Behaviour of Spatial Patterns

2024-05-23, AXIOMA 2024

Robbin Bastiaansen (r.bastiaansen@uu.nl)


Greenland Sea
Barents Sea
Norwegian Sea
Iceland
Sweden
Finland
Norway
Oslo
Stockholm
Baltic Sea
Estonia
Latvia
Lithuania
North Sea
Denmark
Copenhagen
Belarus
United Kingdom
Isle of Man
Ireland
Dublin
Manchester
Birmingham
London
Amsterdam
Netherlands
Hamburg
Berlin
Poland
Warsaw
Ukraine
Germany
Belgium
Luxembourg
Prague
Czechia
Kraków
Slovakia
Moldova
Vienna
Budapest
Paris
Munich
Austria
Hungary
Romania
Switzerland
Slovenia
France
Croatia
Belgrade Београд
Serbia
Lyon
Milan
Sarajevo
Image Landsat / Copernicus
Kosovo
Bulgaria
Florence
Monaco
Italy
Podgorica Подгорица
Google Earth
55°25'06.59" N 5°29'24.24" E elev -45 m eye alt 3210.21 km

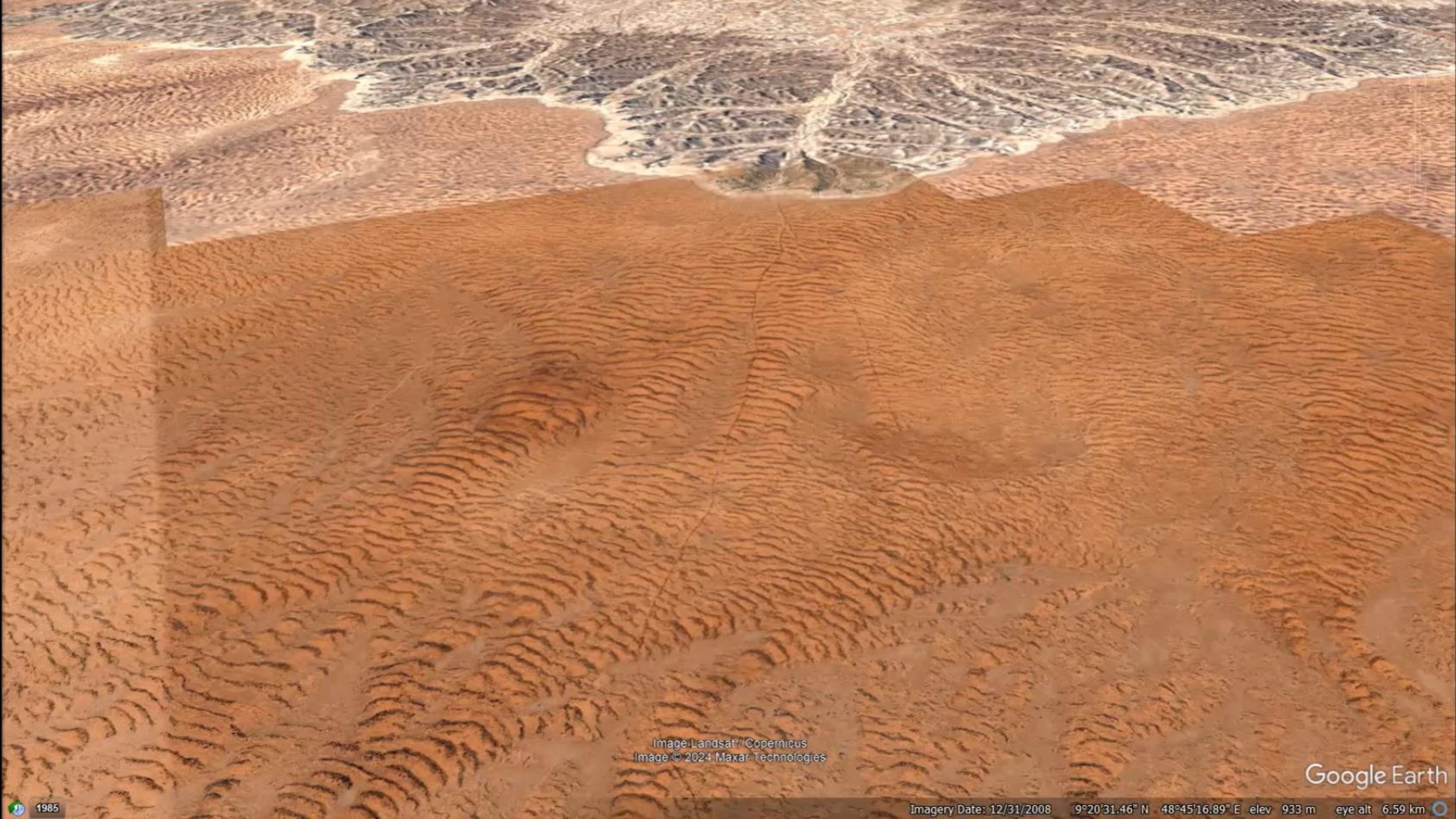


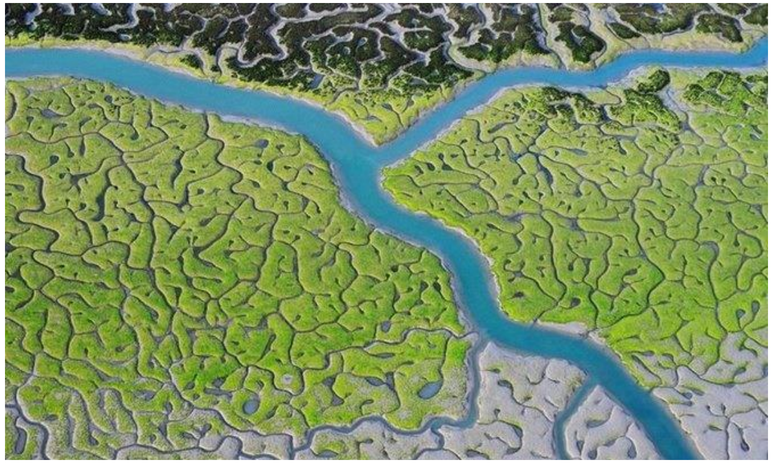
Image Landsat / Copernicus
Image © 2024 Maxar Technologies
Google Earth
1985
Imagery Date: 12/31/2008
9°20'31.46" N 48°45'16.89" E elev 933 m eye alt 6.59 km


Image © 2024 Maxar Technologies
Google Earth
1985
1°19'47.14" S 13°52'48.66" E elev 594 m eye alt 1.73 km

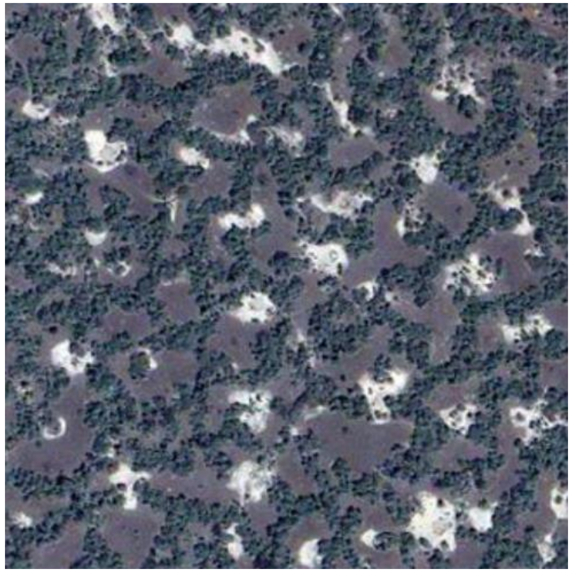
Examples of spatial patterning – ecosystems

