How spatial heterogeneities affect the behaviour of front solutions

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Robbin Bastiaansen
(r.bastiaansen@uu.nl)

Front Solutions

Bistable (Allen-Cahn/Nagumo) equation:

$$\frac{\partial y}{\partial t} = D\frac{\partial^2 y}{\partial x^2} + y(1-y^2) + \mu$$

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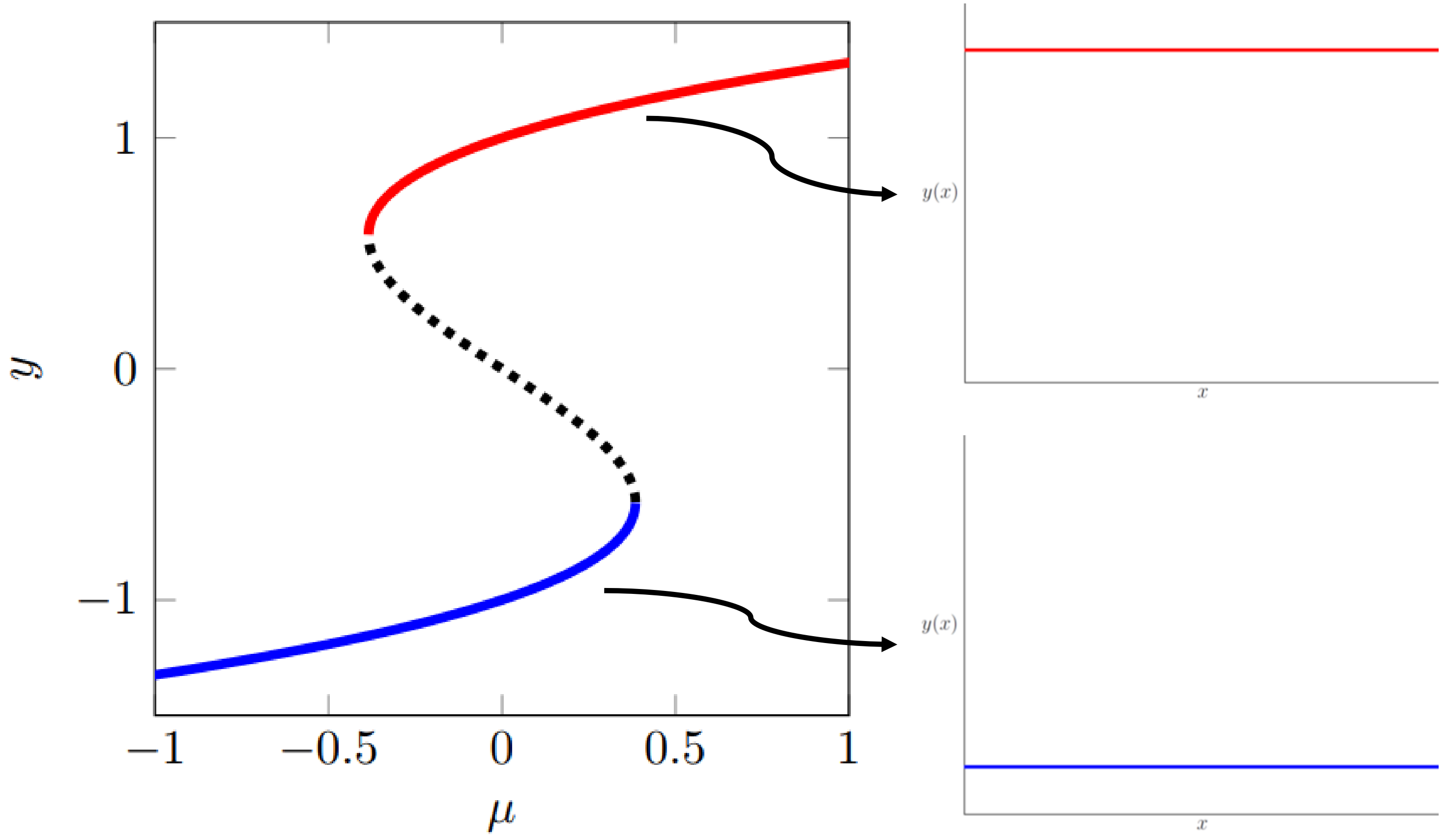
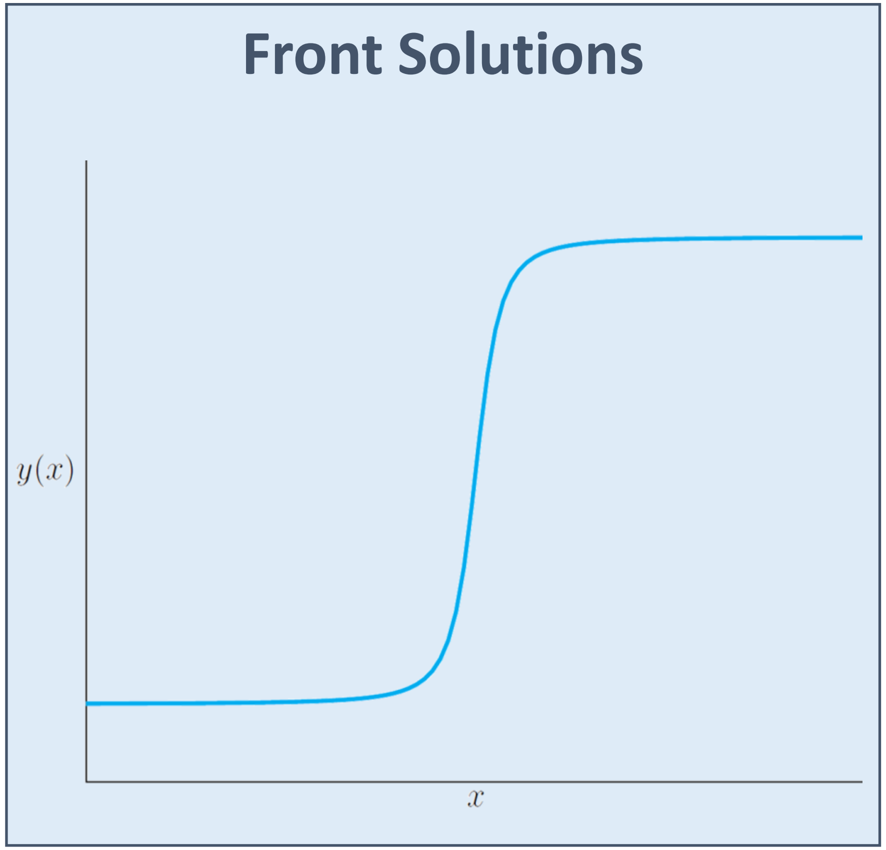
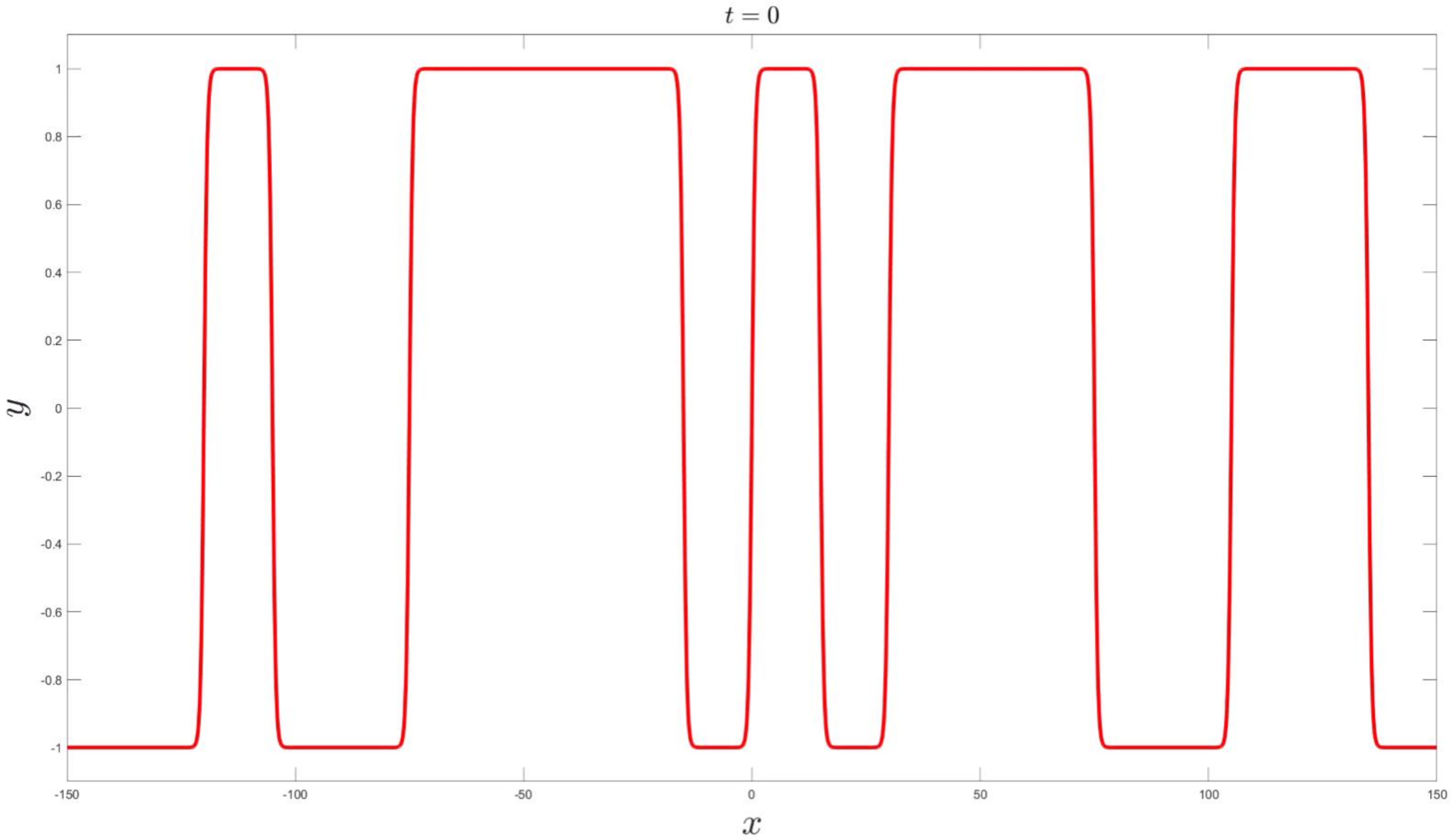




