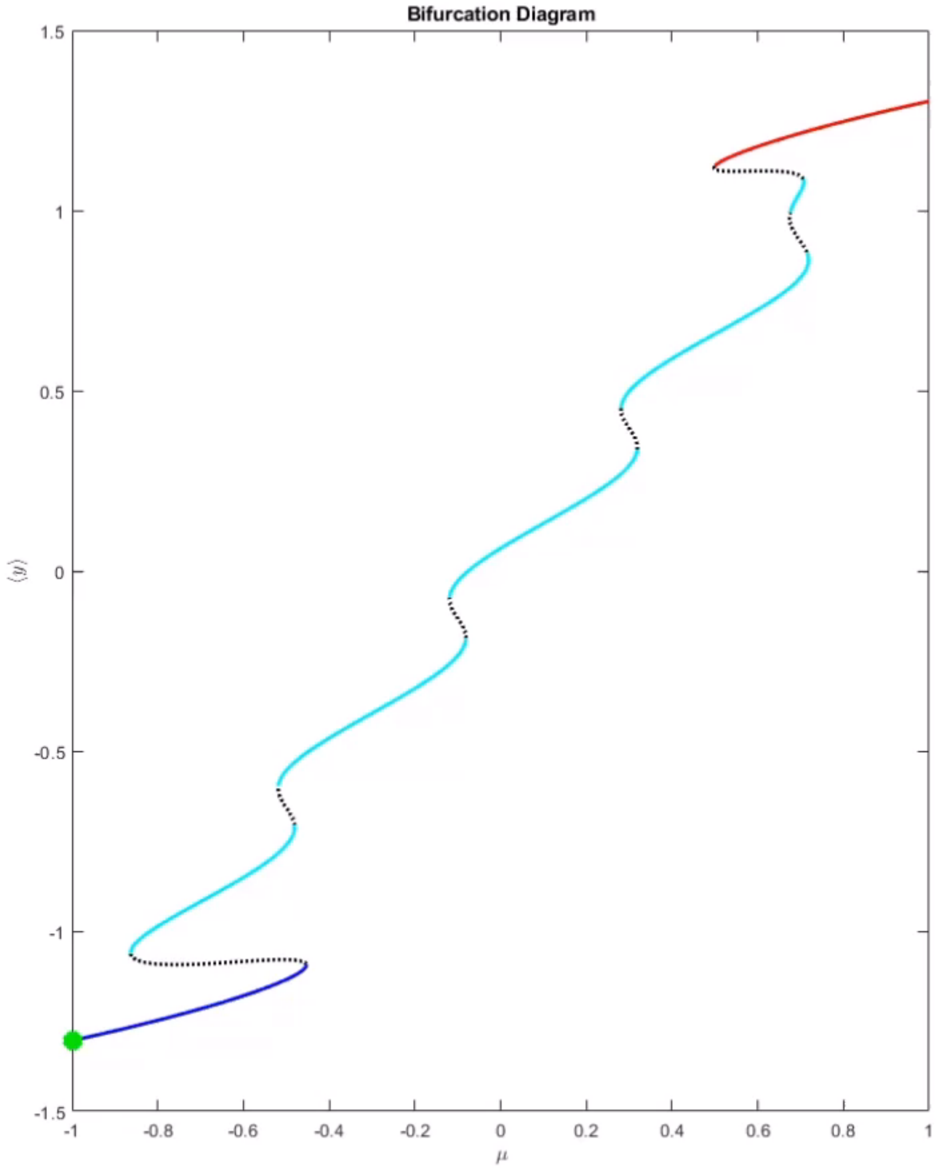
Niger, 2011

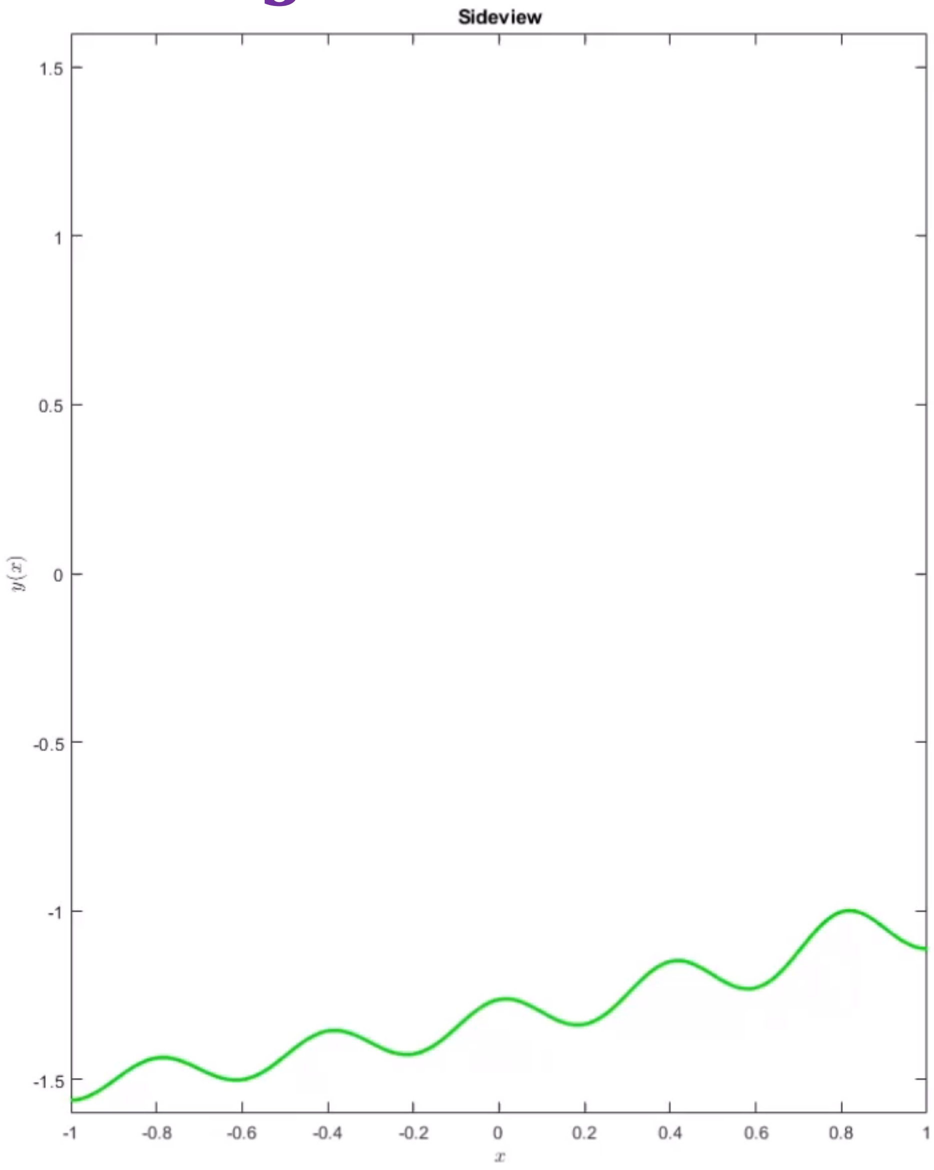


