Tipping in Spatially Extended Systems

2023-11-21, EGU-NP8 CAMPFIRE
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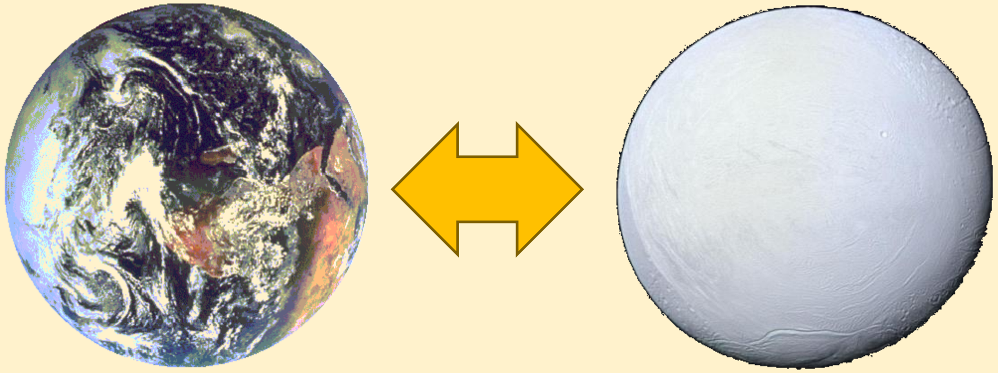
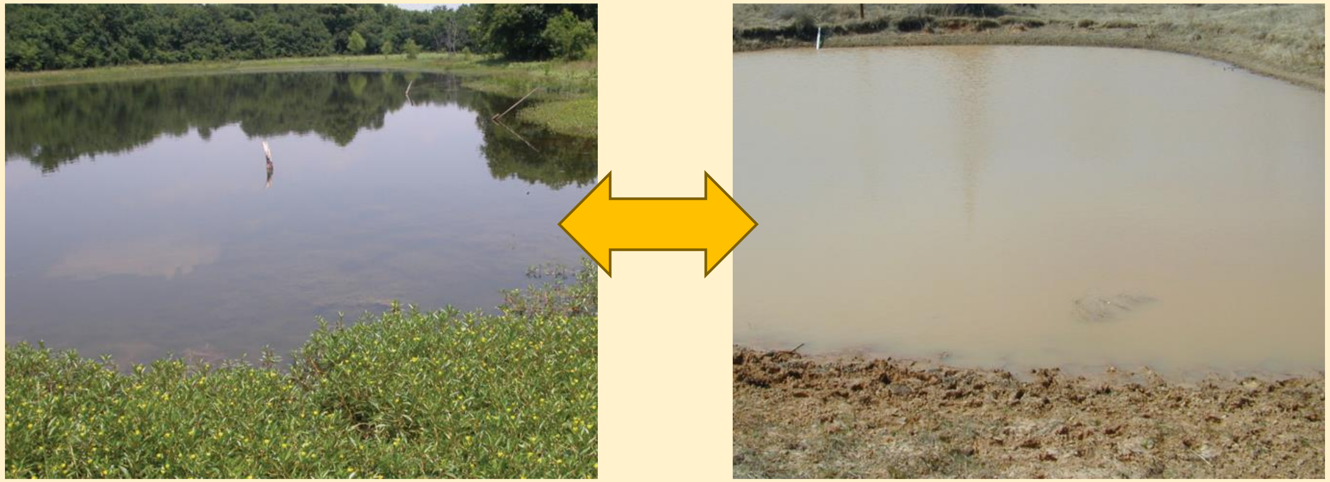
Tipping Points

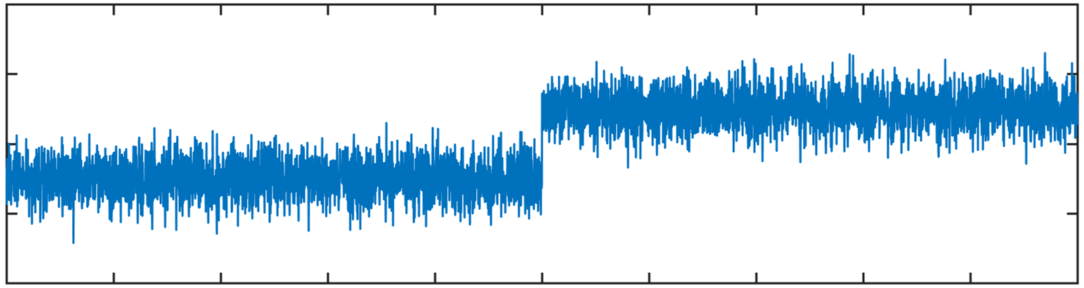
IPCC AR6 (2021) : “a critical threshold beyond which a system reorganizes, often abruptly and/or irreversibly”



Planetary transitions

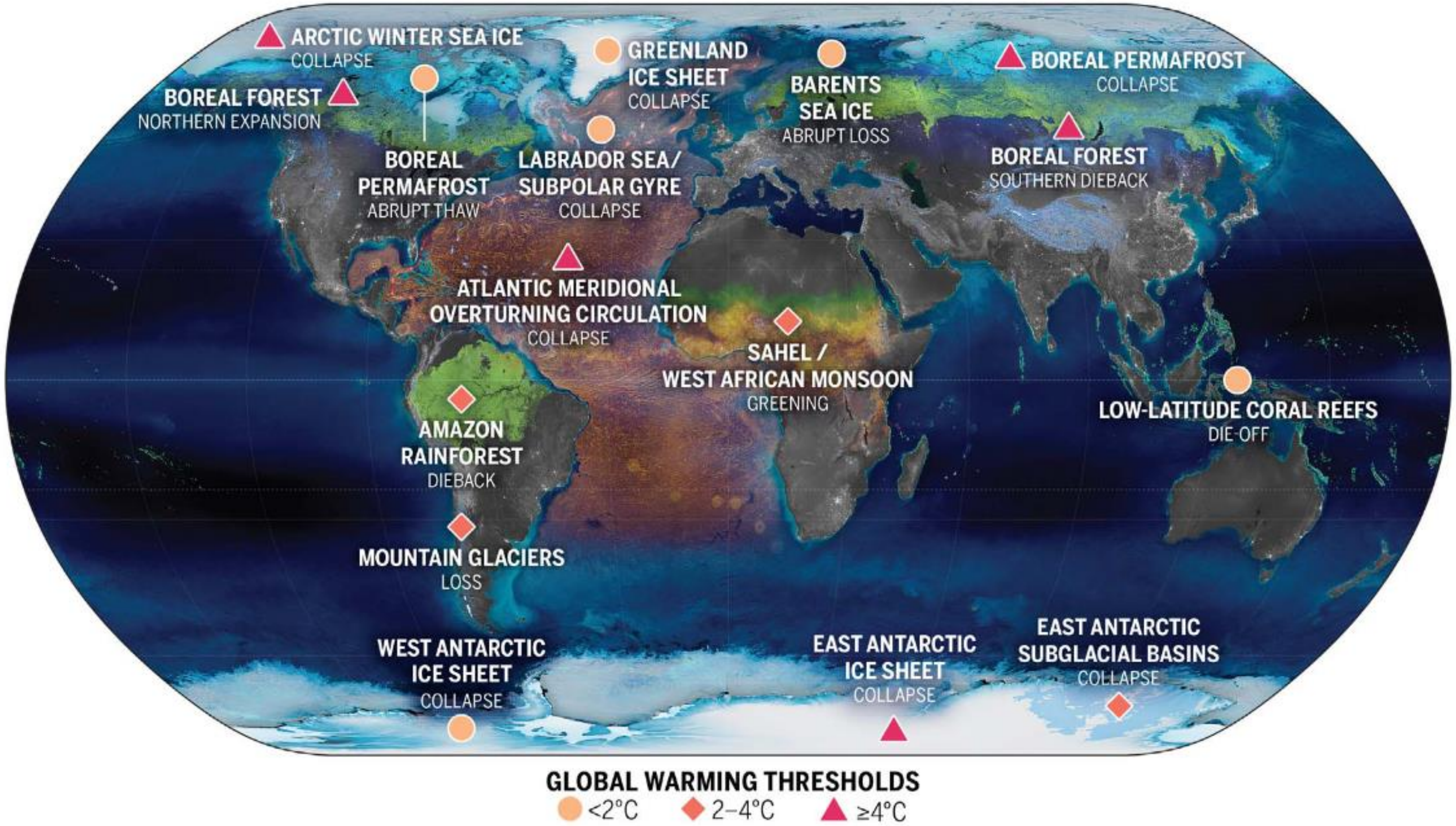
Ecosystem shifts





Tipping Points

IPCC AR6 (2021) : "a critical threshold beyond which a system reorganizes, often abruptly and/or irreversibly"

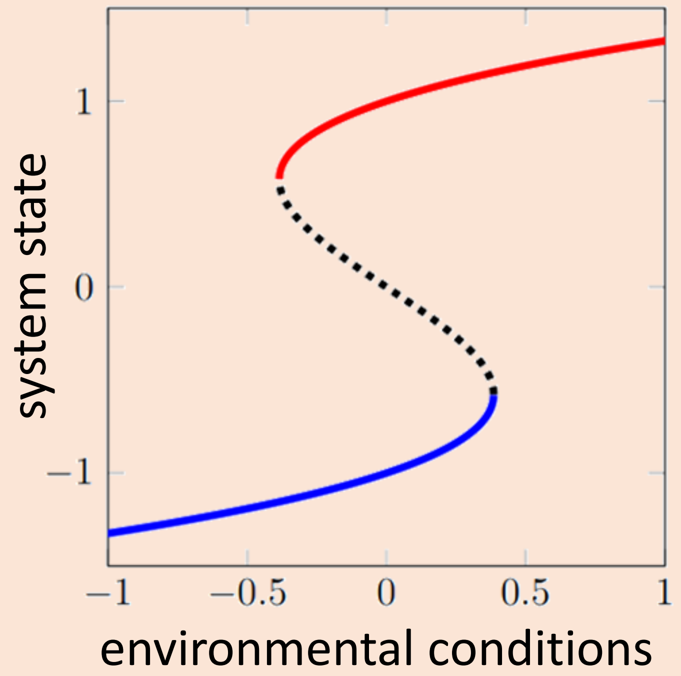


source: McKay et al, 2022

Mathematics

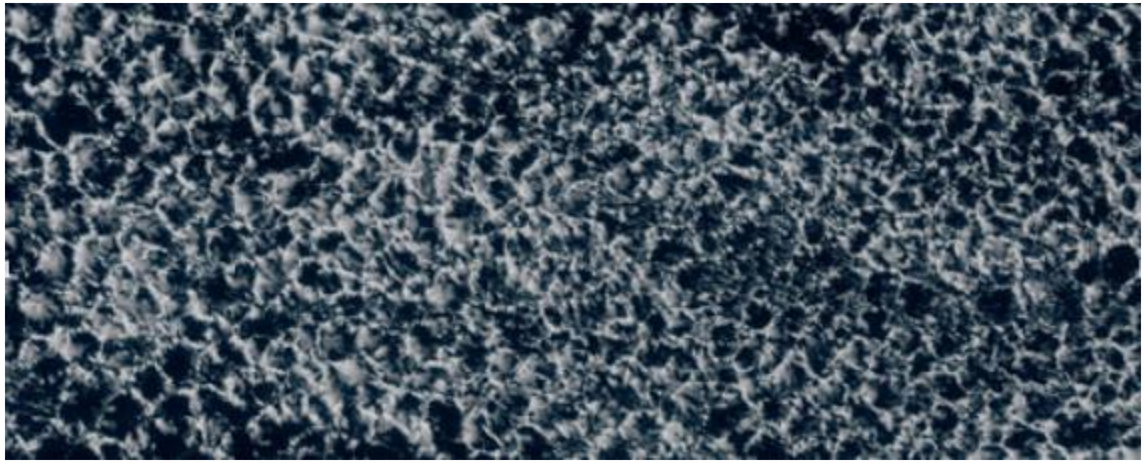
Tipping points ↔ Bifurcations

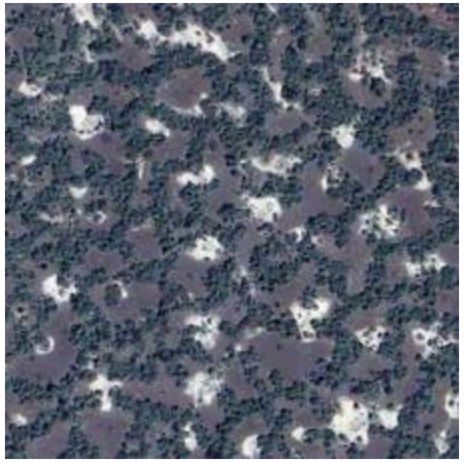
$$\frac{dy}{dt} = f(y, \mu)$$



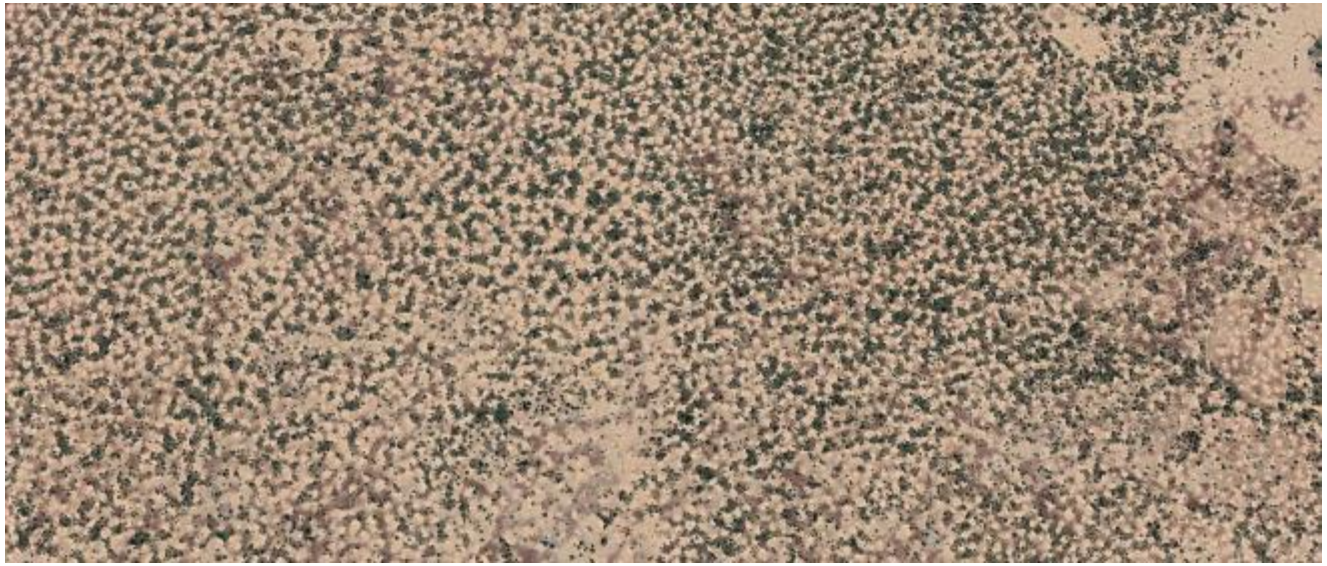
Examples of spatial patterning – regular patterns