Dynamics of $\frac{\partial y}{\partial t} = D \frac{\partial^2 y}{\partial x^2} + y(1 - y^2) + \mu$

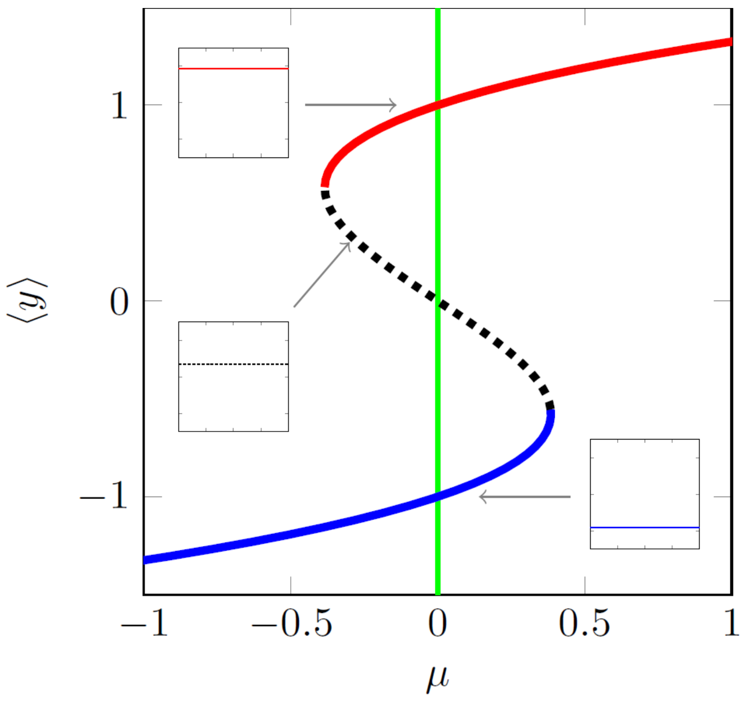


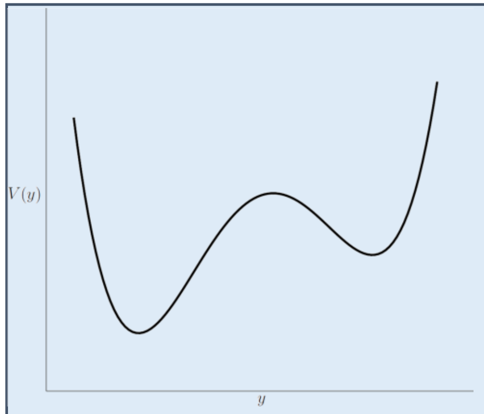
Front Dynamics

$$\frac{\partial y}{\partial t} = D\frac{\partial^2 y}{\partial x^2} + f(y;\mu)$$