Niger, 2014

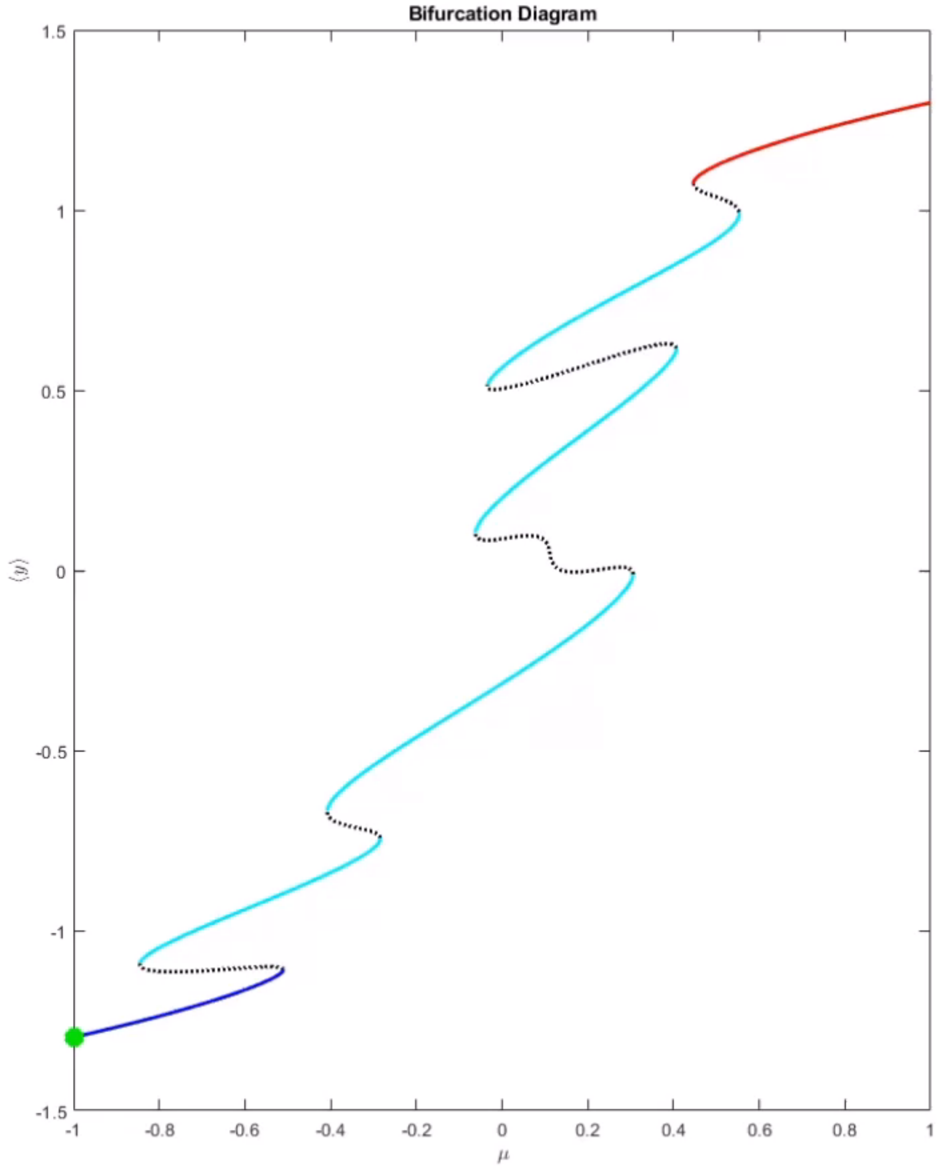
Niger, 2016

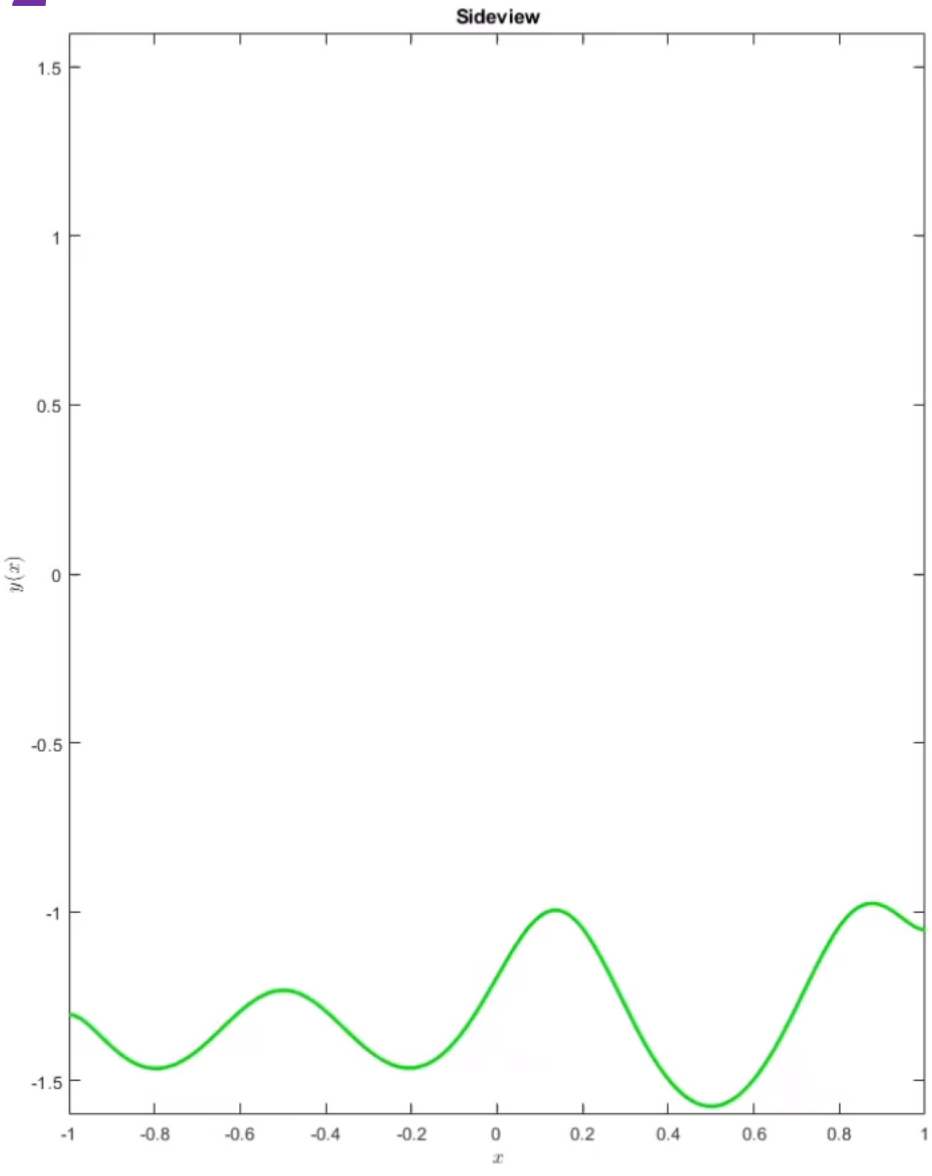
$$y_t = D\, y_{xx} + y(1 - y^2) + \mu + x + \frac{2}{5}\cos(5\pi x)$$