mussel beds


clouds


savannas

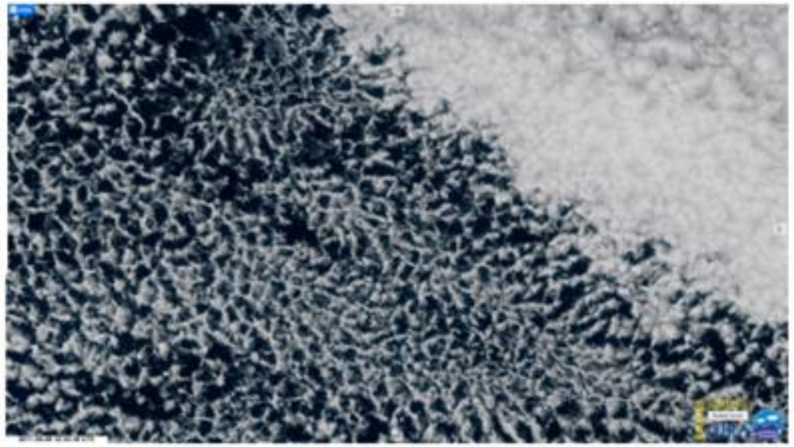

melt ponds


drylands

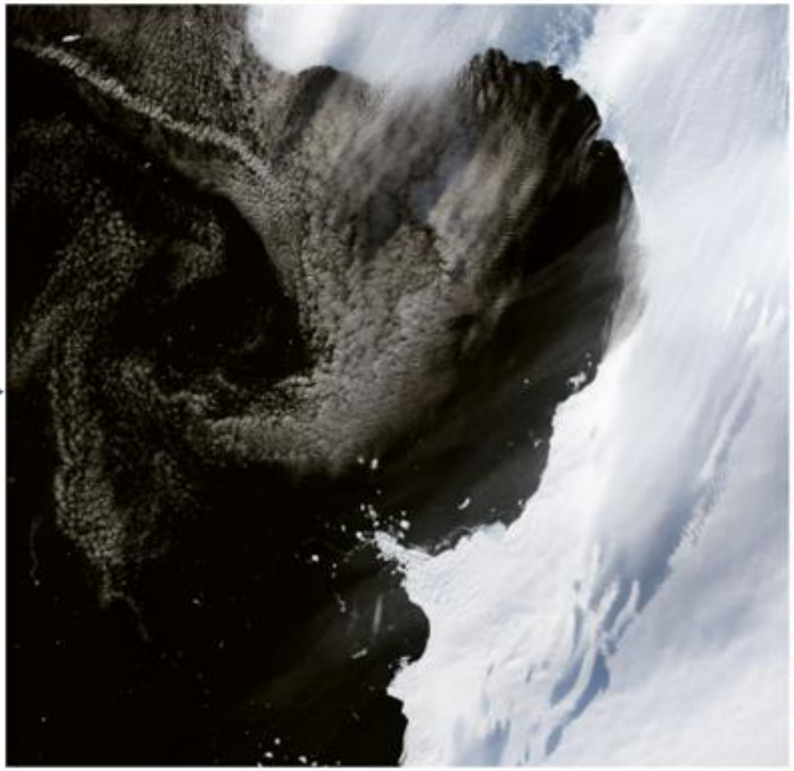
Examples of spatial patterning – spatial interfaces

tropical forest & savanna ecosystems
[Google Earth]

types of stratocumulus clouds
[RAMMB/CIRA SLIDER]



sea-ice & water at Eltanin Bay
[NASA's Earth observatory]



algae bloom in Lake St. Clair
[NASA's Earth observatory]

Part 1:
Turing Patterns

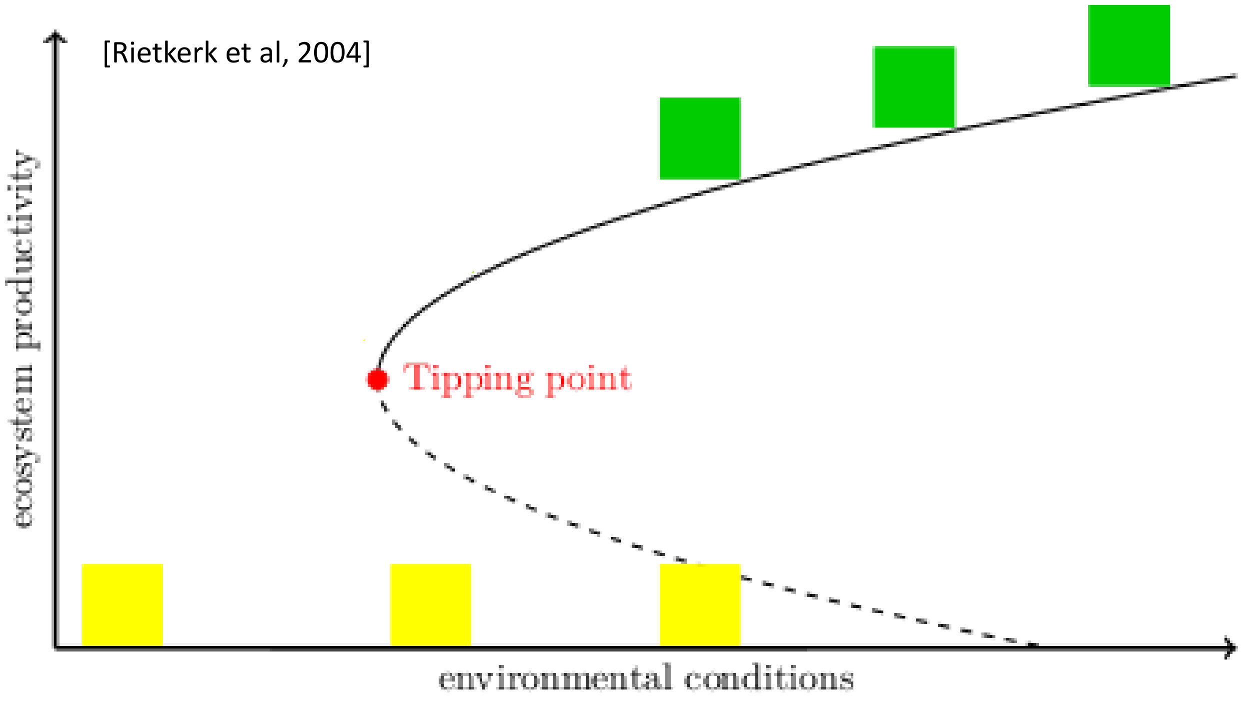
image courtesy: National Science Foundation

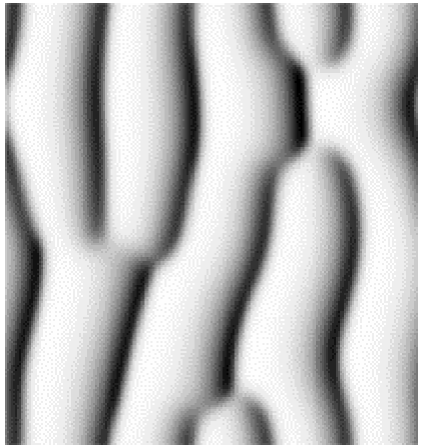
Patterns in models

Add spatial transport:

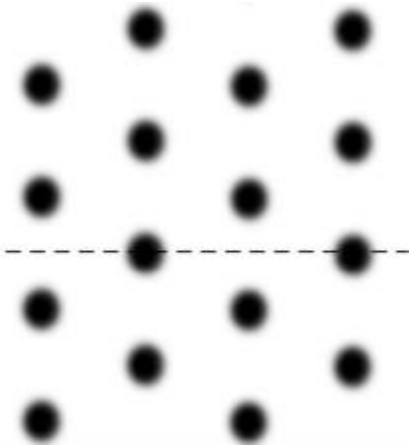
Reaction-Diffusion equations:

$$\begin{cases} \dfrac{du}{dt} = f(u,v) + D_u\,\Delta u \\ \dfrac{dv}{dt} = g(u,v) + D_v\,\Delta v \end{cases}$$



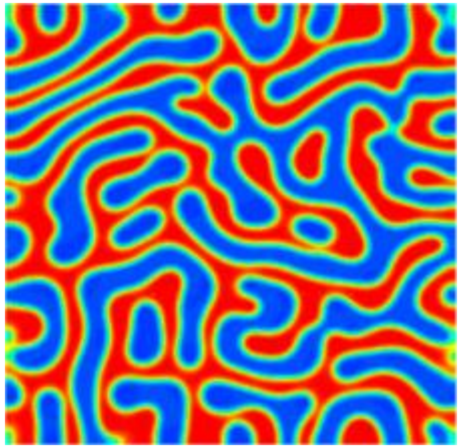


[Klausmeier, 1999]



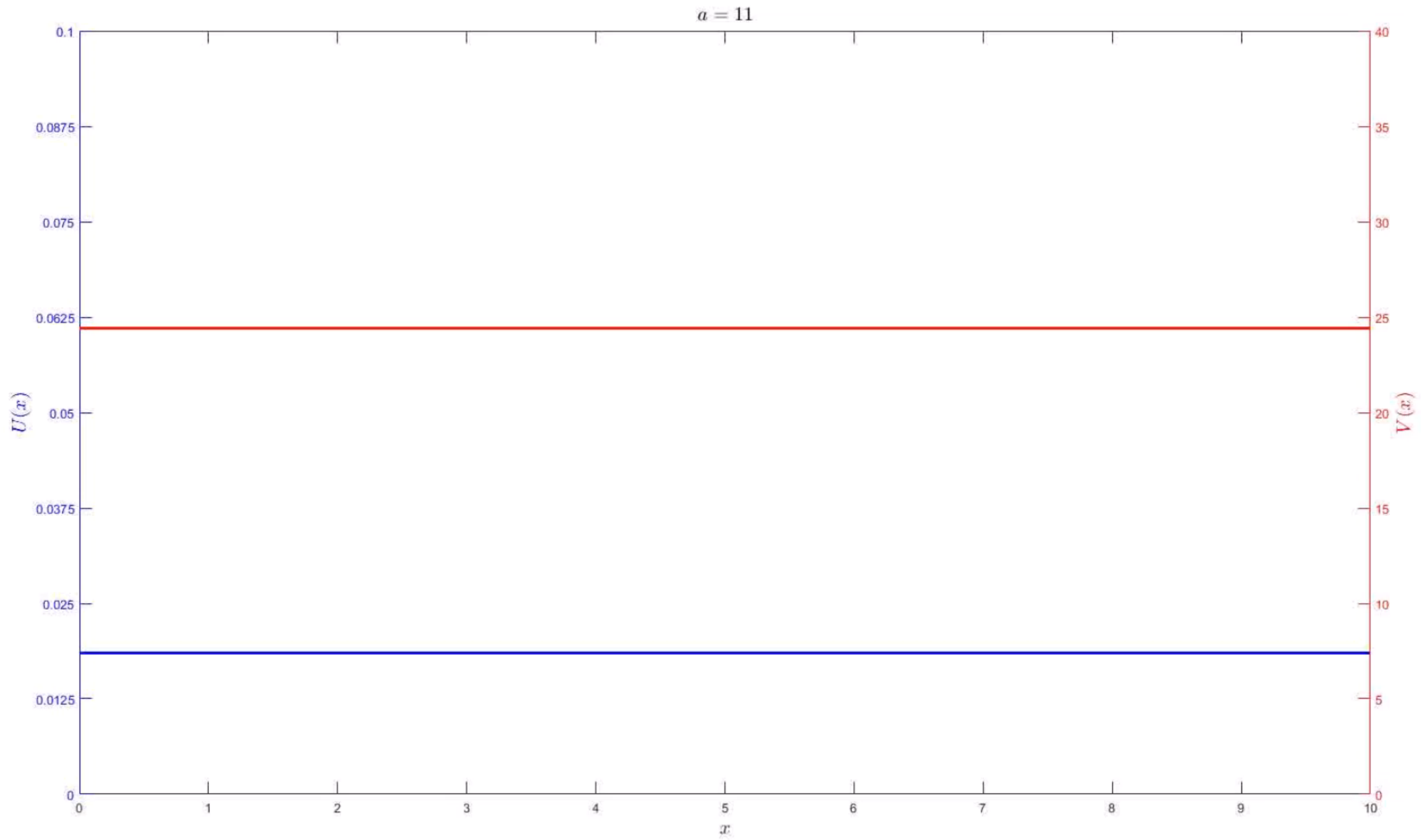
[Gilad et al, 2004]

[Rietkerk et al, 2002]



[Liu et al, 2013]

Behaviour of PDEs



Turing patterns

$$\begin{cases} \frac{du}{dt} = f(u,v) + D_u \Delta u \\ \frac{dv}{dt} = g(u,v) + D_v \Delta v \end{cases}$$

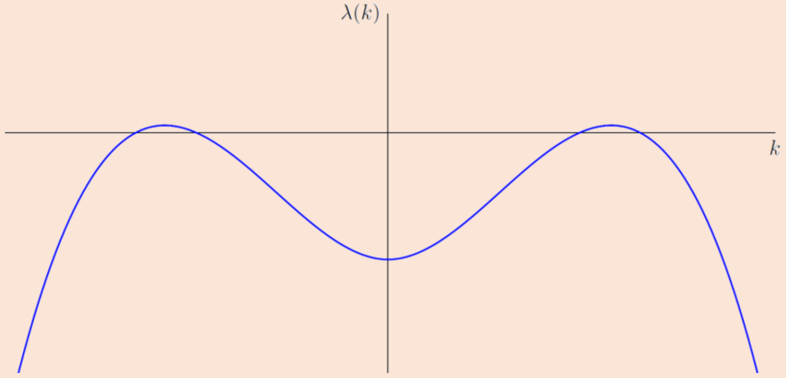
Turing bifurcation

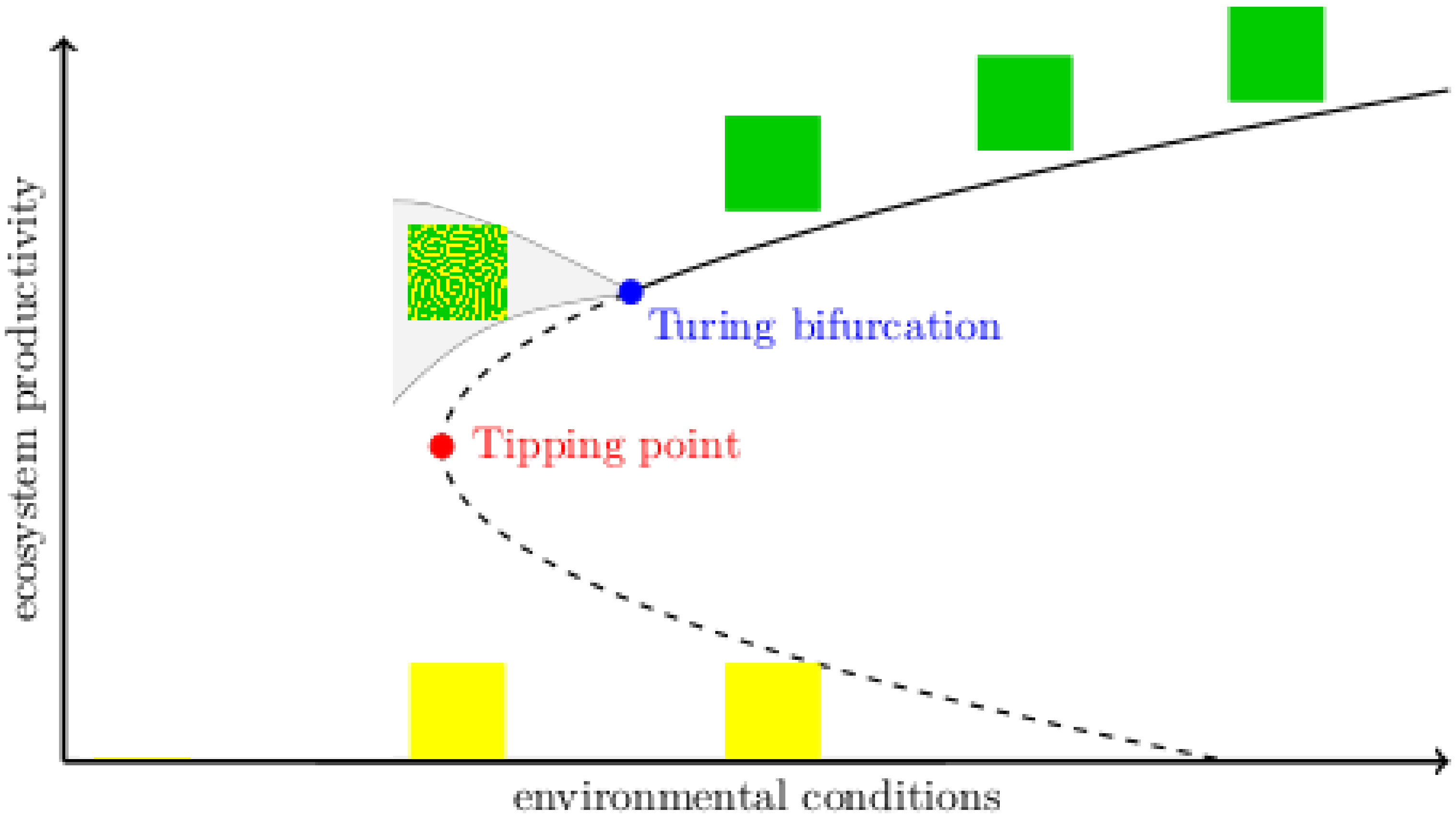
Instability to non-uniform perturbations