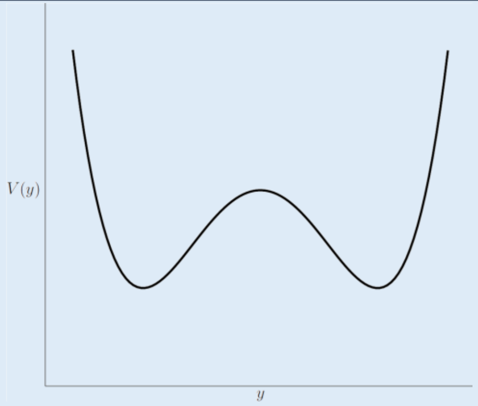
Potential function $V(y;\mu)$:

$$\frac{\partial V}{\partial y}(y;\mu) = -f(y;\mu)$$

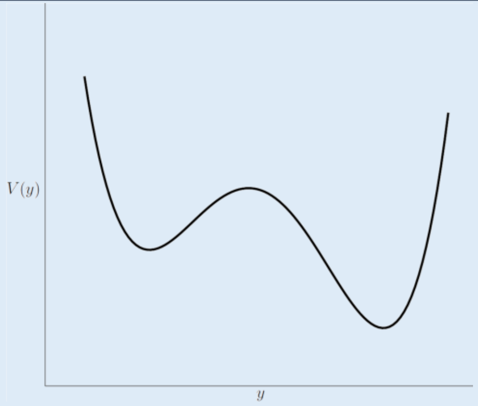




moves right

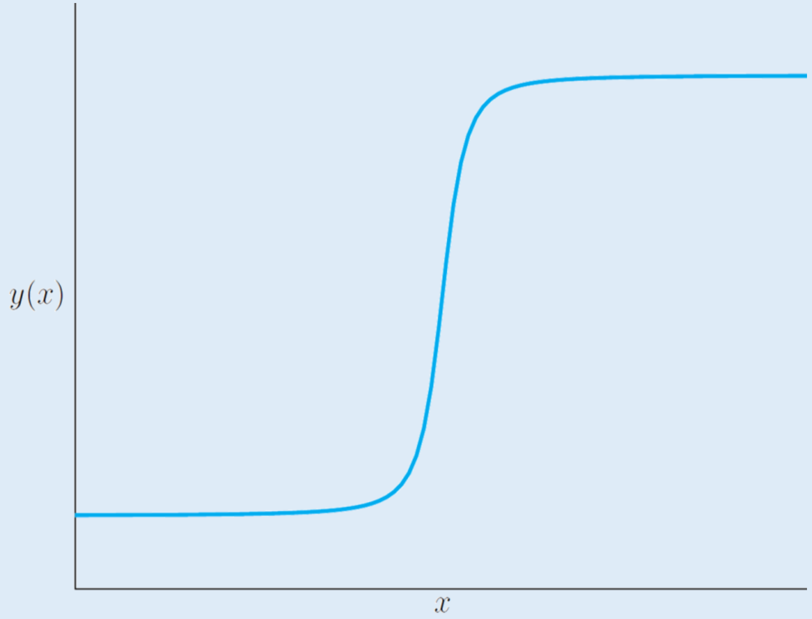


stationary



moves left

Maxwell Point $\mu_{maxwell}$



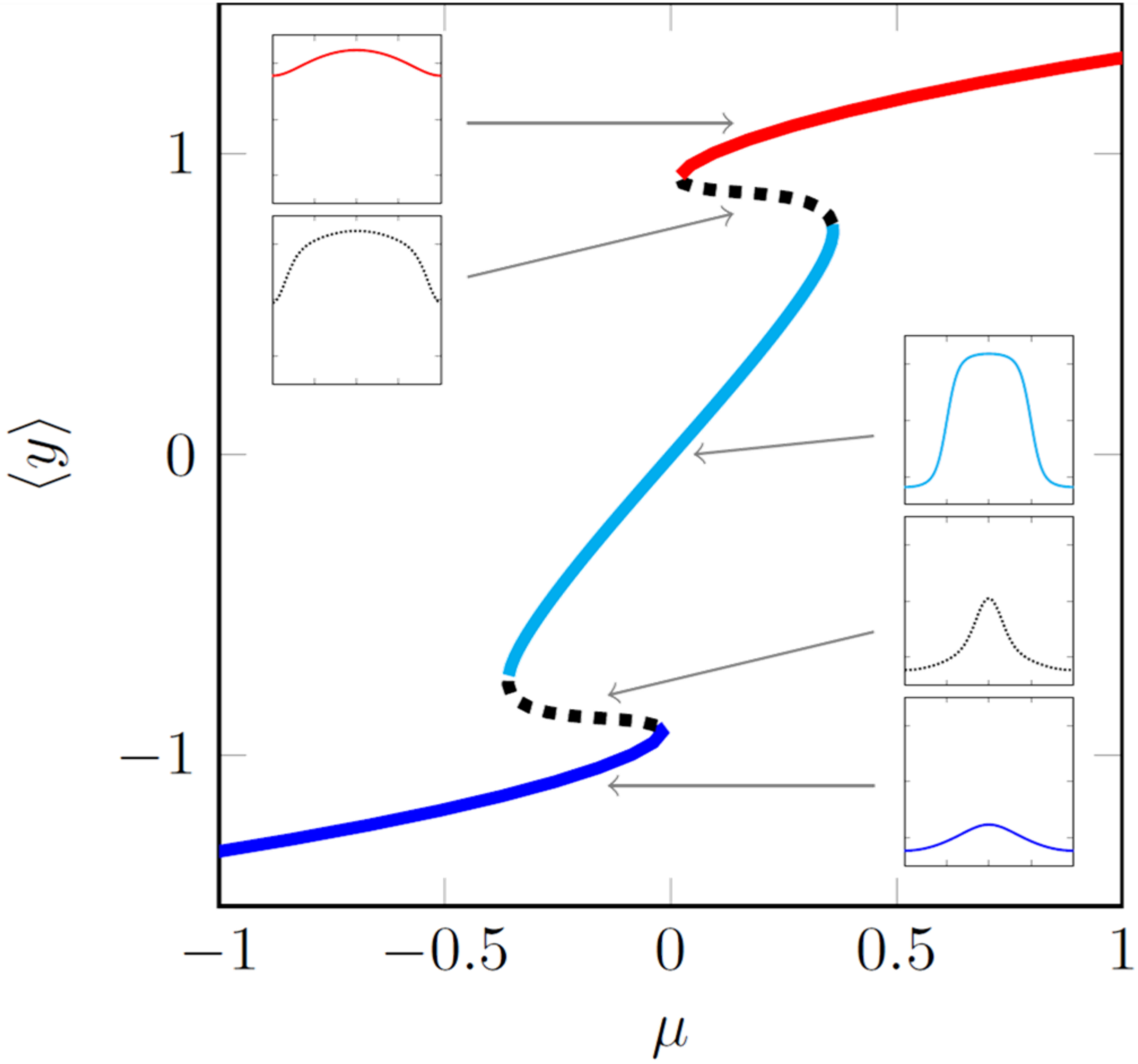
Adding Spatial Heterogeneity

$$\frac{\partial y}{\partial t} = D\frac{\partial^2 y}{\partial x^2} + f(y, x; \mu)$$