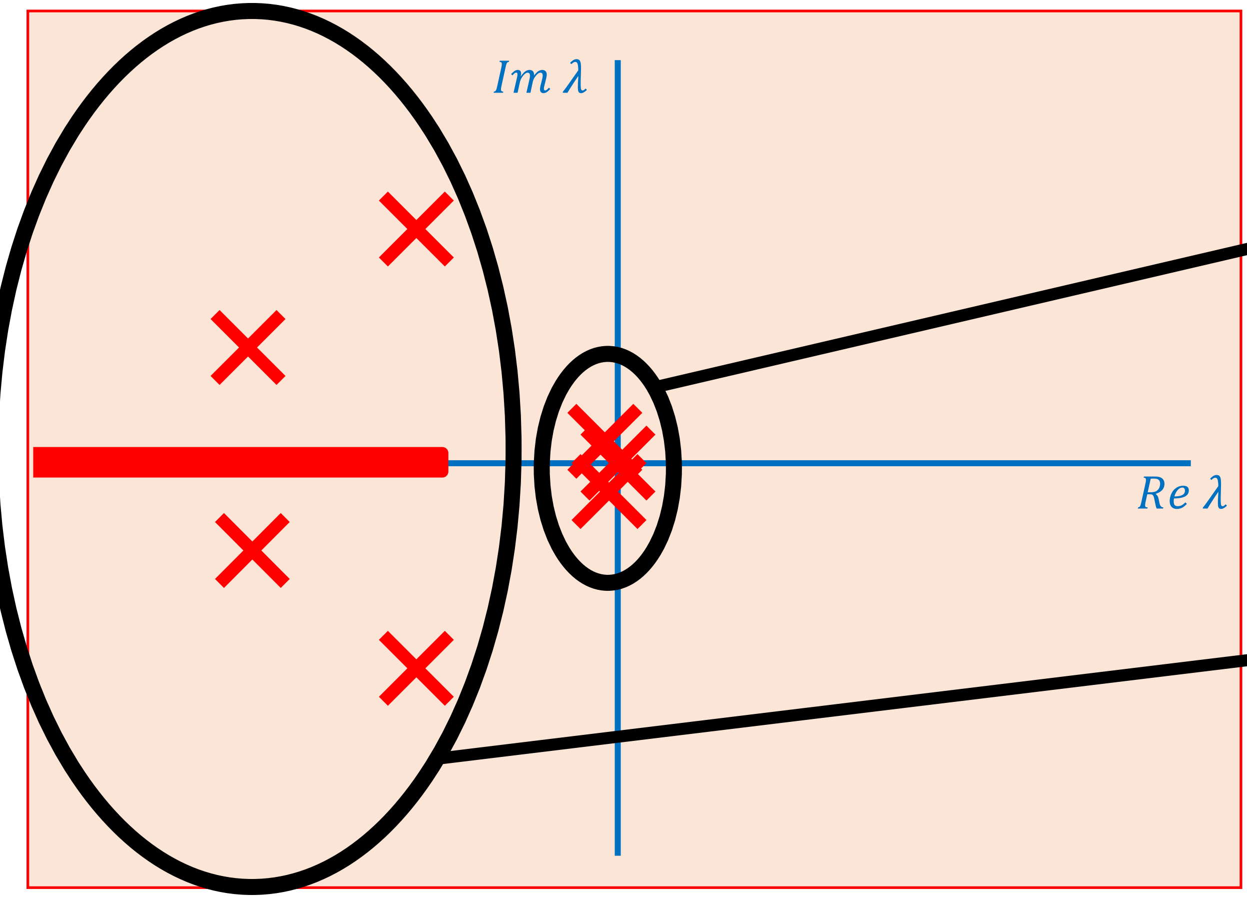


$$y_t = D\,y_{xx} + y(1-y^2) + \mu + \frac{1}{2}\cos(2\pi x) + \sin(3\pi x)$$





Bifurcations



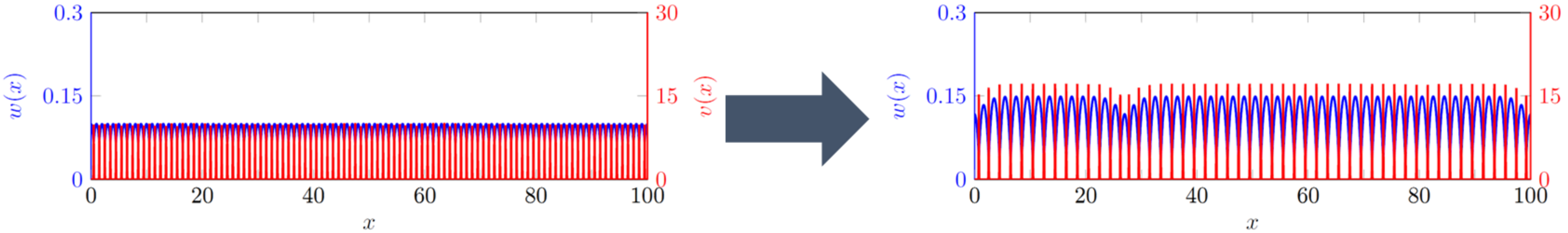