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u_* \\ v_* \end{pmatrix} + e^{\lambda t} e^{ikx} \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix}$$

→ Dispersion relation

$$\lambda(k) = \cdots$$





Weakly non-linear analysis

Ginzburg-Landau equation / Amplitude Equation

& Eckhaus/Benjamin-Feir-Newel criterion

[Eckhaus, 1965; Benjamin & Feir, 1967; Newell, 1974]

Busse balloon

$$\begin{cases} \frac{du}{dt} = f(u,v) + D_u \Delta u \\ \frac{dv}{dt} = g(u,v) + D_v \Delta v \end{cases}$$

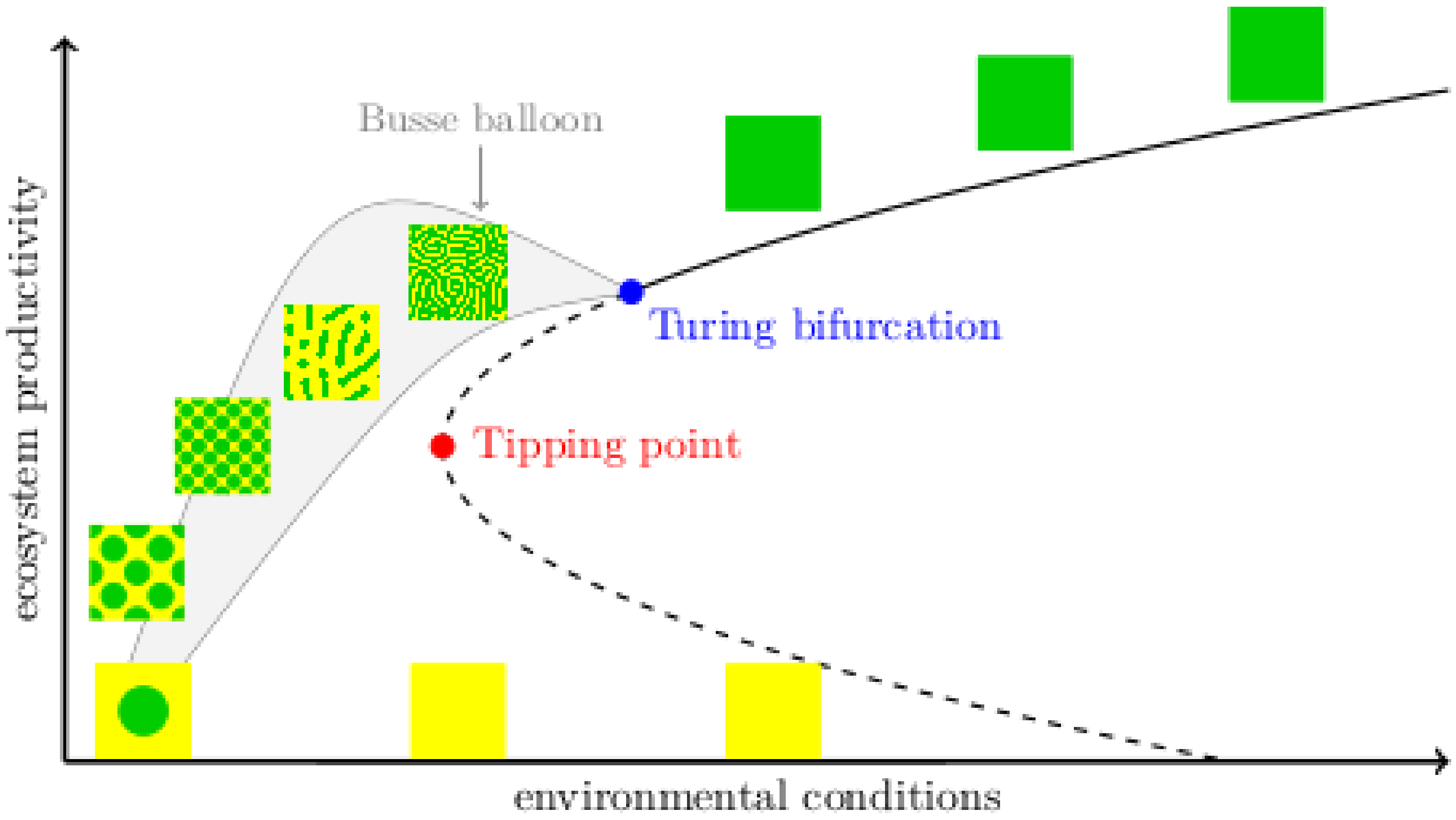
Busse balloon

A model-dependent shape in *(parameter, observable)* space that indicates all stable patterned solutions to the PDE.

Construction Busse balloon

Via numerical continuation

few general results on the shape of Busse balloon

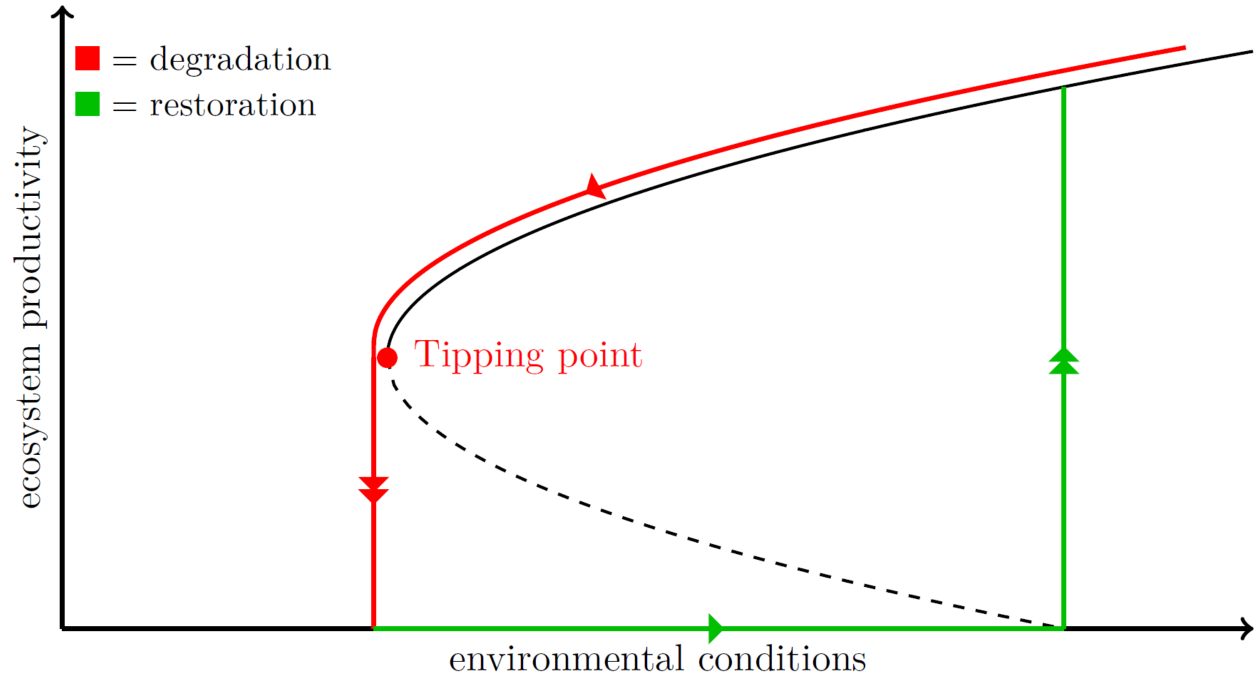


Busse balloon

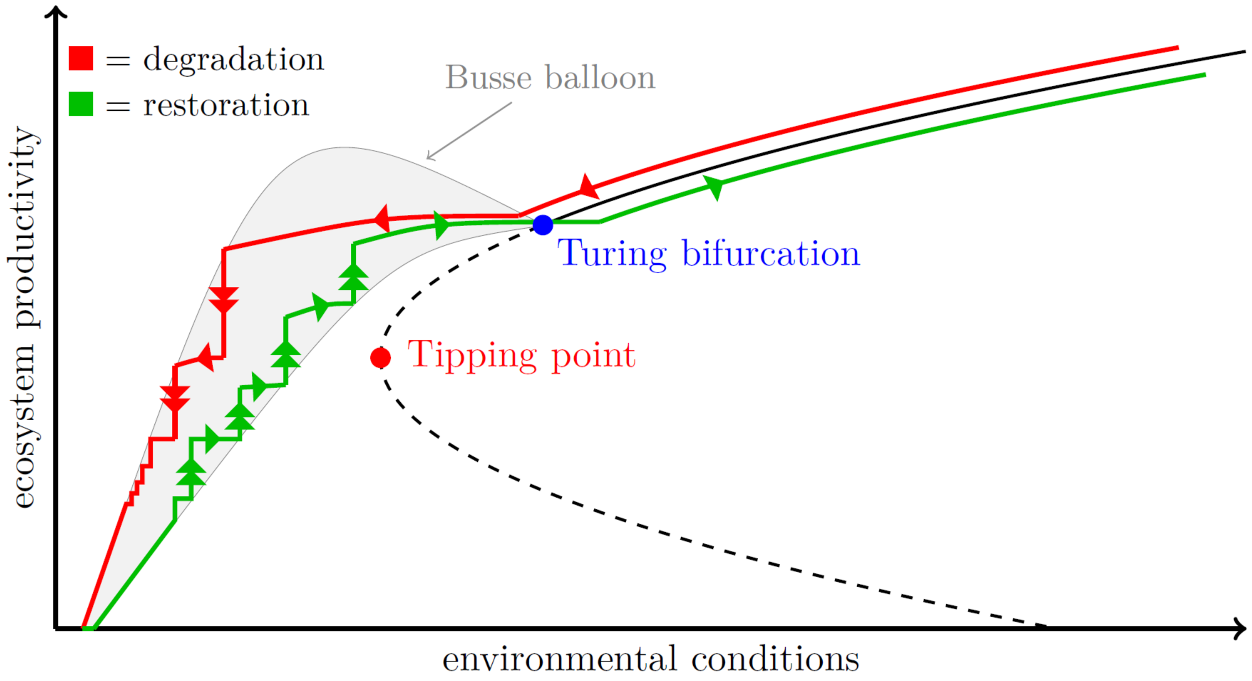
Idea originates from thermal convection

[Busse, 1978]