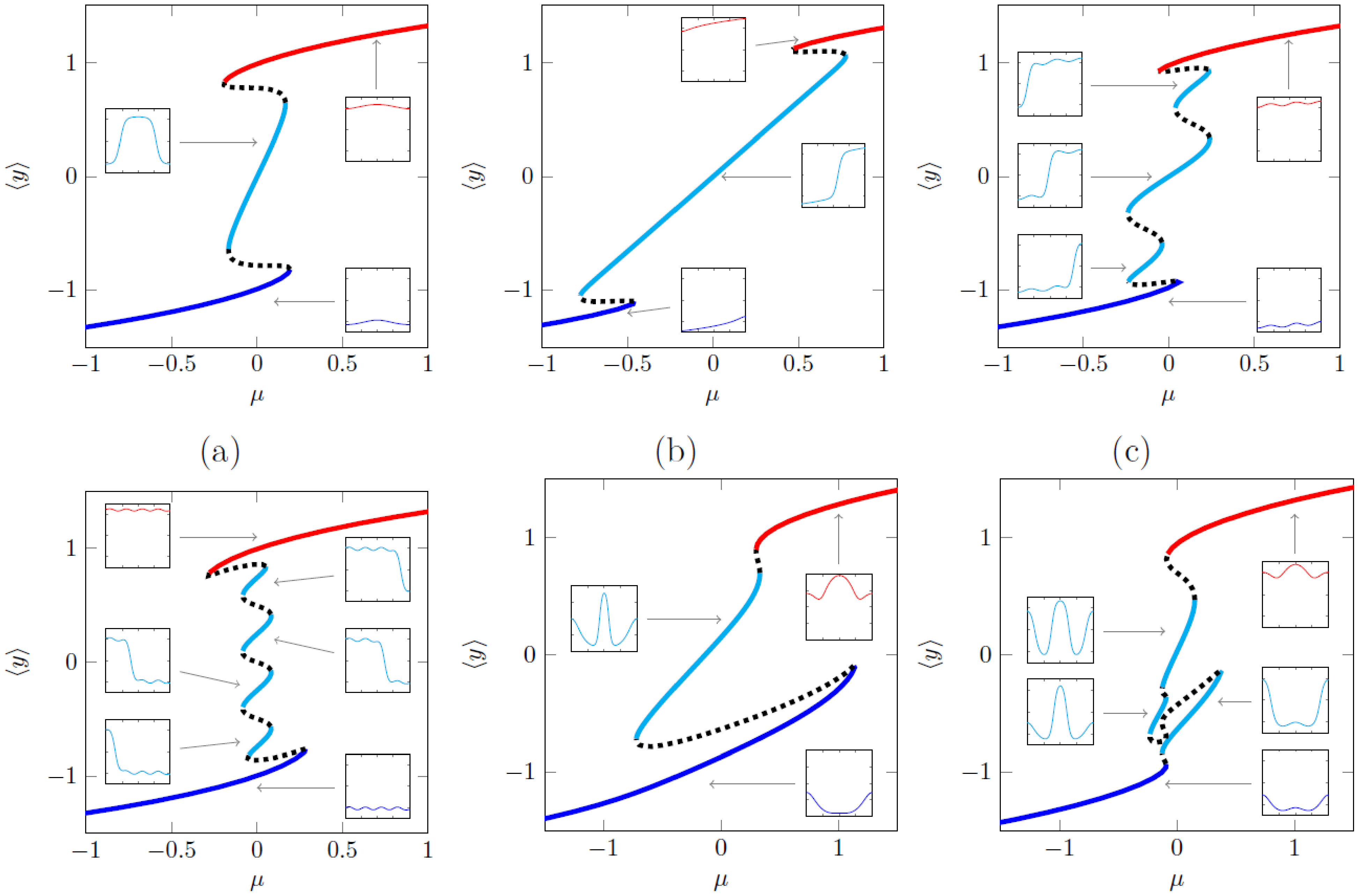
Now, the **local** difference in potentials determines the front movement

New behaviour:

- Multi-fronts can be stationary
- Maxwell point is smeared out



Other Spatial Heterogeneities



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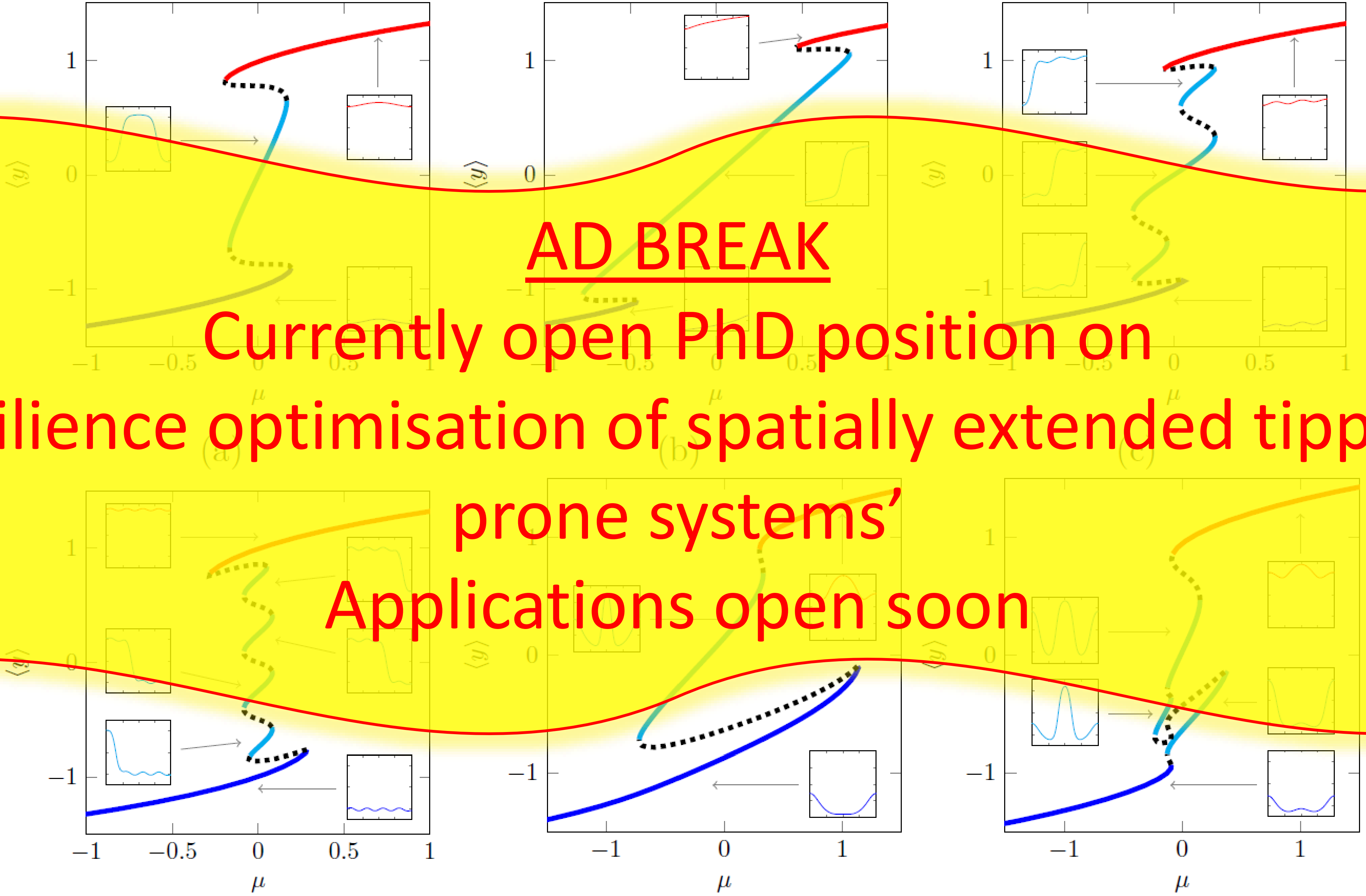
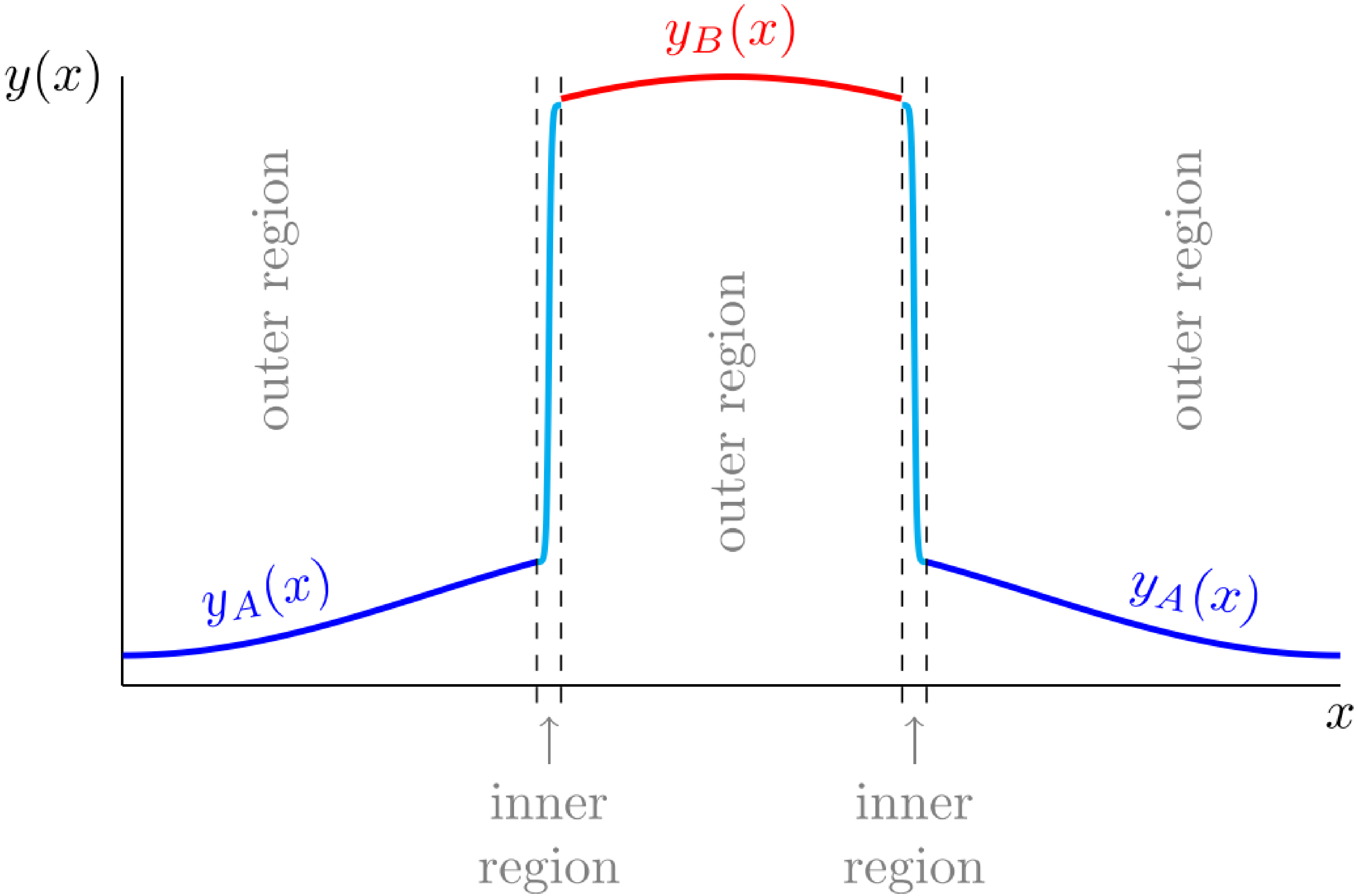


Mathematical Construction of multi-fronts



Small spatial heterogeneity

A spatially inhomogeneous Allen-Cahn equation:

$$\frac{\partial u}{\partial t} = D\,\frac{\partial^2 u}{\partial x^2} + u\left(1 - u^2\right) + \boldsymbol{\varepsilon}\, \boldsymbol{F}\left(\boldsymbol{u}, \frac{\boldsymbol{\partial u}}{\boldsymbol{\partial x}}, \boldsymbol{x}\right)$$

$0 < \varepsilon \ll 1$

Concrete examples:

Spatially periodic forcing

$$F(U,V,x) = \alpha_1\ \cos(kx)\ U + \alpha_2\ \sin(kx)\ V + \alpha_3\ \sin(kx)$$