What happens at bifurcation?

1. SLOW Pattern Adaptation

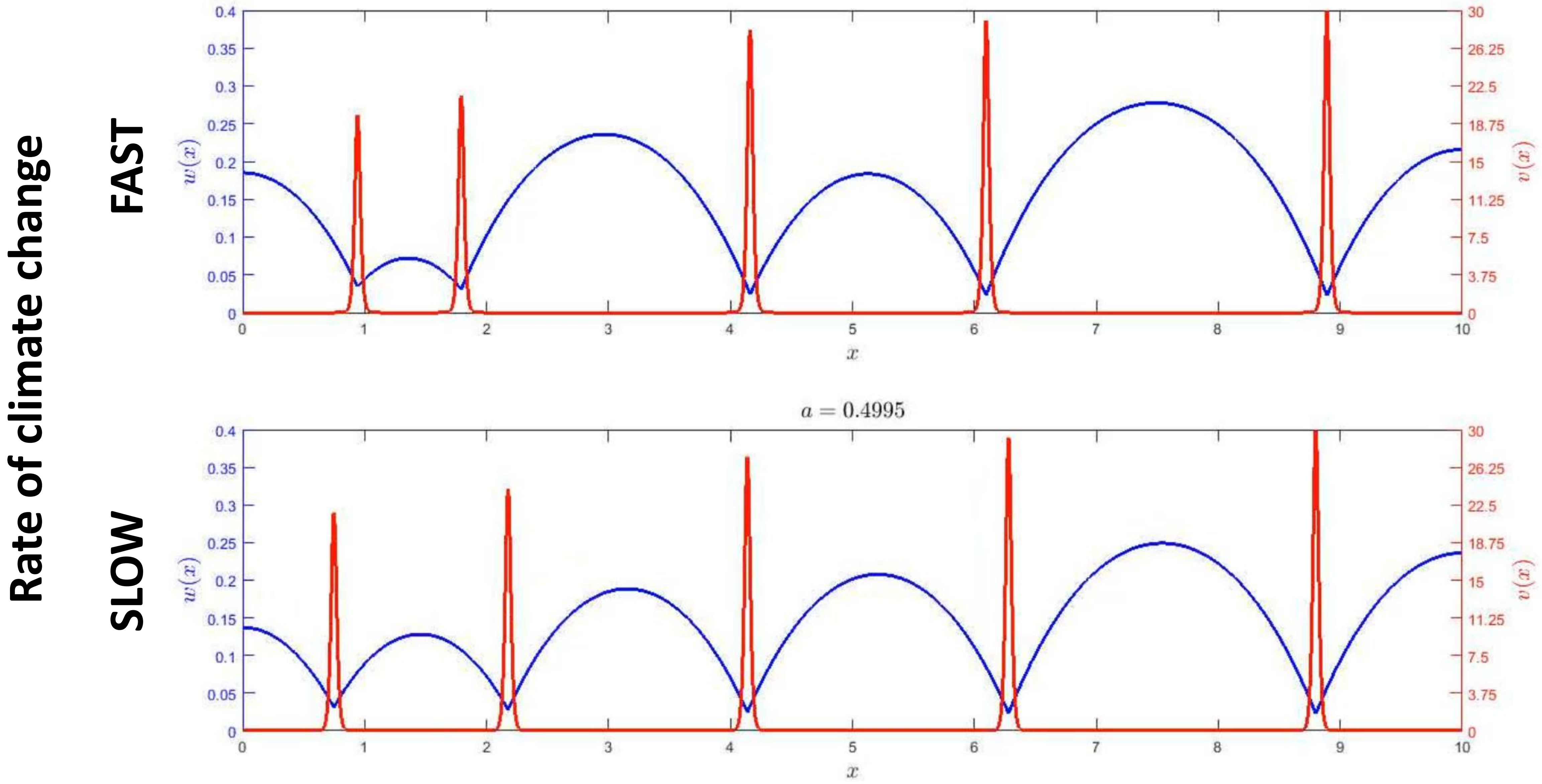
At bifurcation:
→ Location of structure changes