Tipping of (Turing) patterns

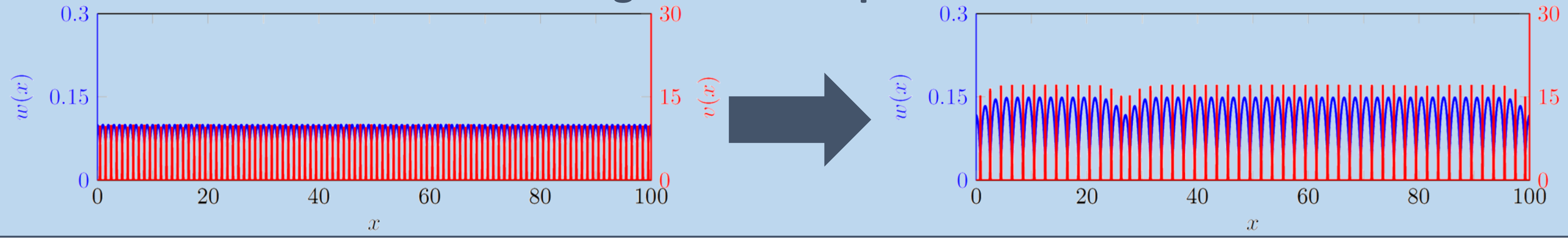


Classic tipping



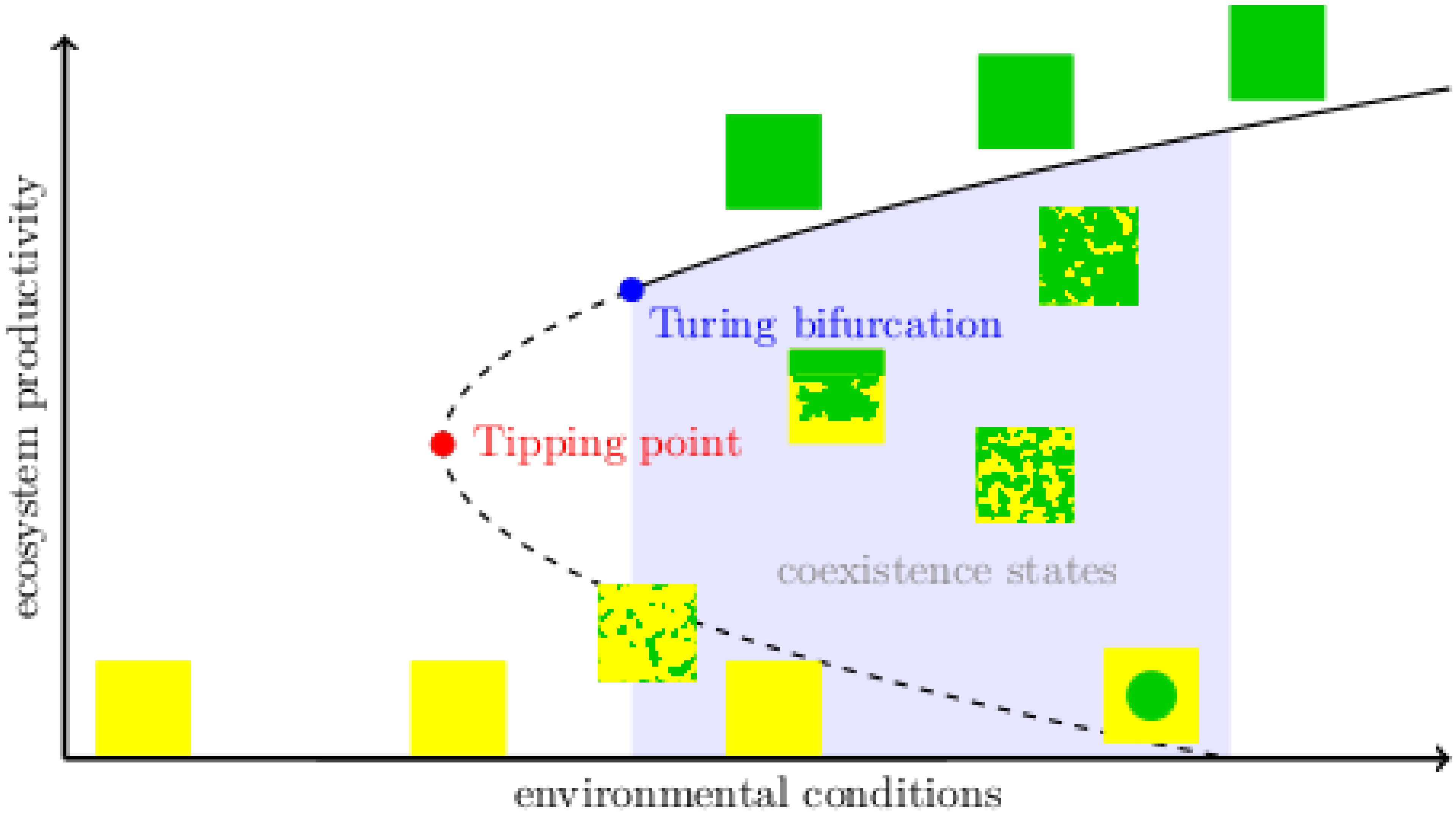
Tipping of patterns

Degradation of patterns



Part 2:
Coexistence States
and spatial heterogeneities

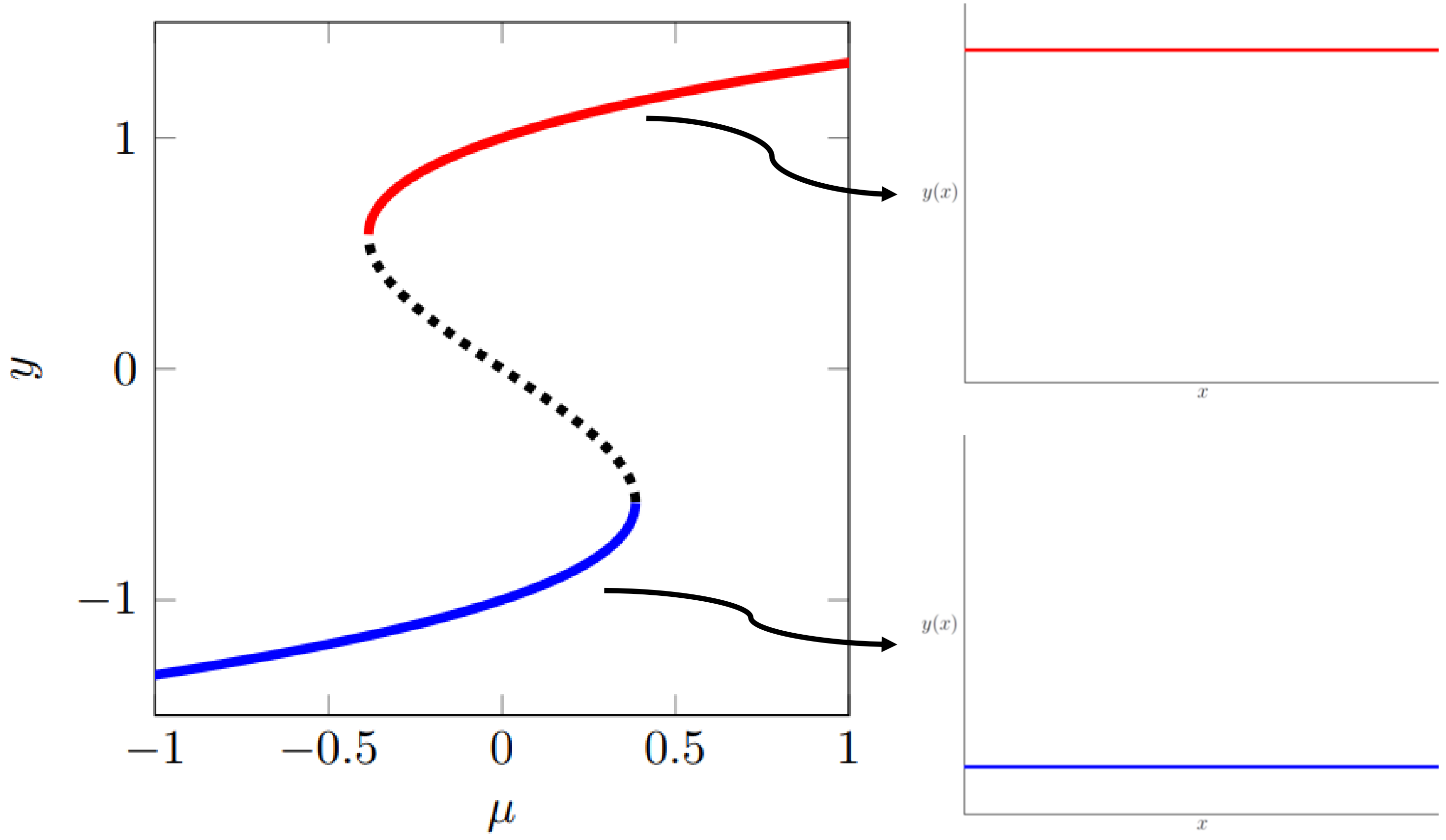
Coexistence states in bifurcation diagram

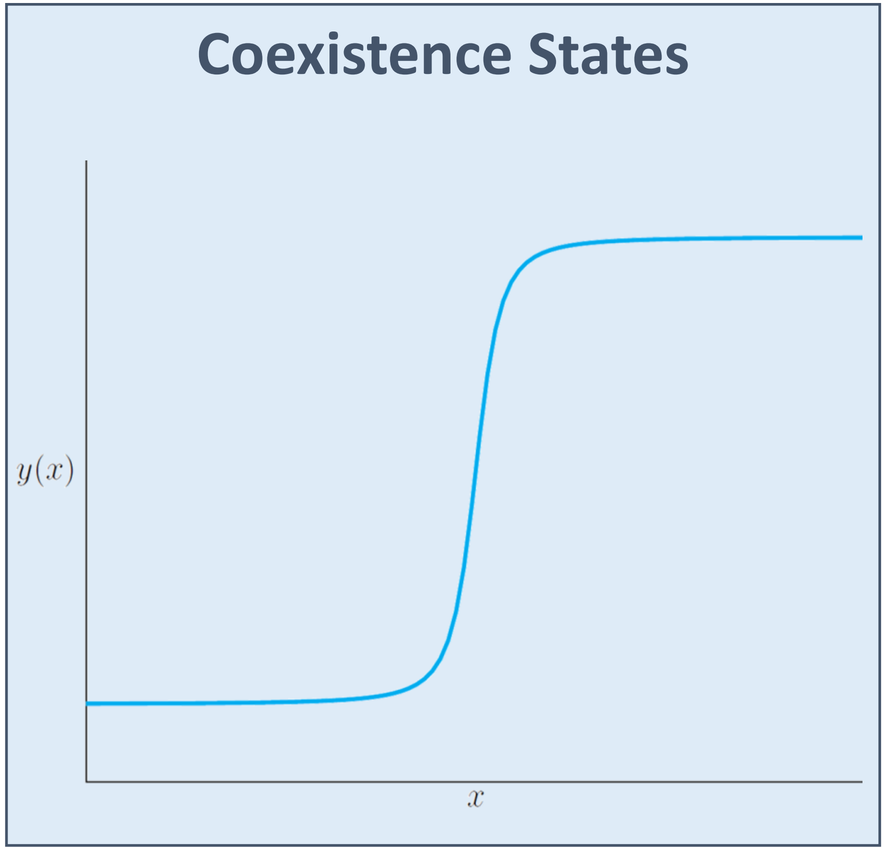


Coexistence states

Bistable (Allen-Cahn/Nagumo) equation:

$$\frac{\partial y}{\partial t} = y(1 - y^2) + \mu + D\,\frac{\partial^2 y}{\partial x^2}$$





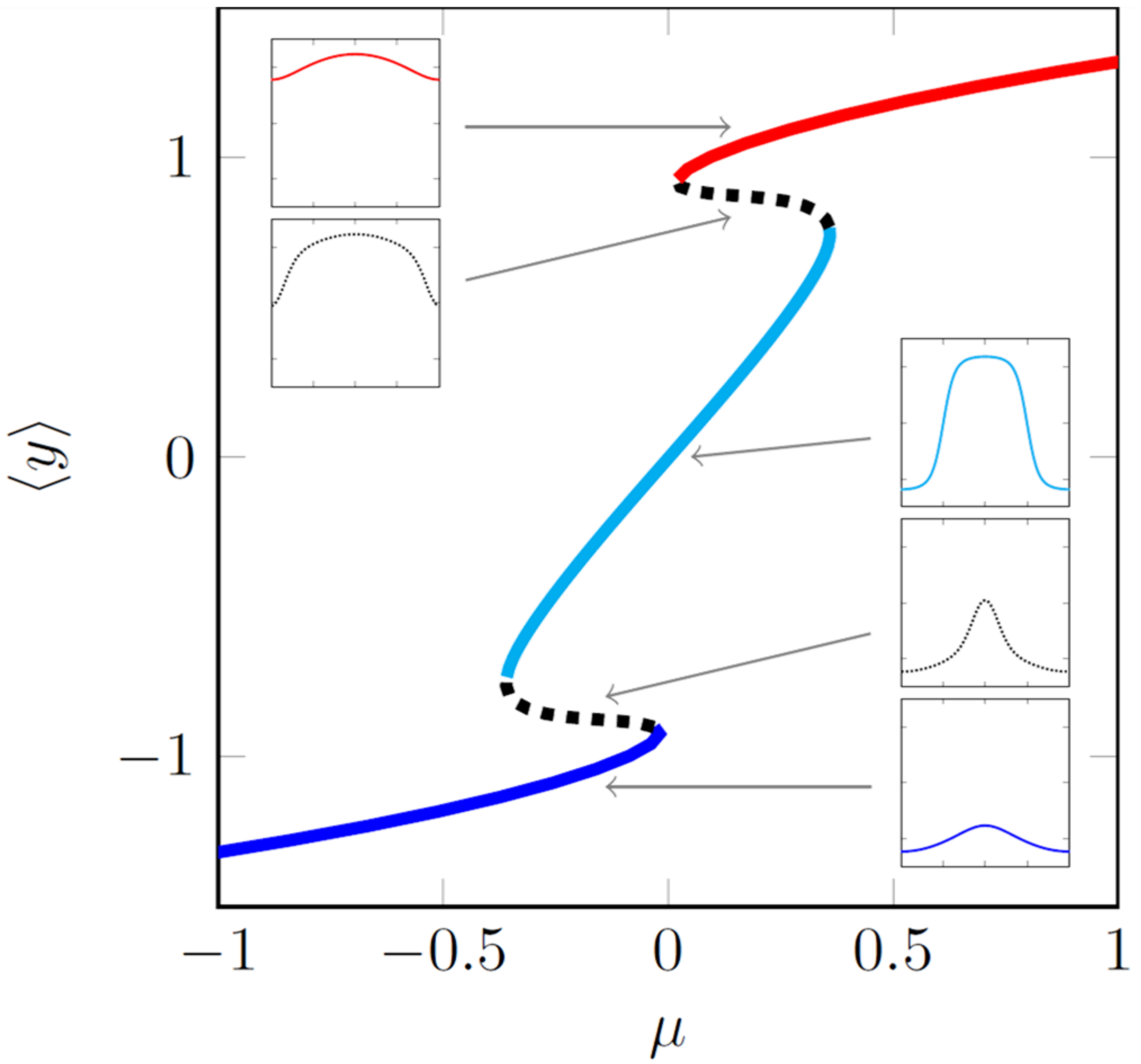
Adding Spatial Heterogeneity

$$\frac{\partial y}{\partial t} = D\frac{\partial^2 y}{\partial x^2} + f(y, x; \mu)$$

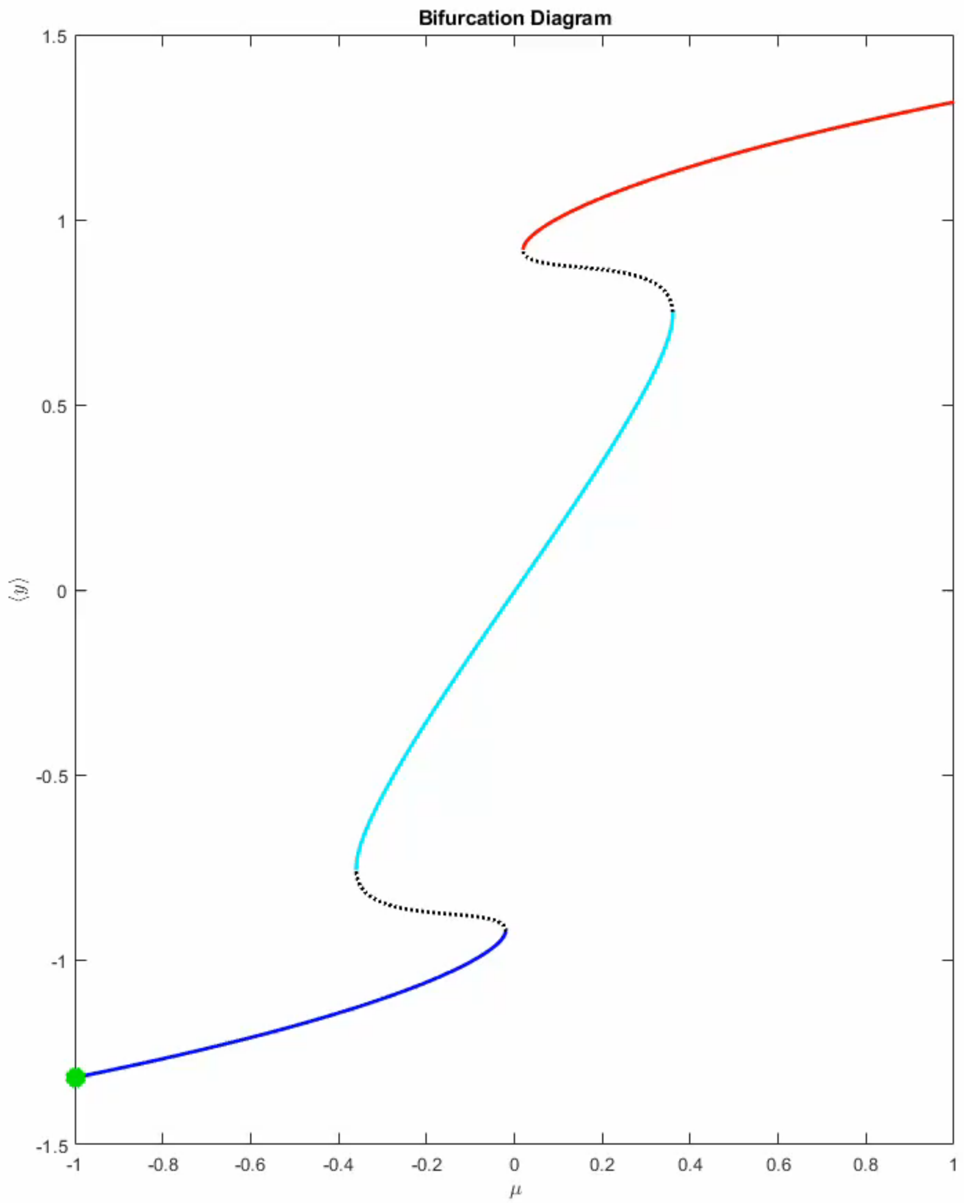
Now, the **local** difference in potentials determines the front movement