Spatially localized forcing

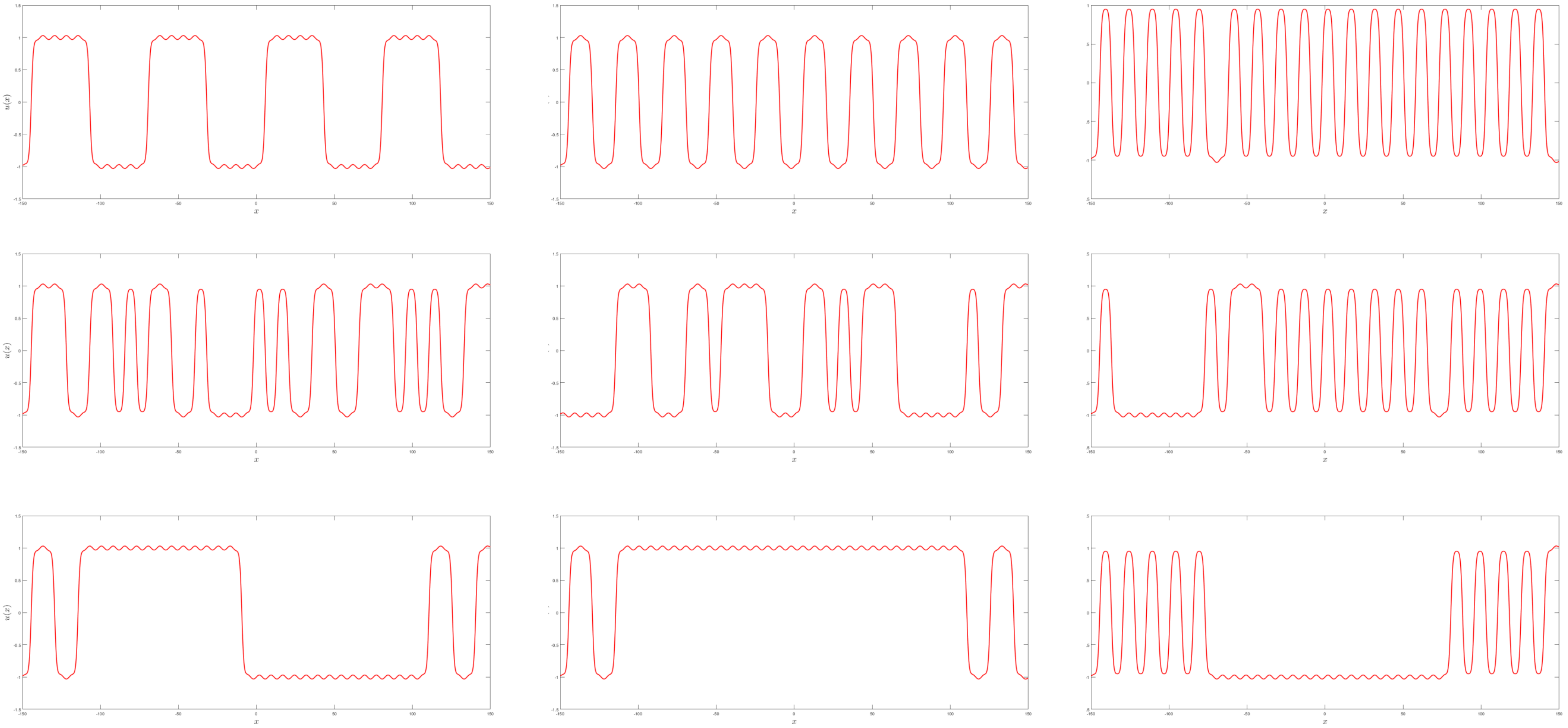
$$F(U,V,x) = H''(x)\,U + H'(x)\,V$$

with

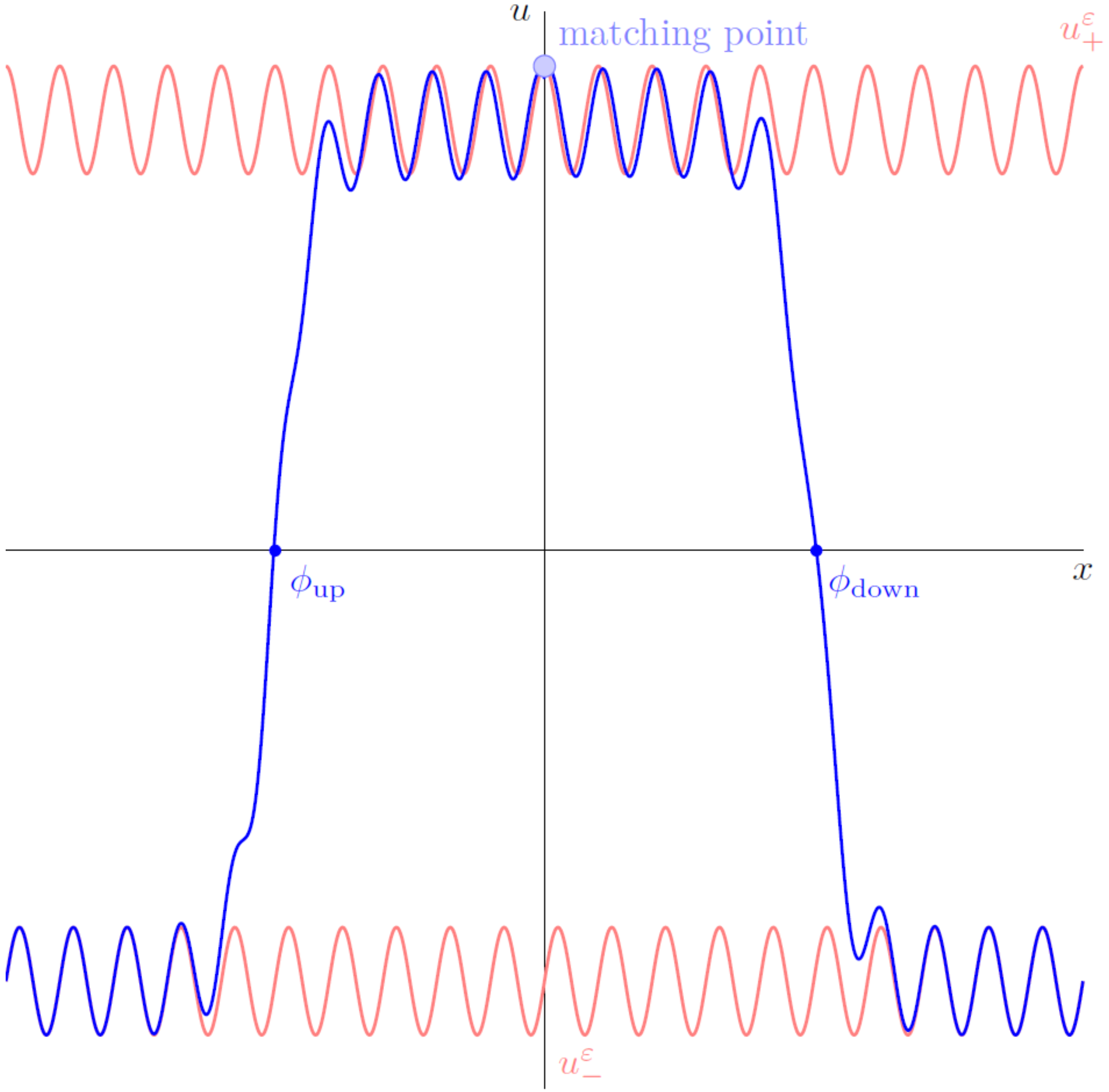
$$H(x) = \operatorname{sech}^2\left(\mu\, x/\sqrt{2}\right)$$

See talk in MS24 by Arjen Doelman TUESDAY 9:45h @ 1D237

Stationary multi-front solutions



Mathematical Construction of 2-front solutions



Construct u in two parts:

For $x < 0$:

$$u_-(x) = u_-^0(x) + \varepsilon\, u_-^1(x) + \mathcal{O}(\varepsilon^2)$$

For $x < 0$

$$u_+(x) = u_+^0(x) + \varepsilon\, u_+^1(x) + \mathcal{O}(\varepsilon^2)$$

Match solutions at $x = 0$ for first two terms of expansion simultaneously at the $\mathcal{O}(\sqrt{\varepsilon})$-level

Similar technique works for construction of N-front solutions and their slow time evolution

N-front dynamics in spatially heterogeneous AC

Let $\phi_1, \ldots, \phi_N$ denote the location/phases of the fronts

They evolve (at leading order) according to a N-dimensional ODE

$$\frac{d\phi_j}{dt} = C\left[-\varepsilon\mathcal{R}_j(\phi_j) + 16\left(e^{-\sqrt{2}\Delta\phi_j} - e^{-\sqrt{2}\Delta\phi_{j-1}}\right)\right]$$