2. FAST Pattern Degradation

At bifurcation:
→ Structures created or destroyed



Vegetation patches under climate change

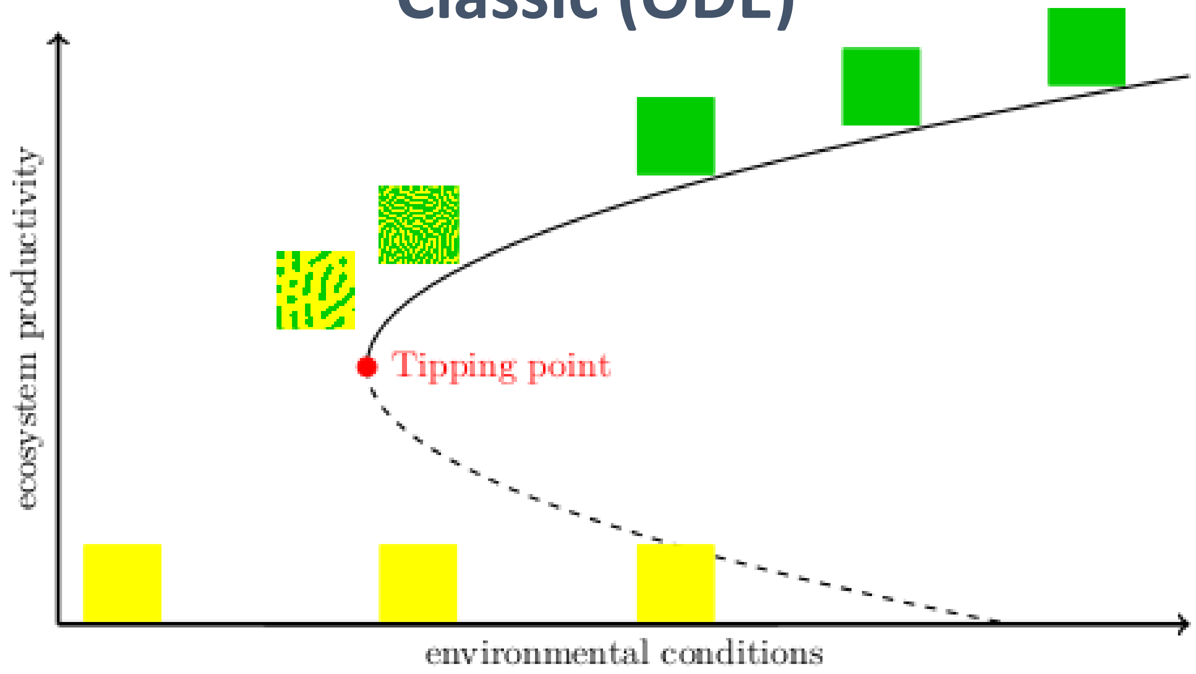


Summary

Tipping in Spatially Extended Systems

What if the system tips?