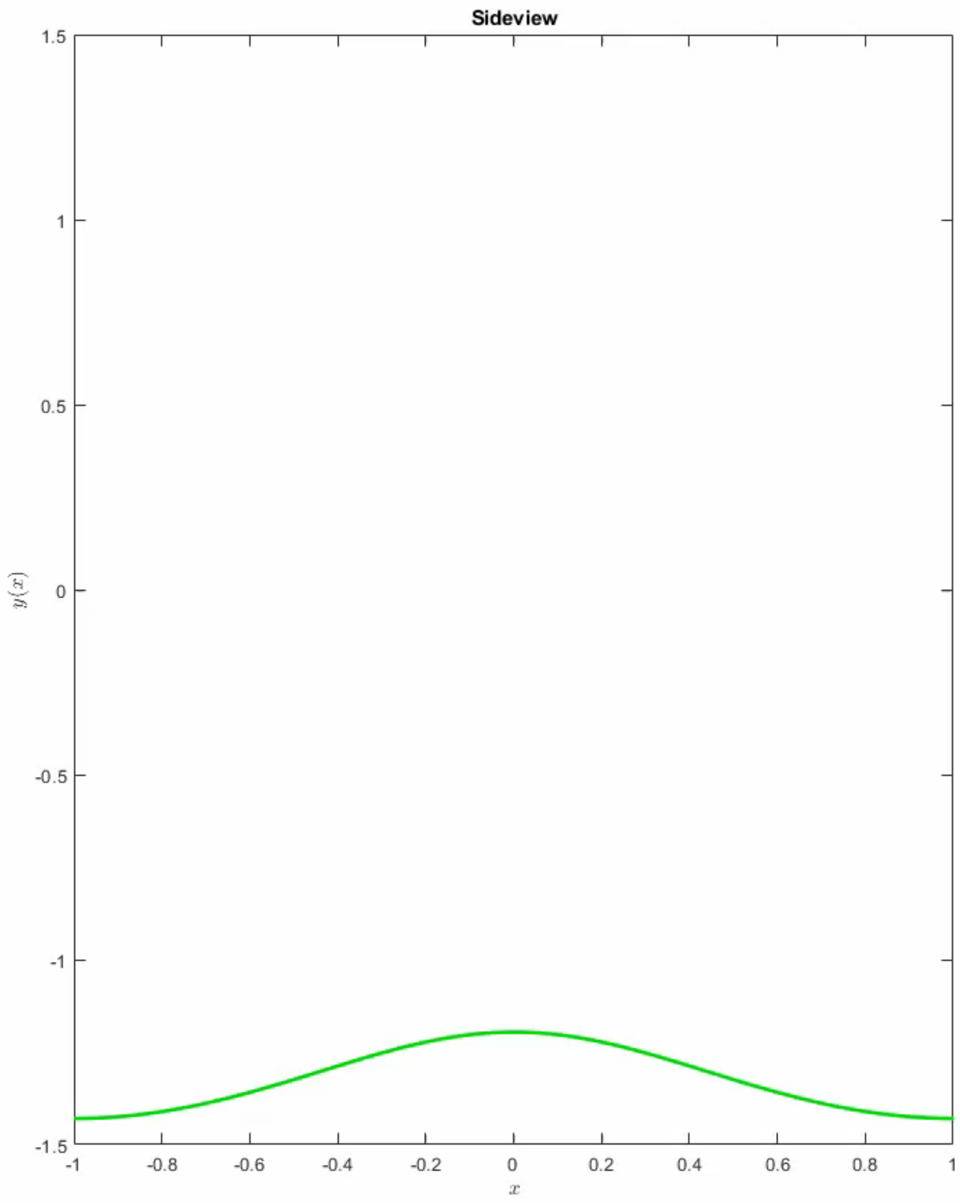
New behaviour:

- Multi-fronts can be stationary
- Maxwell point is smeared out

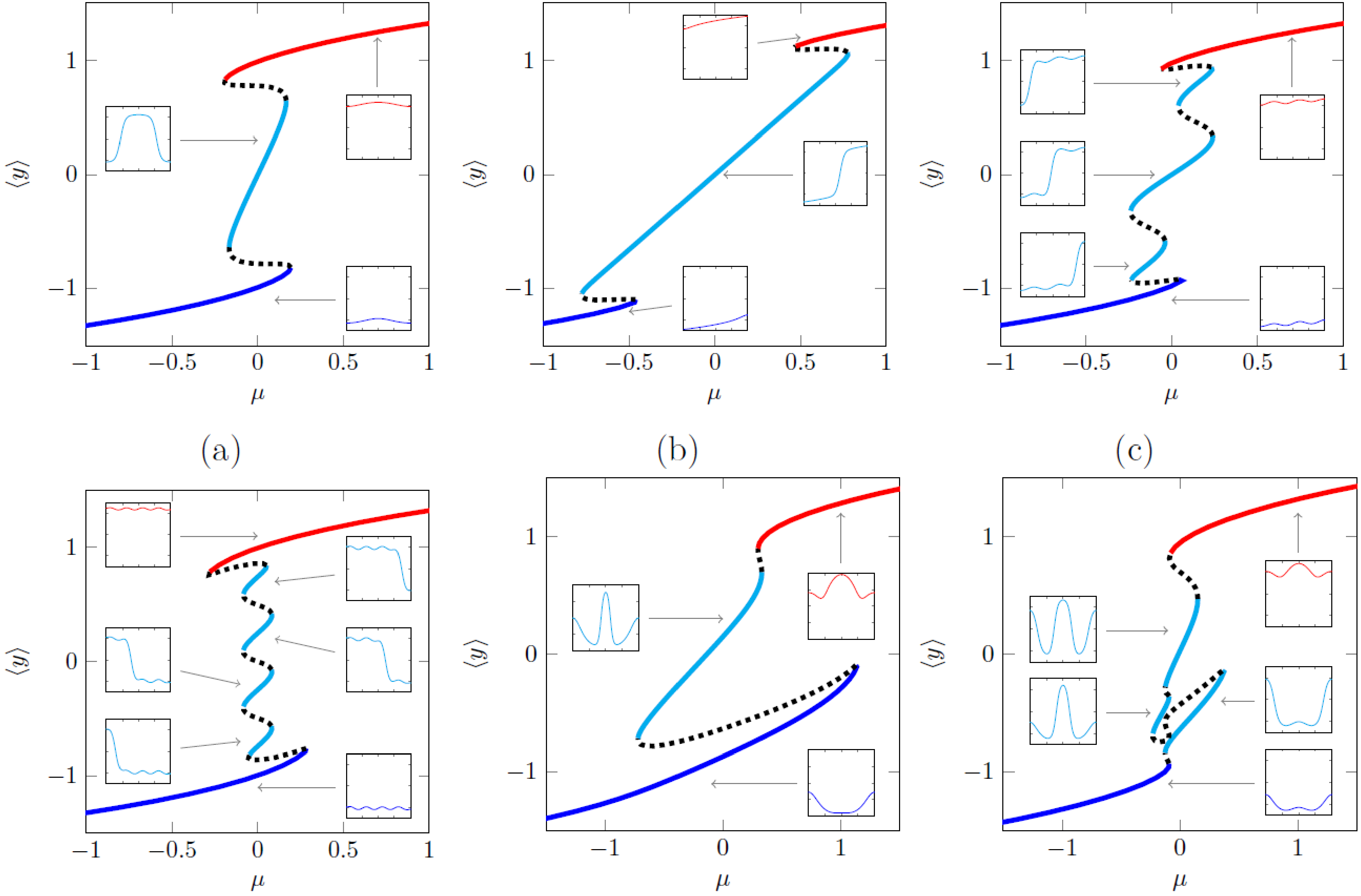


Fragmented Tipping



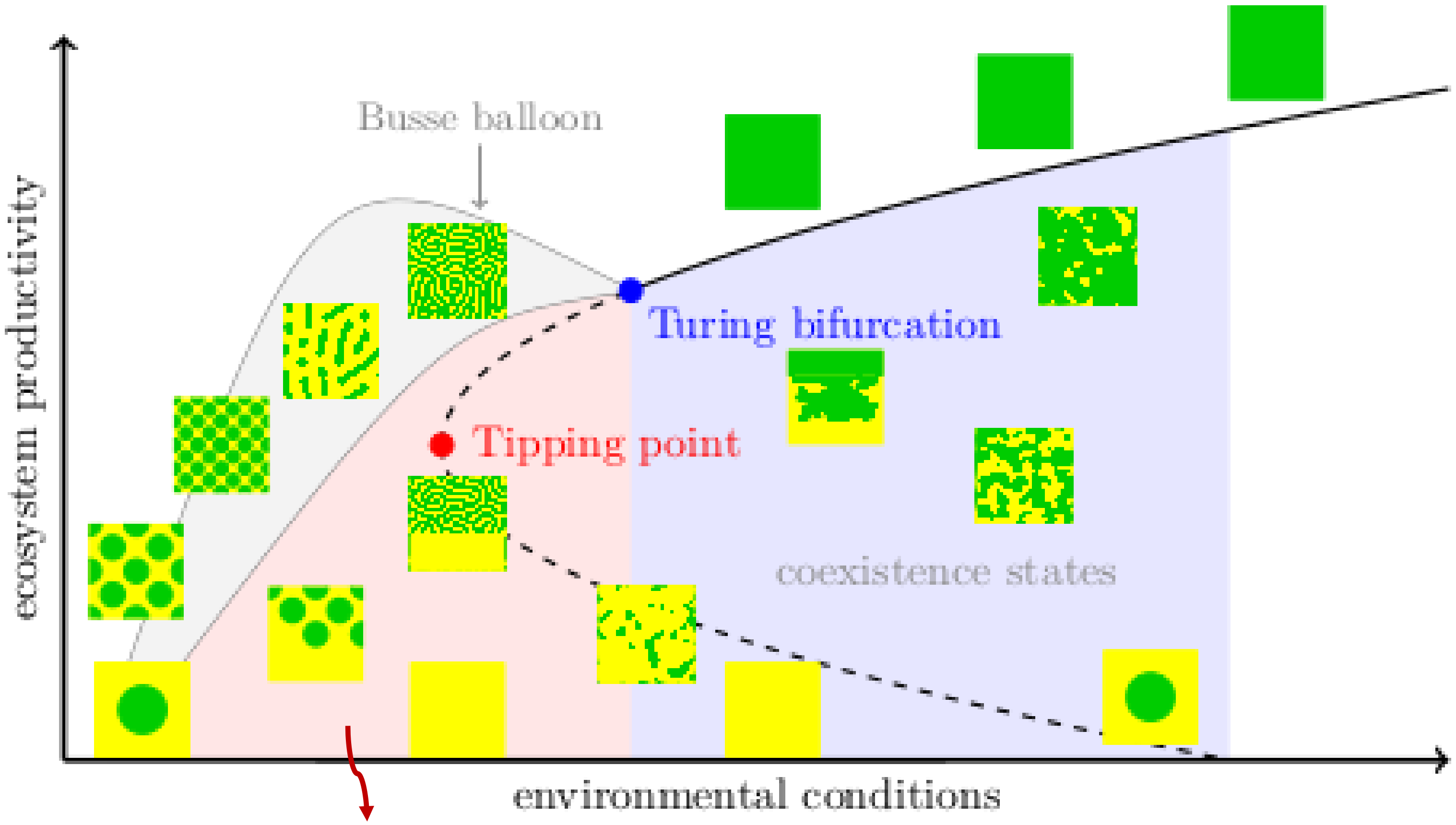


Other Spatial Heterogeneities

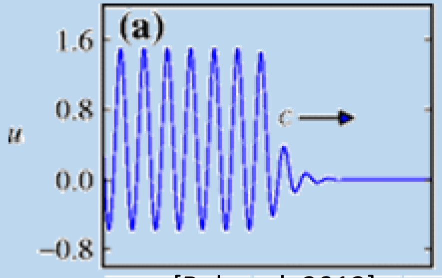


Part 3: Tipping in Spatially Extended Systems?

“Bifurcation Diagram” for spatially extended systems



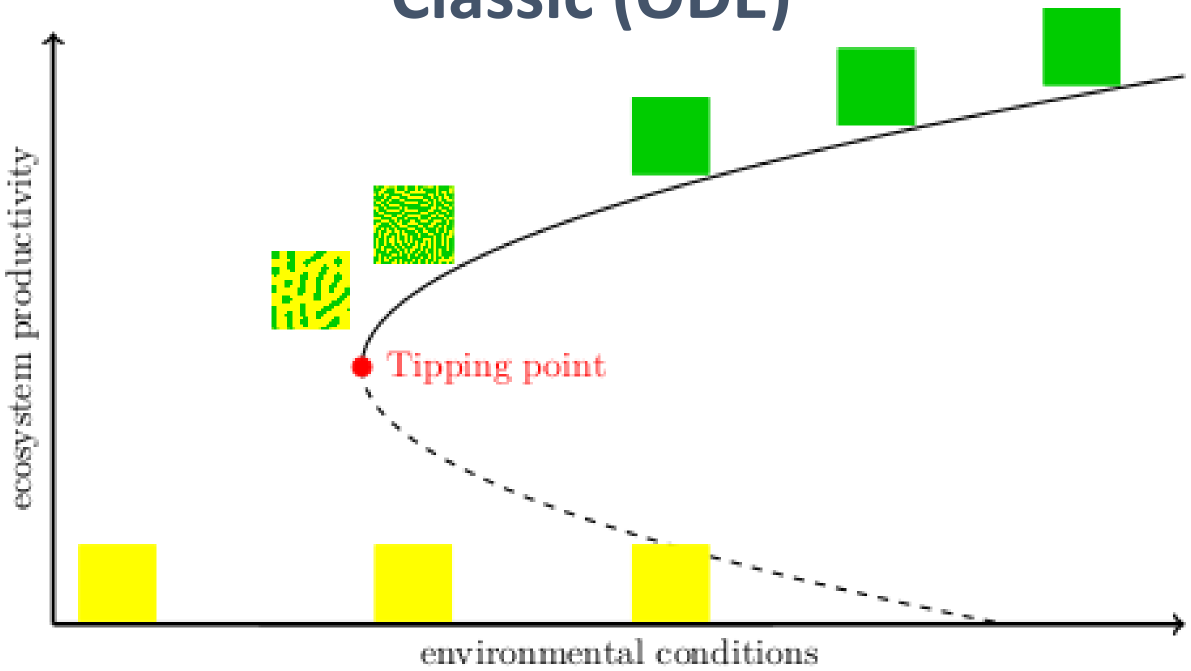
Coexistence states between patterned and uniform states also exist



[Bel et al, 2012]

What if the system tips?