Movement due to spatial heterogeneity

Front interactions

where

$\Delta\phi_j$ denotes the distance between front j and $j+1$ (infinite if it does not exist)

and

$\mathcal{R}_j(\phi)$ the Melnikov integral

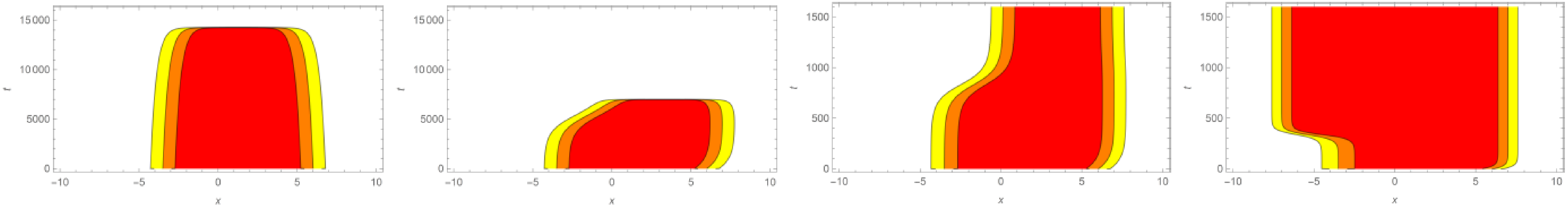
$$\mathcal{R}_j(\phi) = \int_{-\infty}^{+\infty} F\{u_j^0(x), \partial_x u_j^0(x), x+\phi\}\partial_x u_j^0(x)\ dx$$

with u_j^0 the appropriate front solution of the unperturbed system

2-front dynamics in spatially periodic forced AC

Let ϕ_1, ϕ_2 denote the location/phases of the fronts
They evolve according to a 2-dimensional ODE

$$\frac{d\phi_1}{dt} = \tilde{C}\left[-\varepsilon(A+B)\sin(k\phi_1) + e^{-\sqrt{2}(\phi_2-\phi_1)}\right]$$
$$\frac{d\phi_2}{dt} = \tilde{C}\left[-\varepsilon(A-B)\sin(k\phi_2) - e^{-\sqrt{2}(\phi_2-\phi_1)}\right]$$

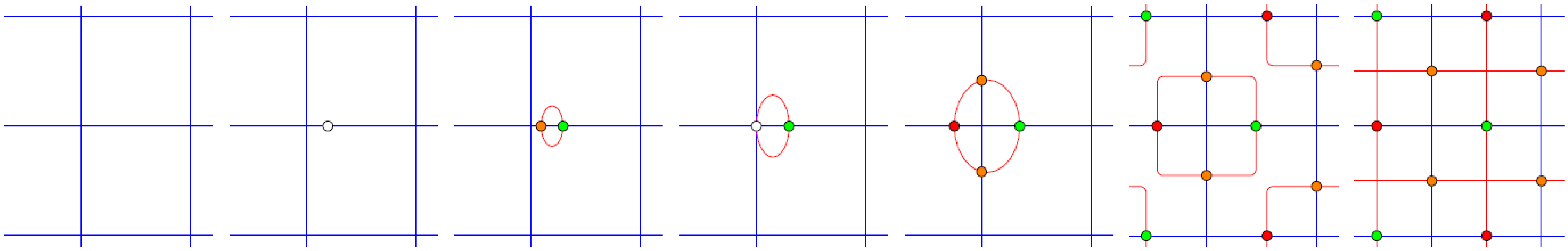


$\varepsilon = 0$ $\varepsilon = 0.01$ $\varepsilon = 0.1$ $\varepsilon = 0.4$

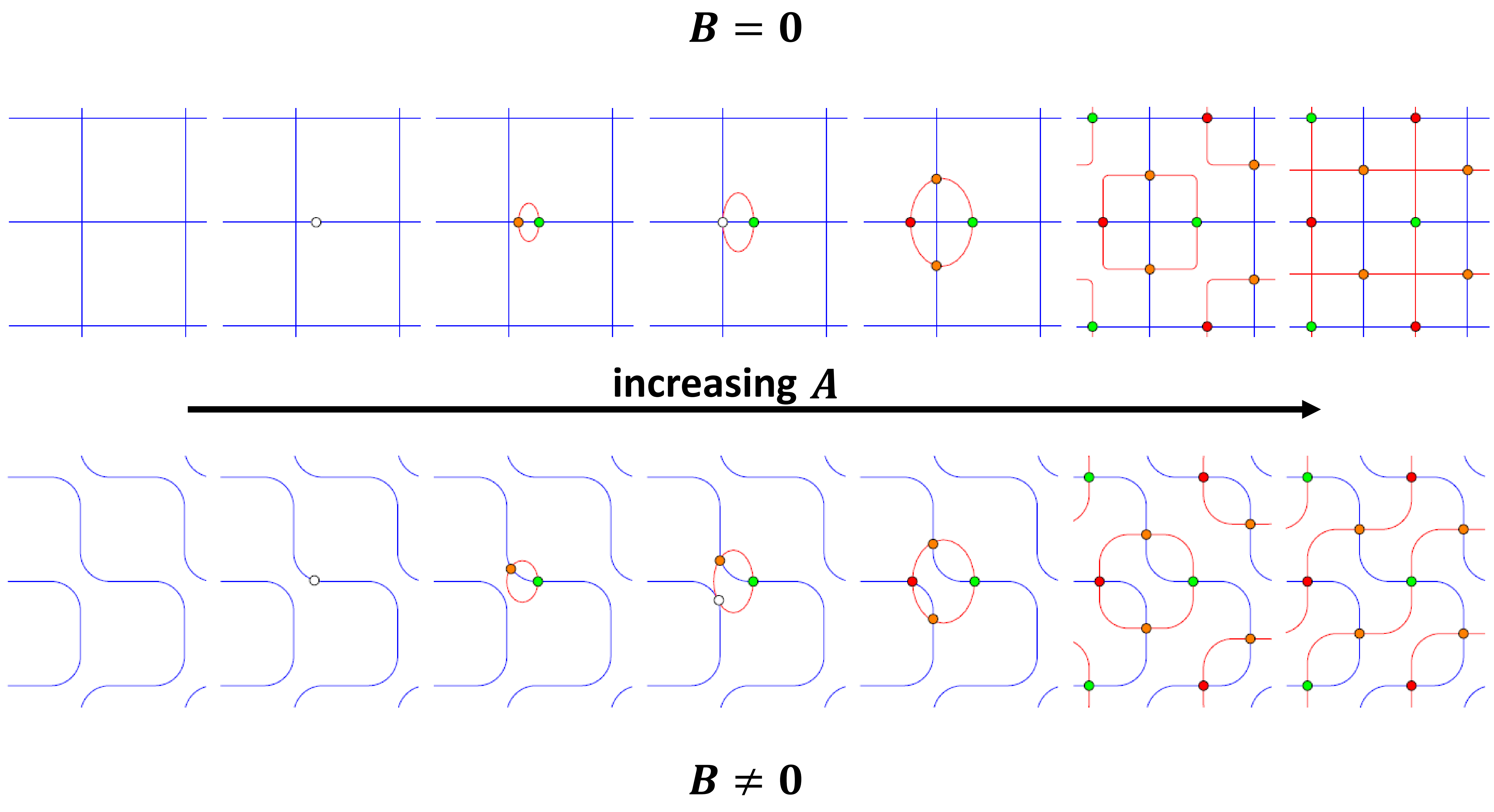
2-front dynamics in spatially periodic forced AC