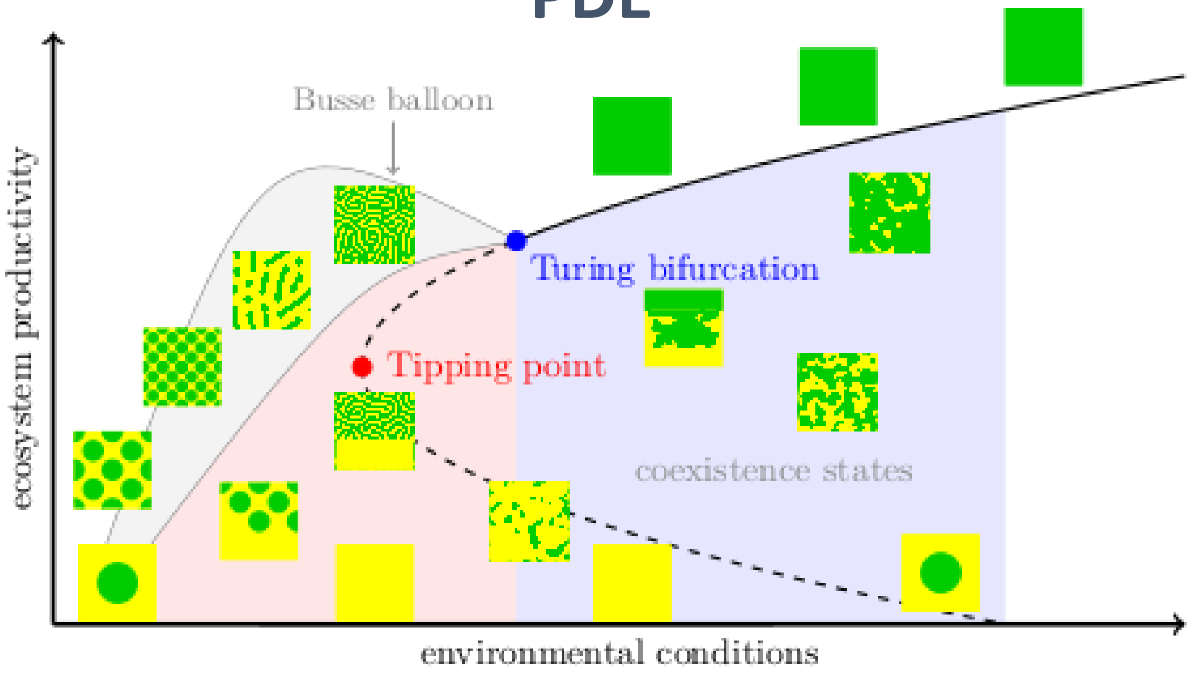
Classic (ODE)



Crossing a Tipping Point:
→ Always full reorganization

Early Warning Signals
signal for WHEN

PDE



Crossing a bifurcation:
Now also possible:
→ Spatial reorganization (Turing patterns)
→ Fragmented tipping (coexistence states)

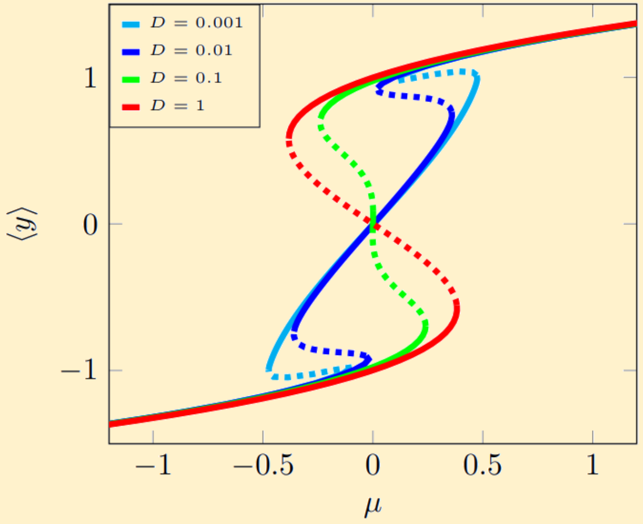
Early Warning Signals
need to signal WHEN & WHAT

Do systems always behave like this? (a.k.a. the small print)

No.

Well-mixed systems

Spatially confined systems



→ Such systems (again) behave like ODEs ←

But even in other systems terms & conditions apply:
System-specific knowledge is required!