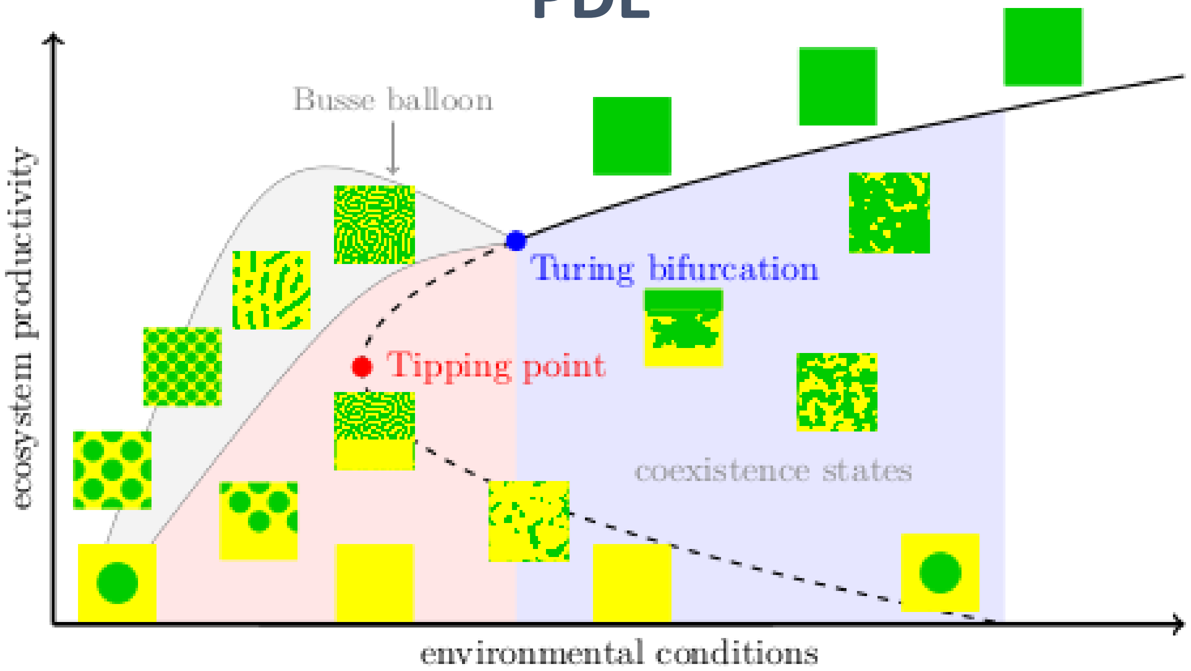
Classic (ODE)



Crossing a Tipping Point:
→ Always full reorganization

Early Warning Signals
signal for <u>WHEN</u>

PDE

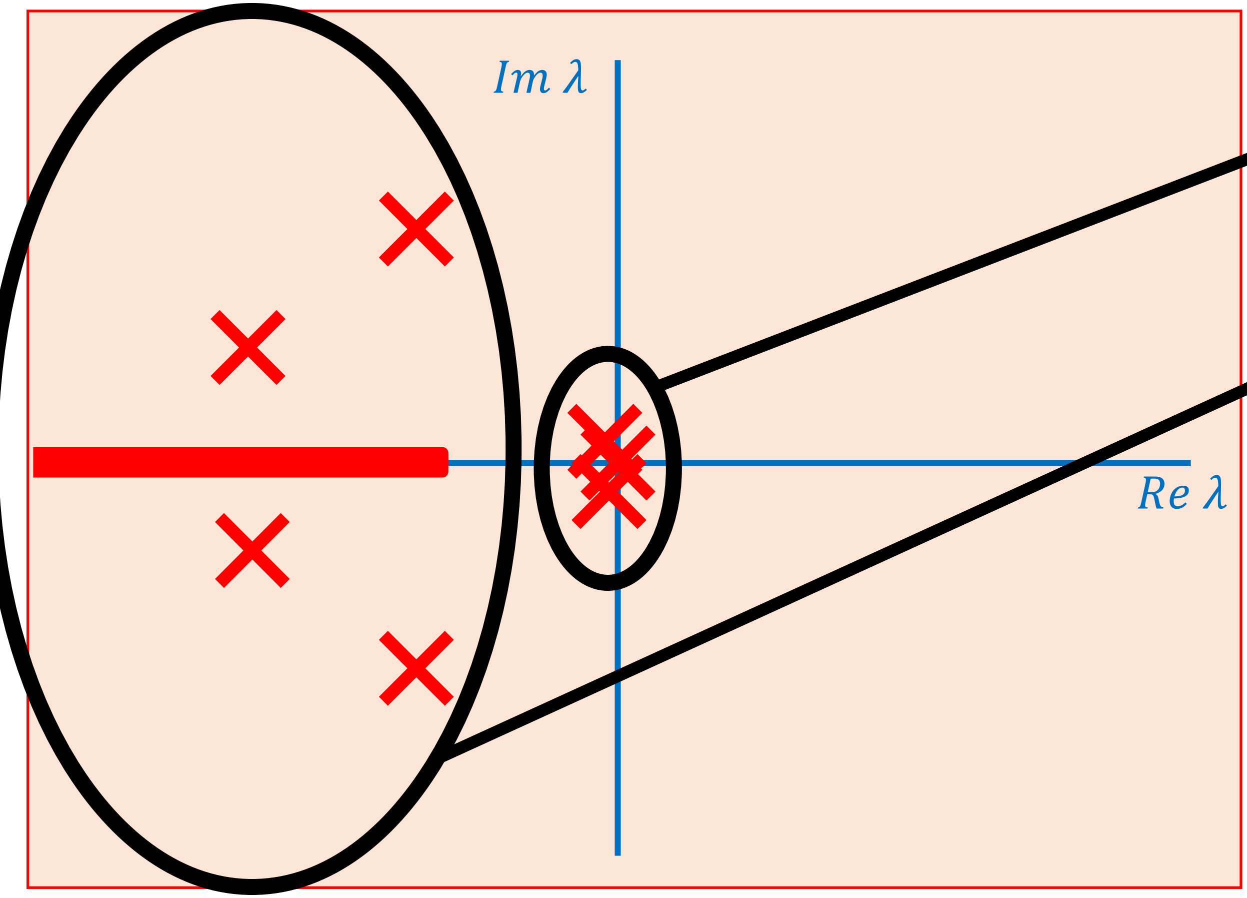


Crossing a bifurcation:
Now also possible:
→ Spatial reorganization (Turing patterns)
→ Fragmented tipping (coexistence states)

Early Warning Signals
need to signal <u>WHEN</u> & <u>WHAT</u>

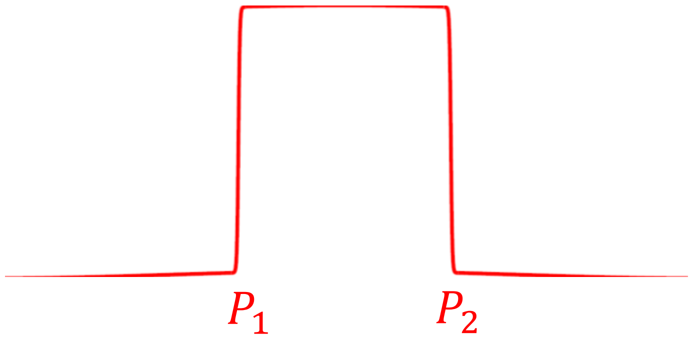
Part 4: Dynamics & Bifurcations of Patterned States

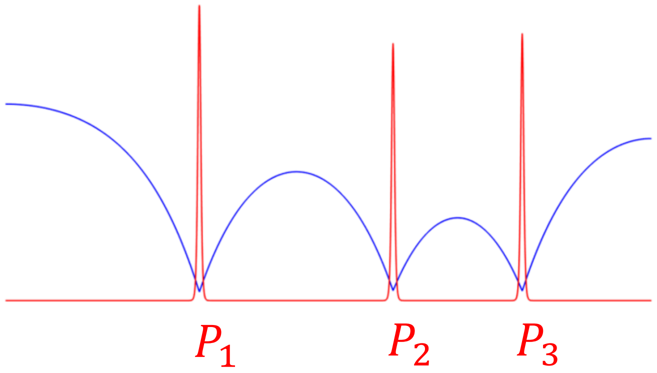
Dynamics of Patterned States

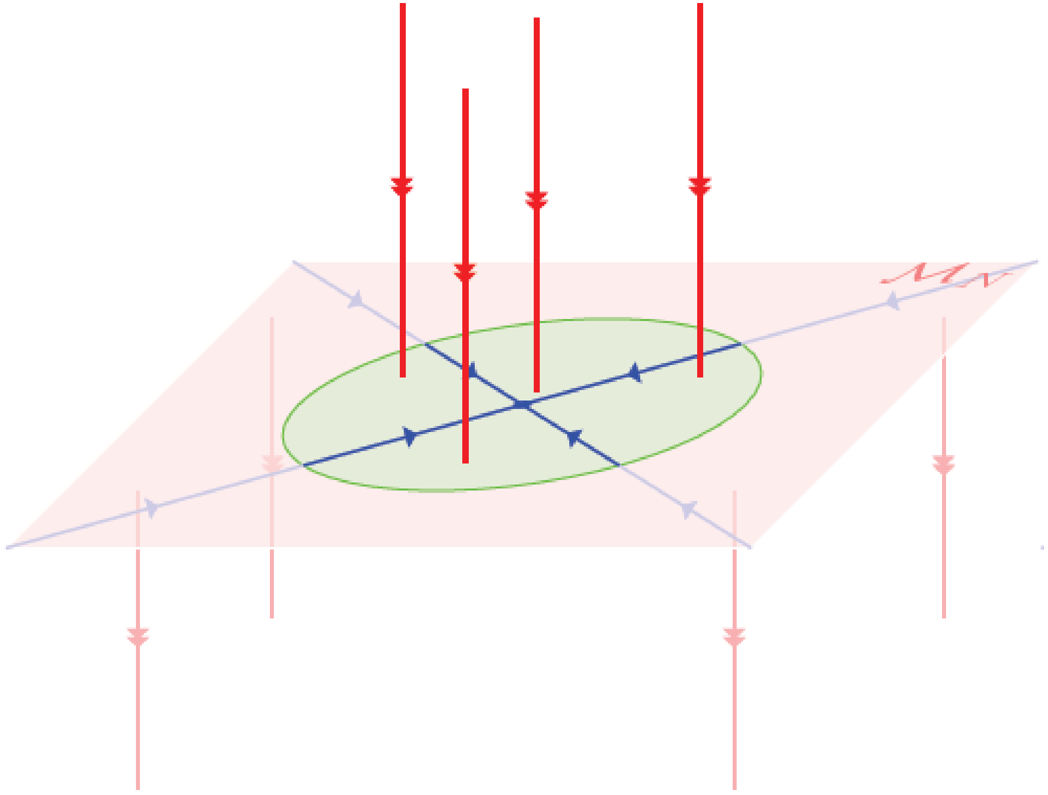


1. SLOW Pattern Adaptation

2. FAST Pattern Degradation





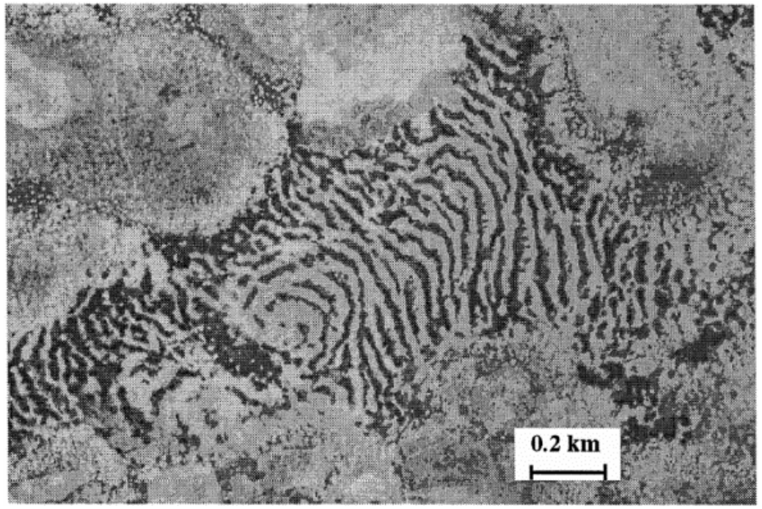


1. SLOW pattern adaptation

Somaliland, 1948 [Macfadyen, 1950]

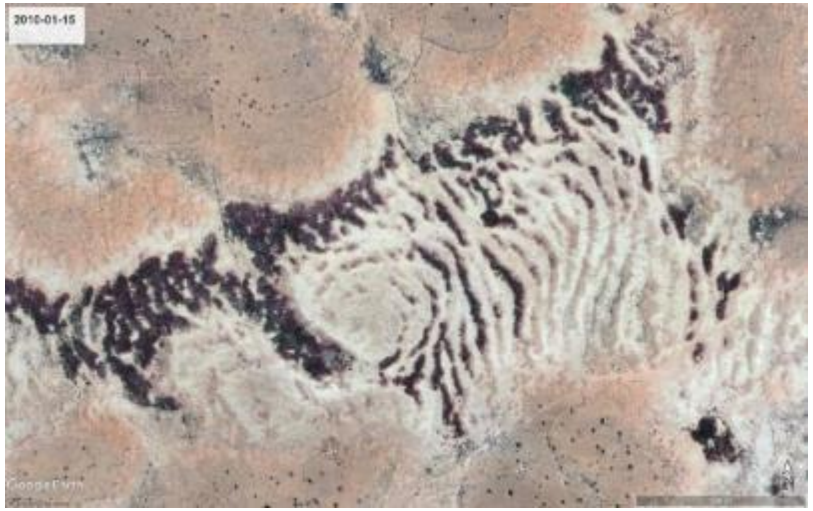
Somaliland, 2008

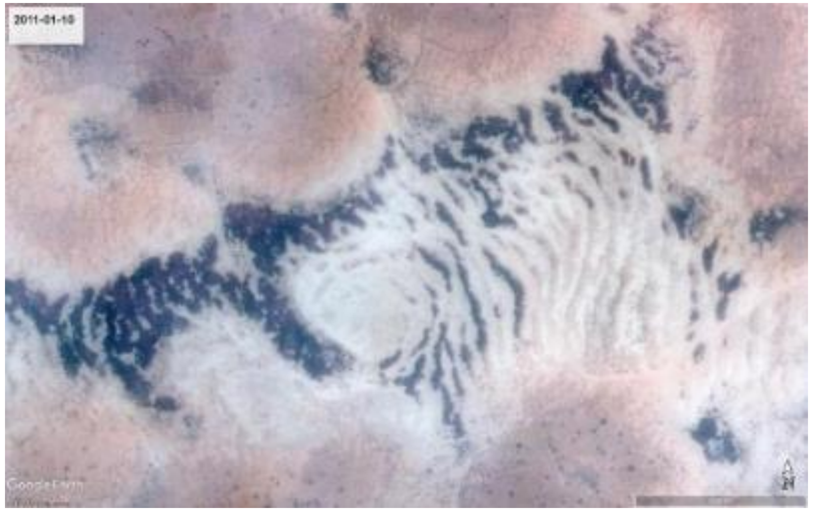
2. FAST Pattern Degradation

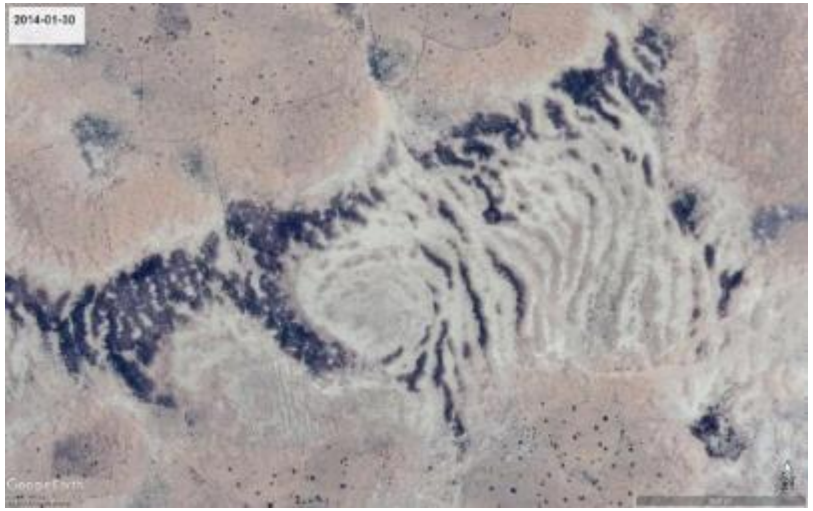


Niger, 1950 [Valentin, 1999]