Let $D := \phi_2 - \phi_1$ and $S := \phi_2 + \phi_1$ and take $B = 0$ for simplicity.
After a bunch of scalings:

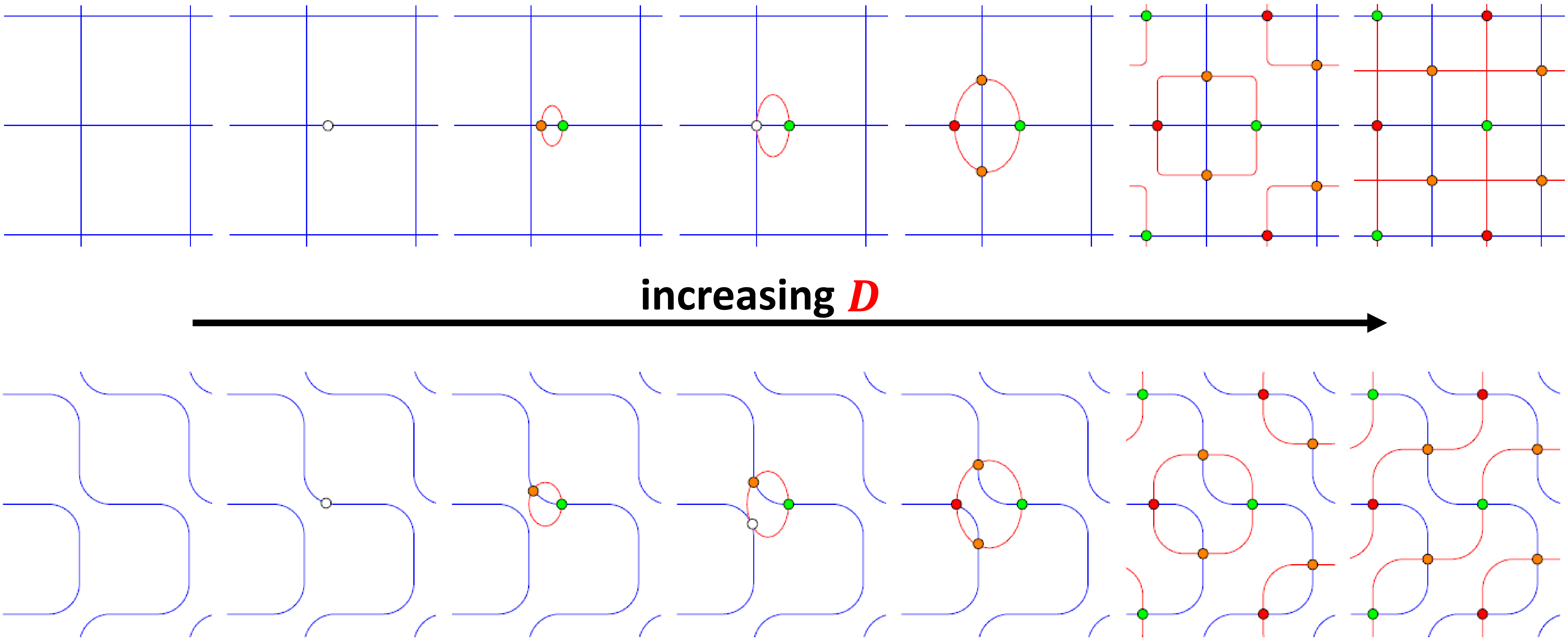
$$\frac{dD}{dt} = -A\ \sin(R + kD/2)\cos(kS/2) - e^{-\sqrt{2}D}$$
$$\frac{dS}{dt} = -A\ \cos(R + kD/2)\sin(kS/2)$$



increasing A


$B = 0$
increasing A
$B \neq 0$

$\boldsymbol{B} = 0$



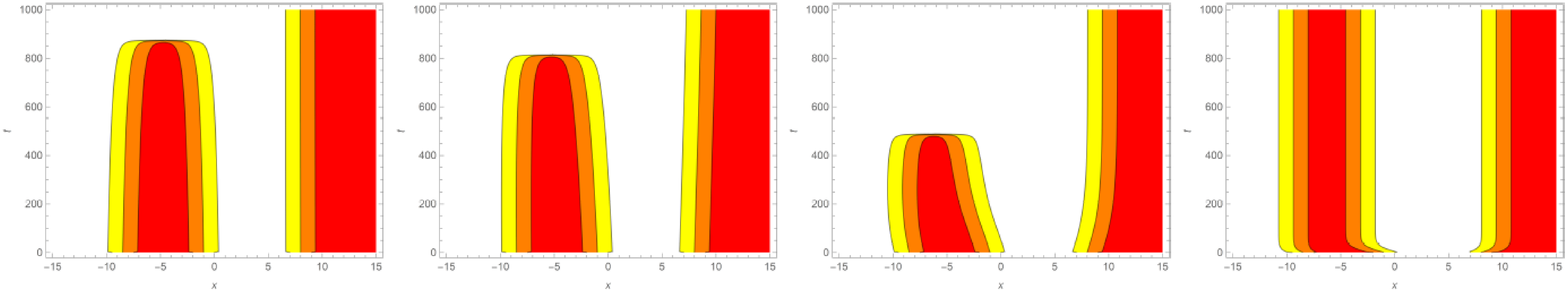
$\boldsymbol{B} \neq 0$

Stable stationary N-fronts in spatially heterogeneous AC

<u>THEOREM</u>

(*spatially periodically forced Allen-Cahn equations*)

Any N-front configuration
with fronts located sufficiently far apart
evolves to a (stable) stationary N-front configuration.



$\varepsilon = 0$ $\varepsilon = 0.001$ $\varepsilon = 0.01$ $\varepsilon = 01$

Summary

<u>Spatially heterogeneous parameters lead to:</u>

- existence of (many different) stationary multi-front solutions
- different bifurcation diagrams due to smearing of Maxwell point
- new front dynamics:
  - combination of front interactions and spatial heterogeneity effects
  - e.g. coarsening dynamics in Allen-Cahn can be stopped!

slides at bastiaansen.github.io

<u>THANKS TO:</u>

Tasso Kaper, Arjen Doelman, Henk Dijkstra, Anna von der Heydt

Bastiaansen, R., Dijkstra, H. A., & von der Heydt, A. S. (2021). Fragmented Tipping in a spatially heterogeneous world. *Environmental Research Letters, 17, 045006*

Bastiaansen, R., Doelman, A., & Kaper, T.J. Multi-front dynamics in spatially inhomogeneous Allen-Cahn Equations. *In progress*