Summary

<u>Spatial Patterns:</u>

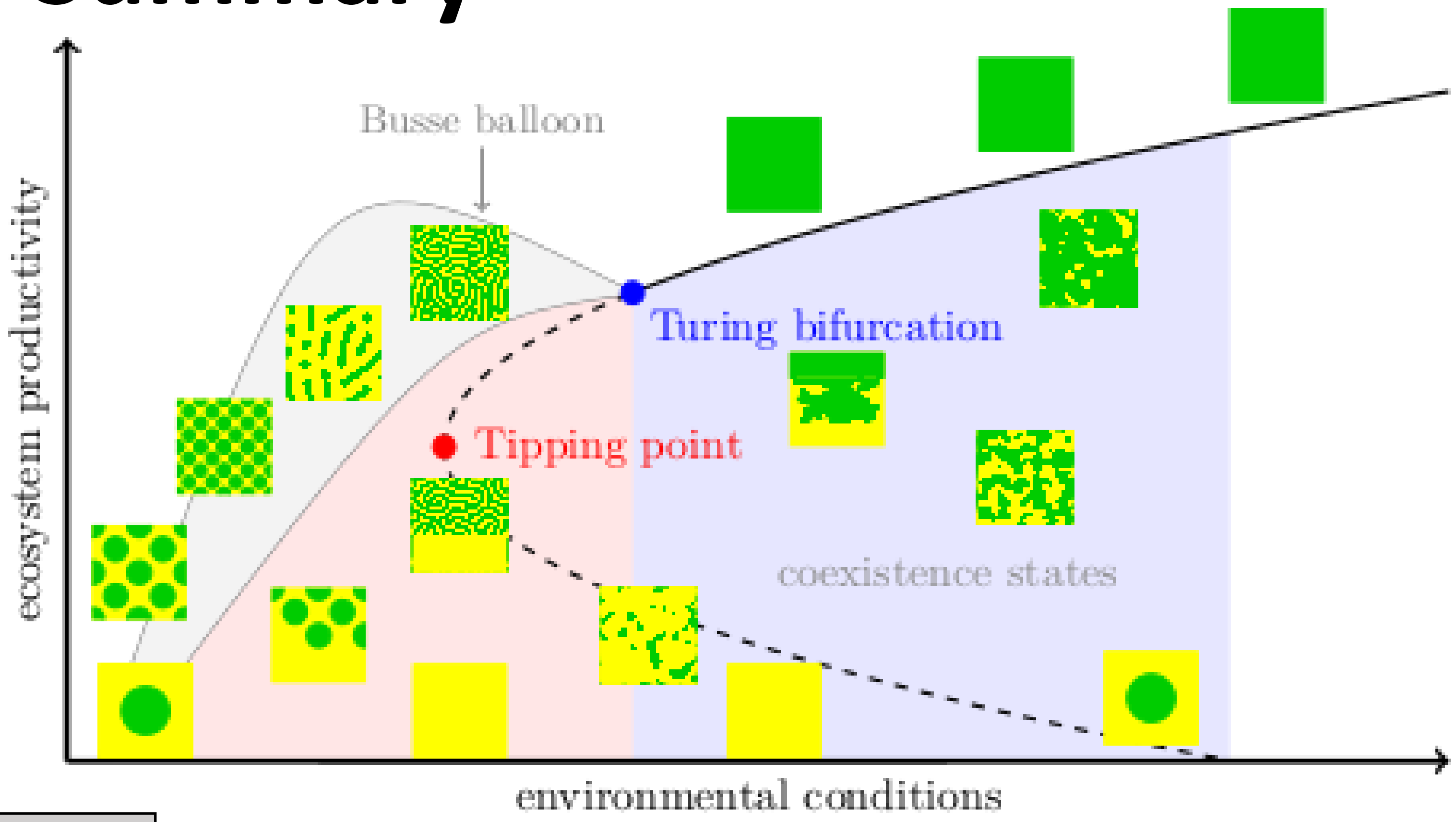
- Turing Patterns
- Coexistence States

<u>Tipping can be more subtle:</u>

- Spatial reorganization
- Fragmented Tipping

<u>Dynamics of Patterns is:</u>

- Slow Pattern Adaptation
- Fast Pattern Degradation



<u>THANKS TO:</u>

Swarnendu Banerjee, Mara Baudena, Alexandre Bouvet, Martina Chirilus-Bruckner, Vincent Deblauwe, Arjen Doelman, Henk Dijkstra, Maarten Eppinga, Anna von der Heydt, Olfa Jaïbi, Johan van de Koppel, Stéphane Mermoz, Max Rietkerk, Eric Siero, Koen Siteur

Rietkerk, M., Bastiaansen, R., Banerjee, S., van de Koppel, J., Baudena, M., & Doelman, A. (2021). Evasion of tipping in complex systems through spatial pattern formation. Science, 374(6564), eabj0359.

Bastiaansen, R., Dijkstra, H. A., & von der Heydt, A. S. (2021). Fragmented Tipping in a spatially heterogeneous world. *Environmental Research Letters, 17, 045006*