Niger, 2008

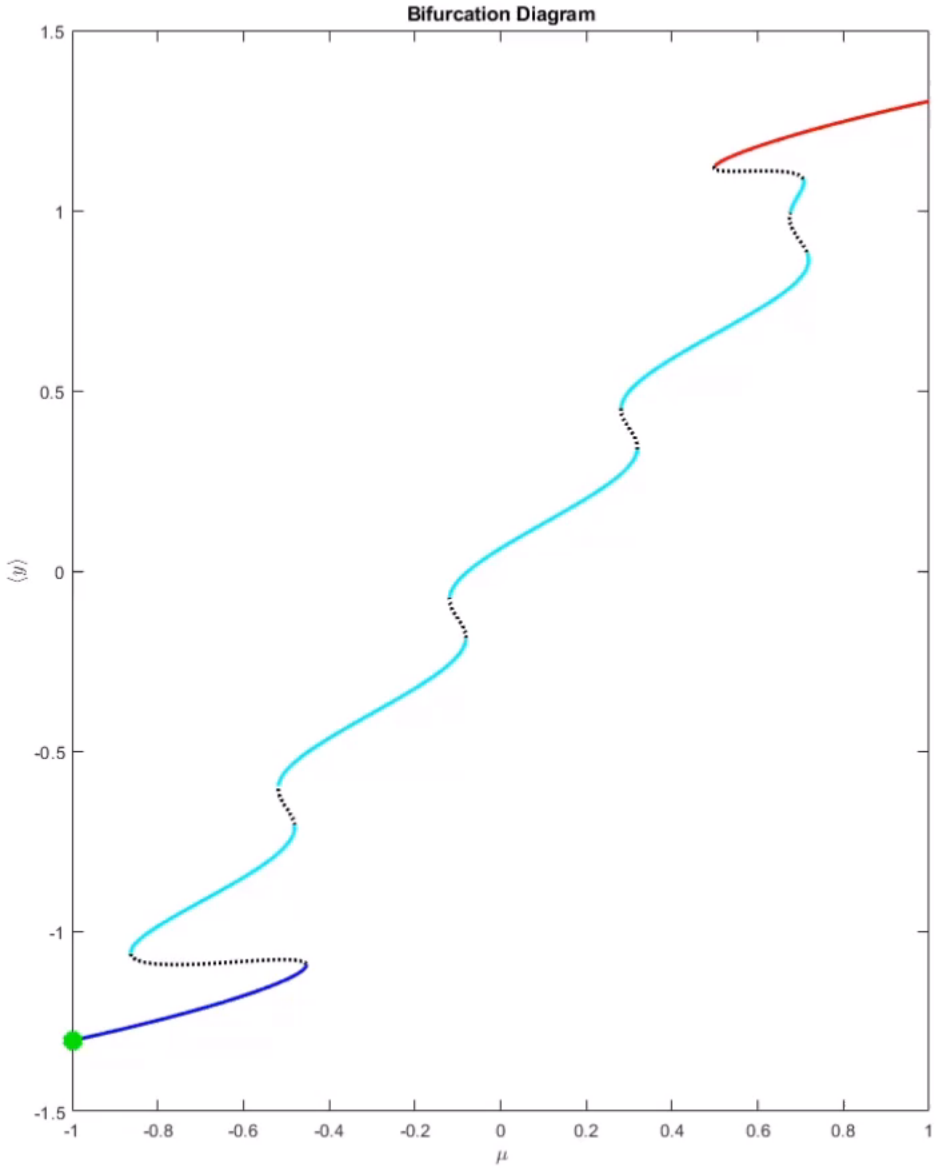

Niger, 2010

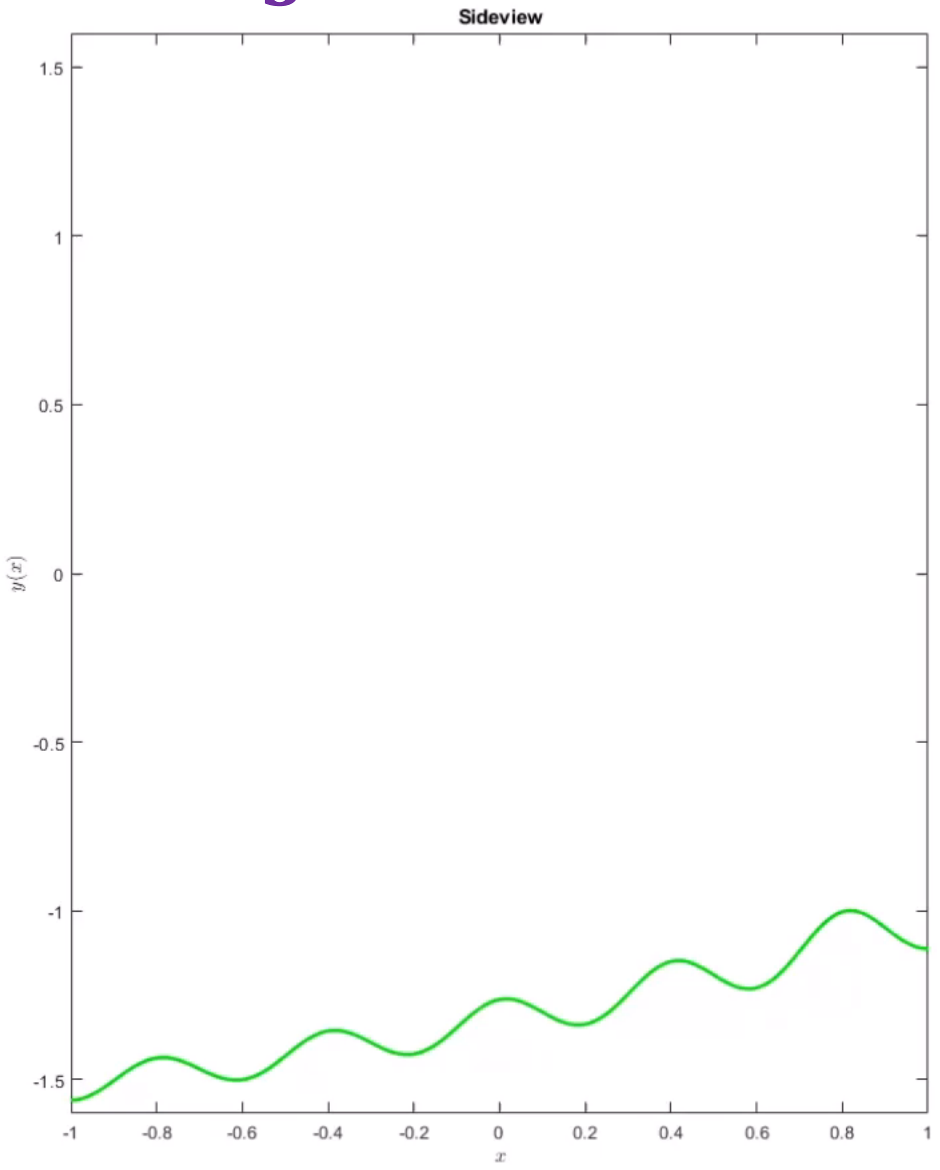

Niger, 2011


Niger, 2014

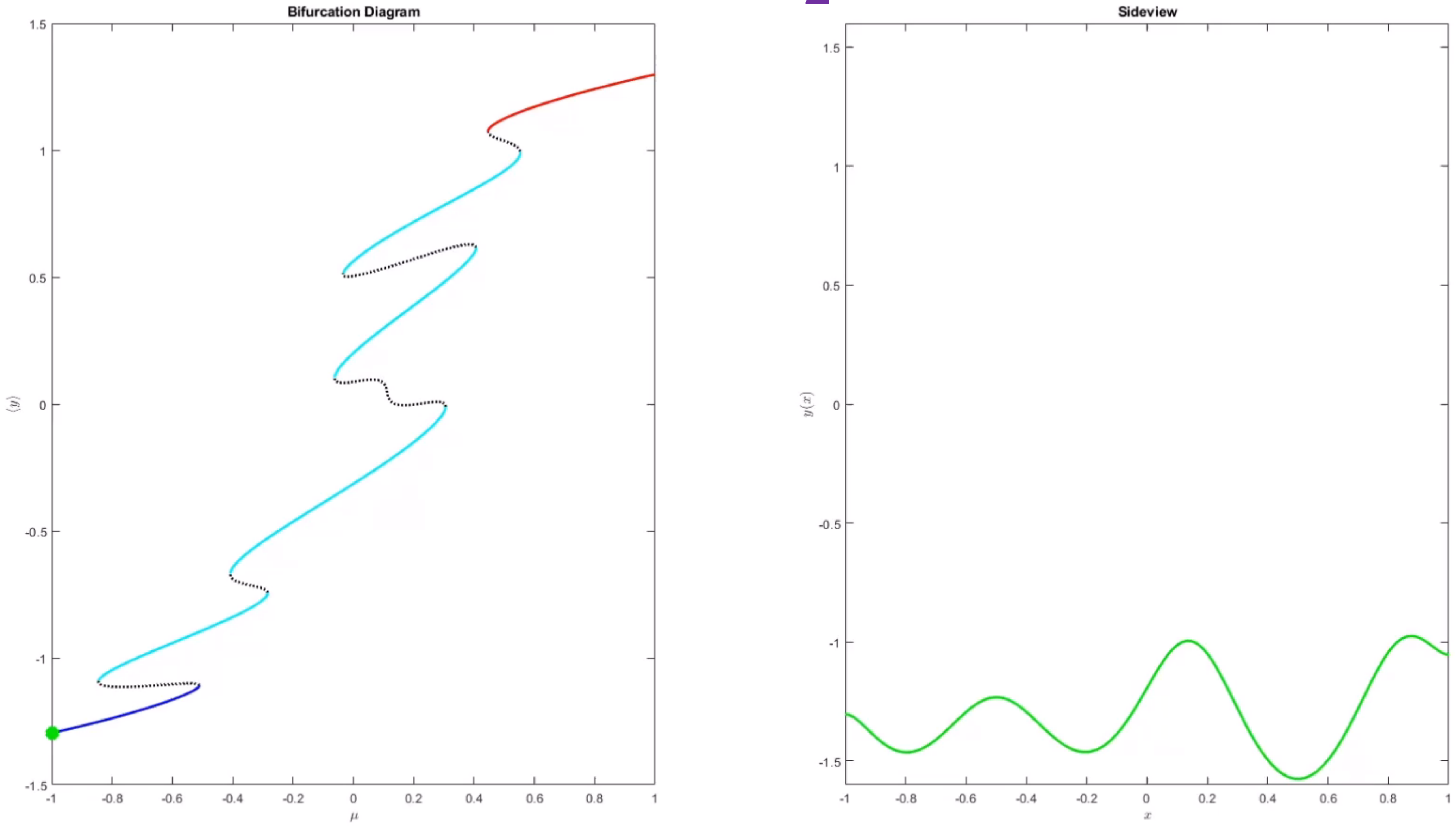

Niger, 2016

$$y_t = D\,y_{xx} + y(1 - y^2) + \mu + x + \frac{2}{5}\cos(5\pi x)$$

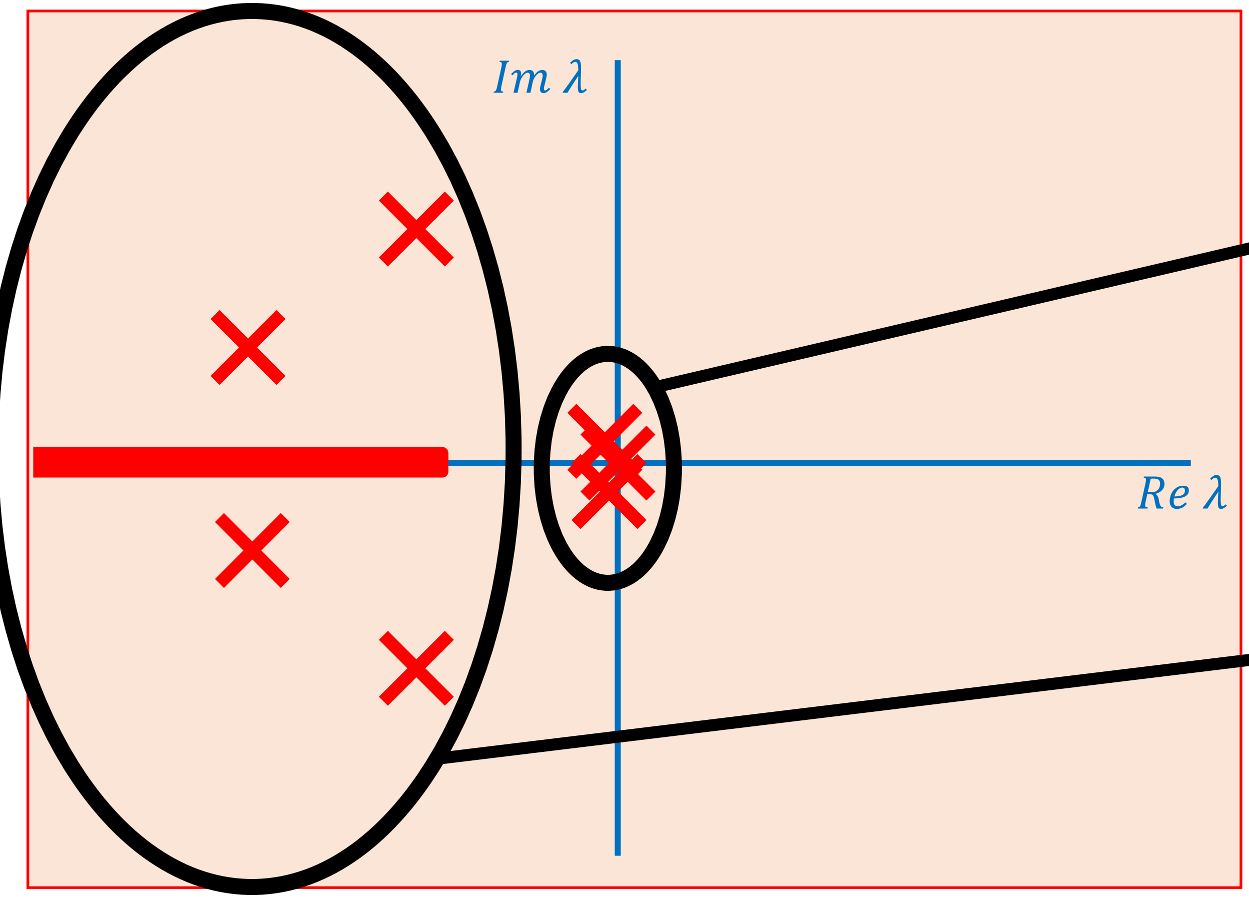




$$y_t = D\,y_{xx} + y(1-y^2) + \mu + \frac{1}{2}\cos(2\pi x) + \sin(3\pi x)$$



Bifurcations



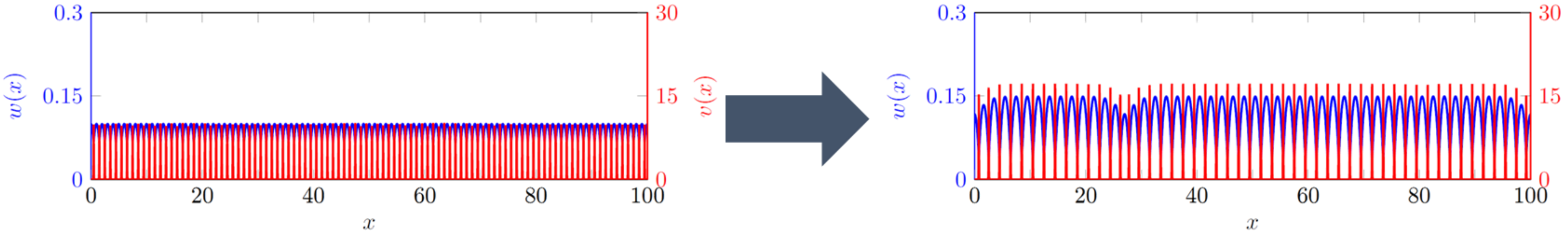
What happens at bifurcation?

1. SLOW Pattern Adaptation

At bifurcation:
→ Location of structure changes

2. FAST Pattern Degradation

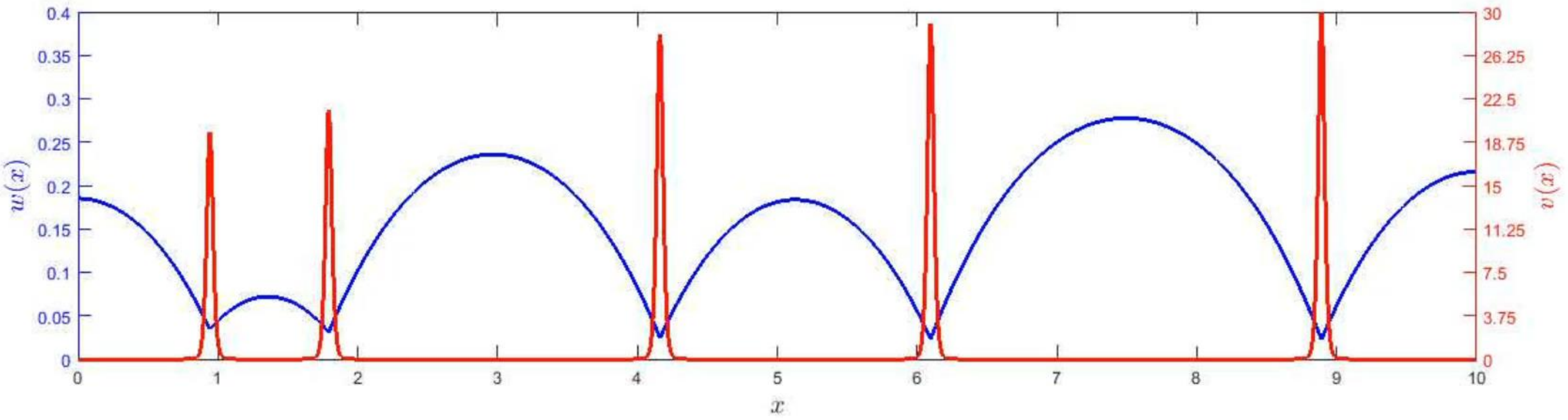
At bifurcation:
→ Structures created or destroyed



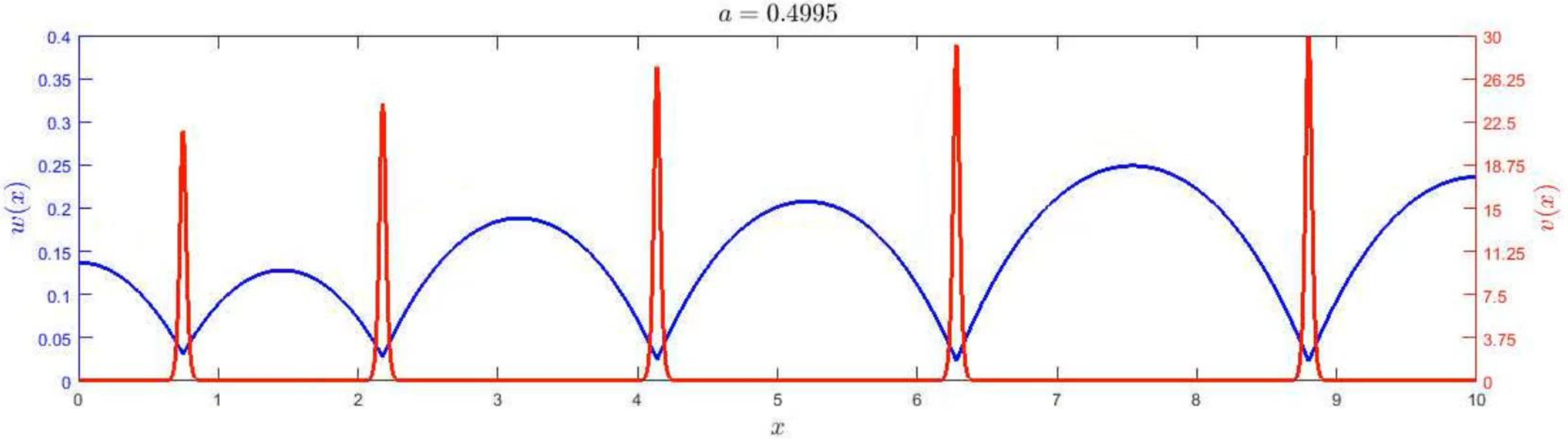
Vegetation patches under climate change

Rate of climate change

FAST



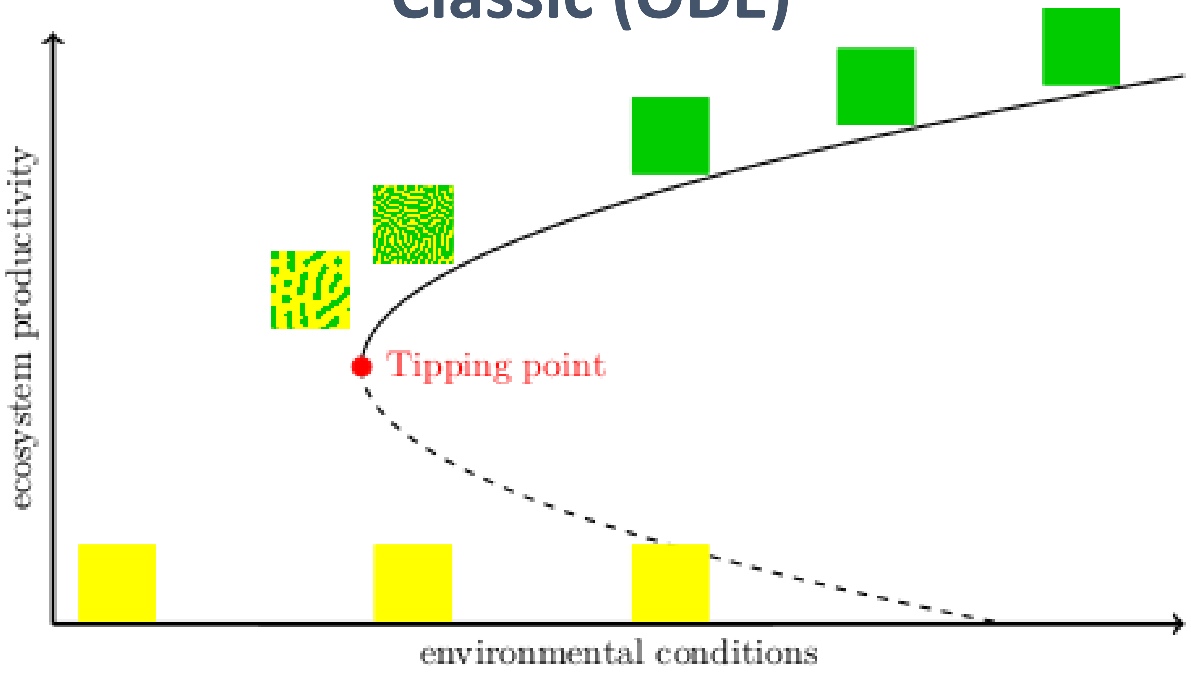
SLOW




Summary
Tipping in Spatially
Extended Systems

What if the system tips?

Classic (ODE)



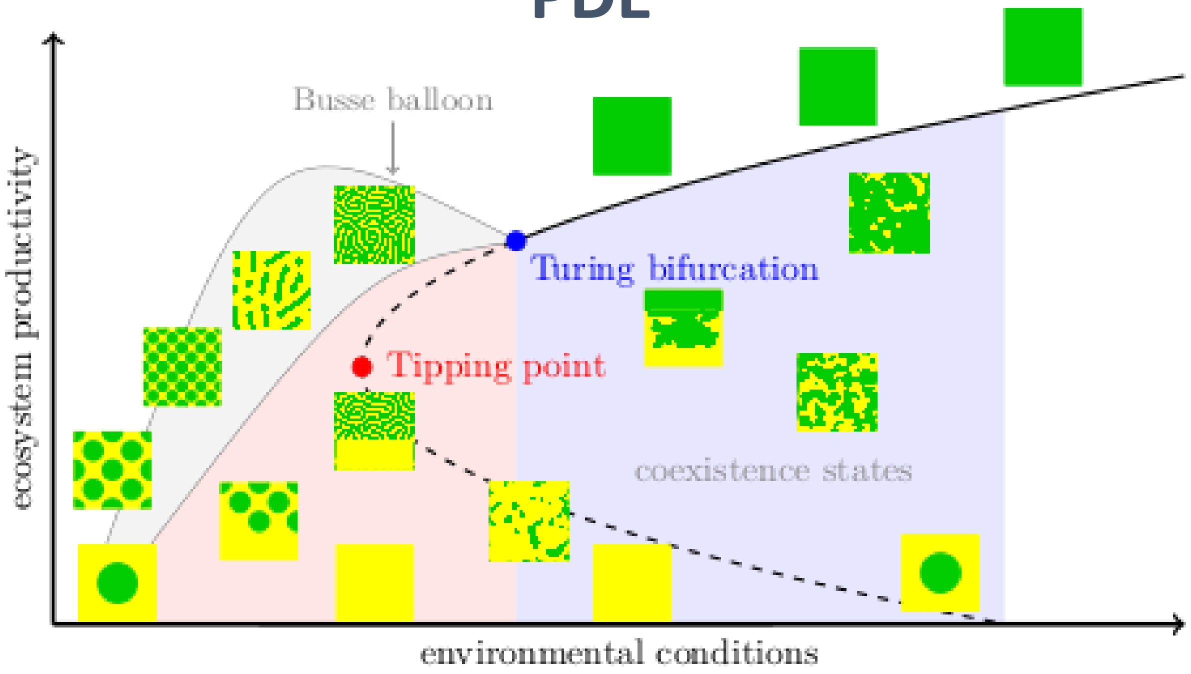
Crossing a Tipping Point:

→ Always full reorganization

Early Warning Signals

signal for WHEN

PDE



Crossing a bifurcation:

Now also possible:

→ Spatial reorganization (Turing patterns)

→ Fragmented tipping (coexistence states)

Early Warning Signals

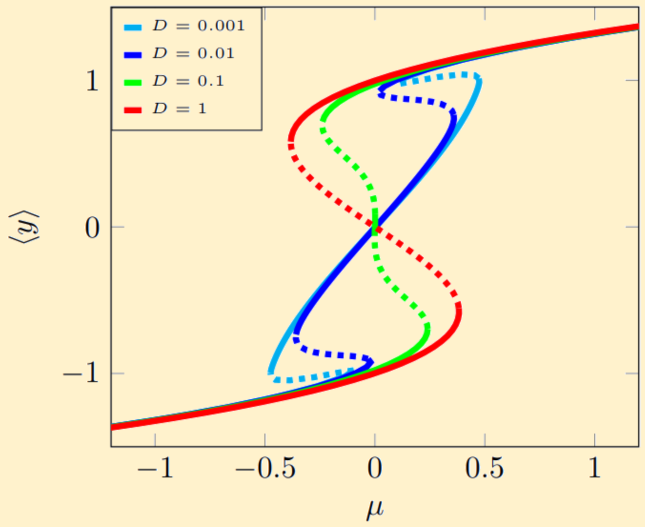
need to signal WHEN & WHAT

Do systems always behave like this? (a.k.a. the small print)

No.

Well-mixed systems

Spatially confined systems



→ Such systems (again) behave like ODEs ←

But even in other systems terms & conditions apply:
System-specific knowledge is required!

Summary

Spatial Patterns:

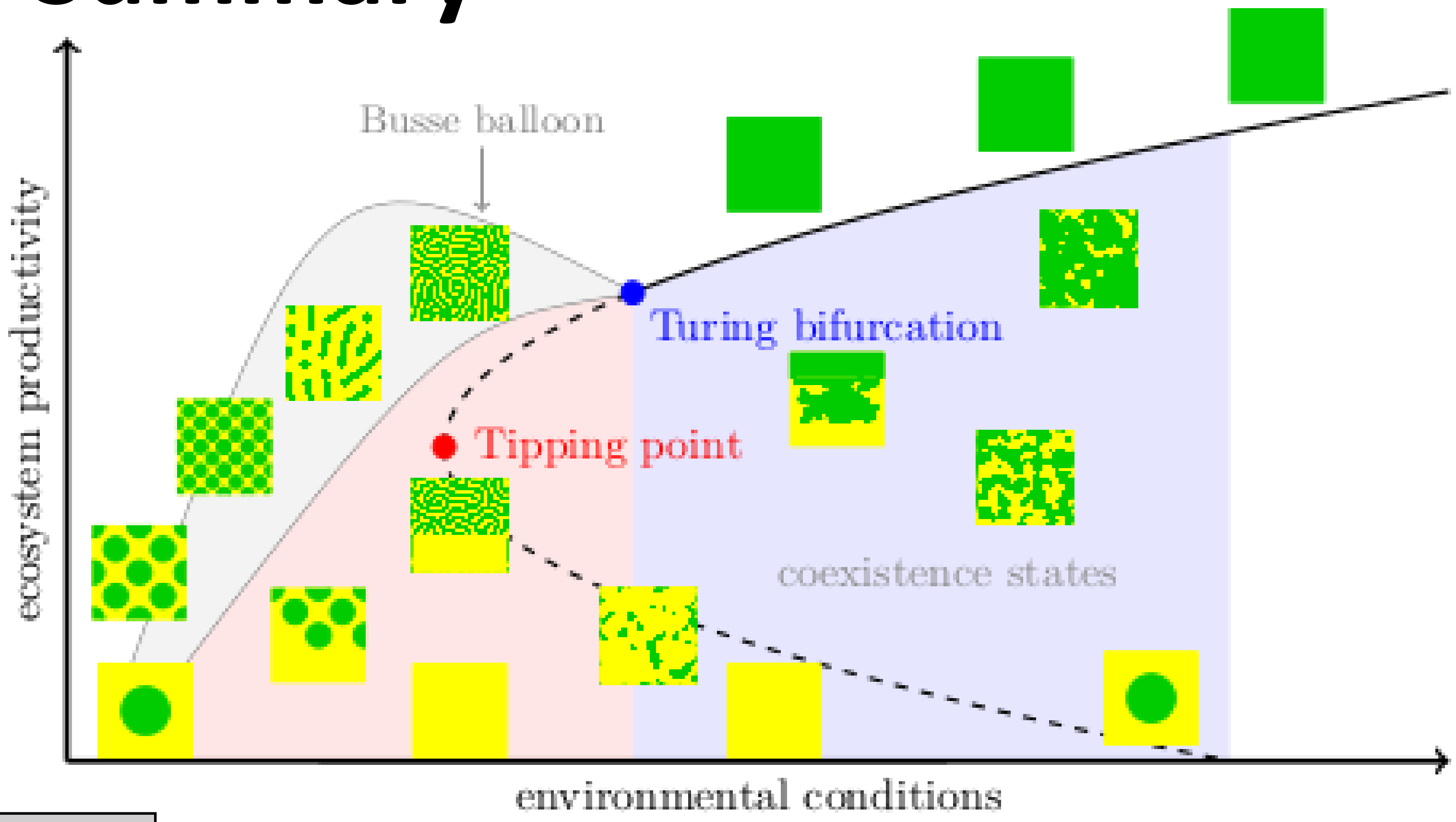
- Turing Patterns
- Coexistence States

Tipping can be more subtle:

- Spatial reorganization
- Fragmented Tipping

Dynamics of Patterns is:

- Slow Pattern Adaptation
- Fast Pattern Degradation



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Bastiaansen, R., Dijkstra, H. A., & von der Heydt, A. S. (2021). Fragmented Tipping in a spatially heterogeneous world. *Environmental Research Letters, 17, 045006*