mussel beds

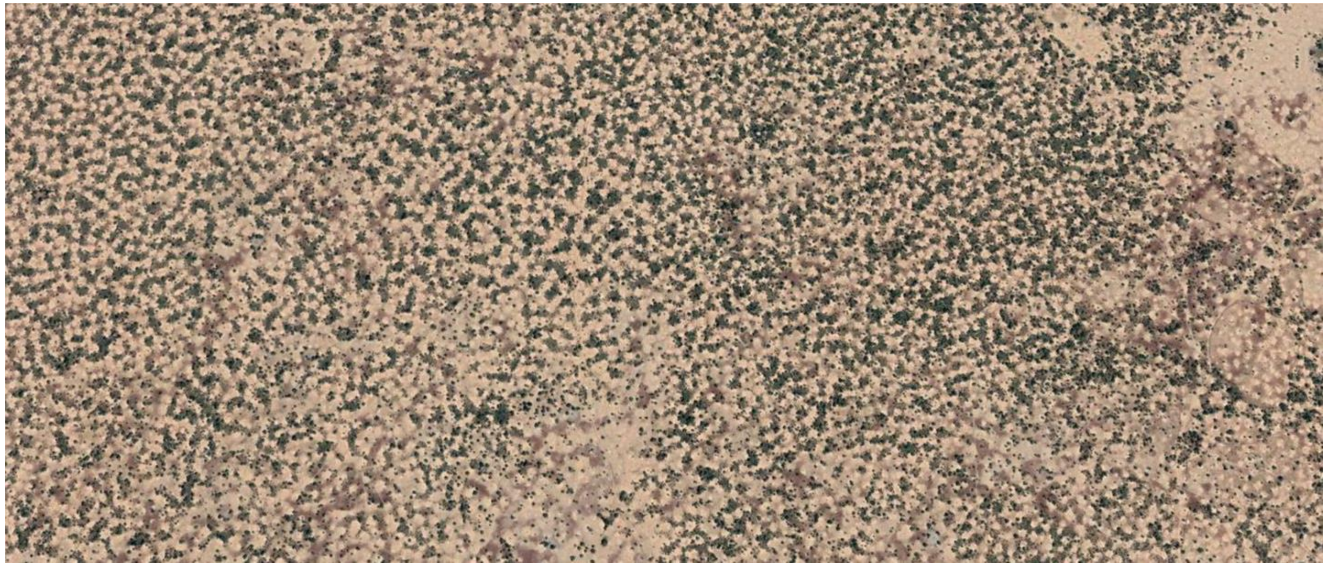
vegetation in coastal systems



marsh formation



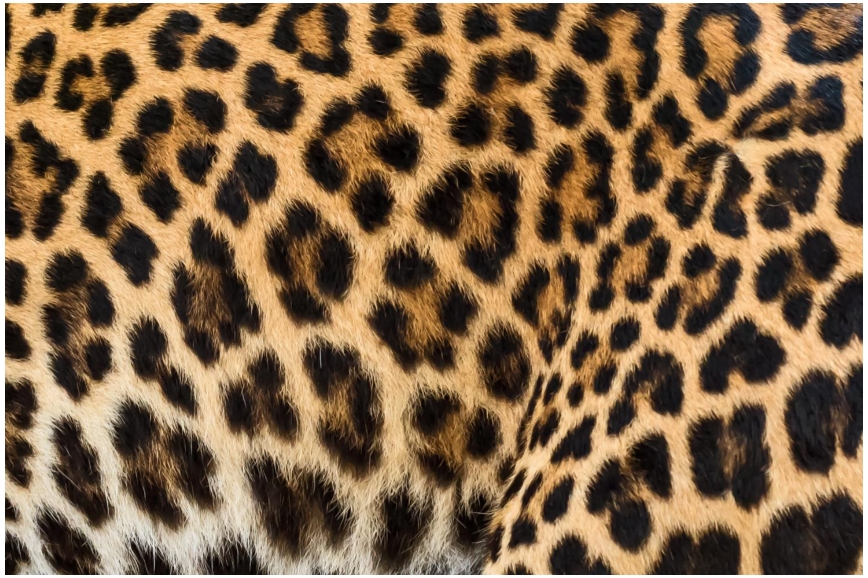
savannas



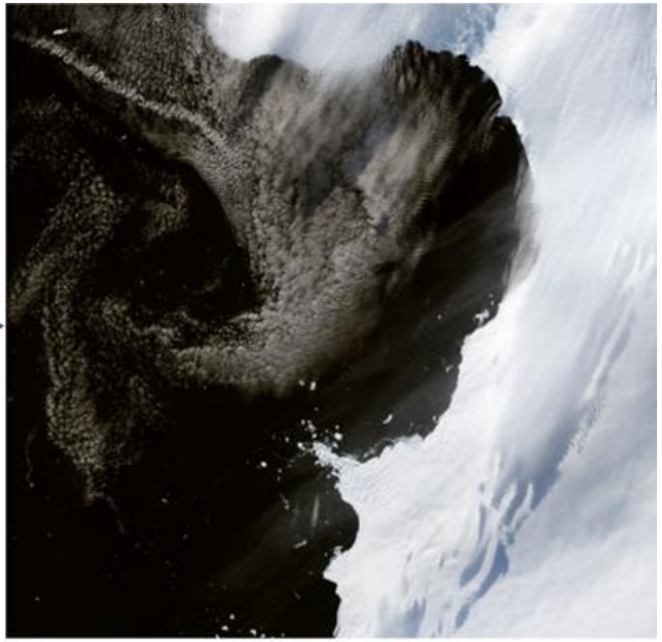
drylands

tropical forests

Examples of spatial patterning – animals



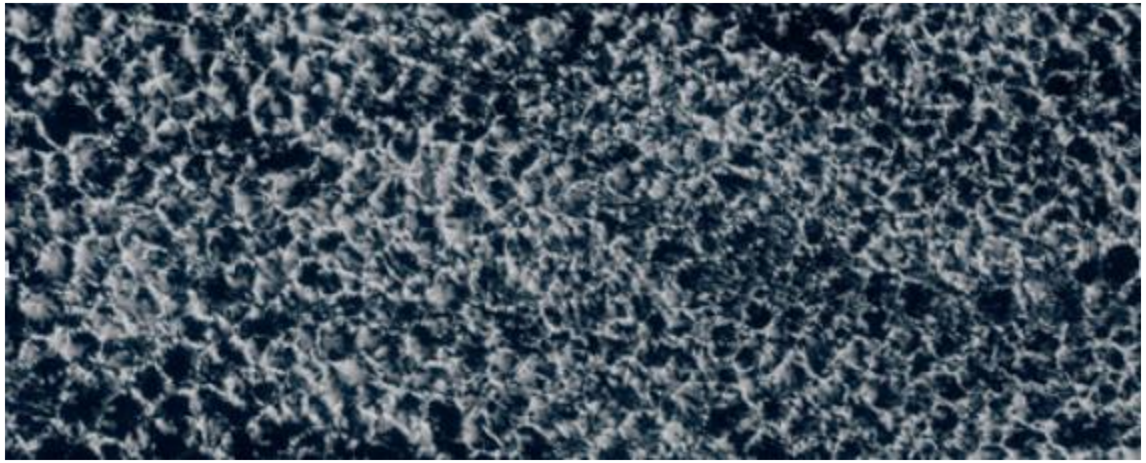
Examples of spatial patterning – climate



sea-ice & water at Eltanin Bay
[NASA's Earth observatory]

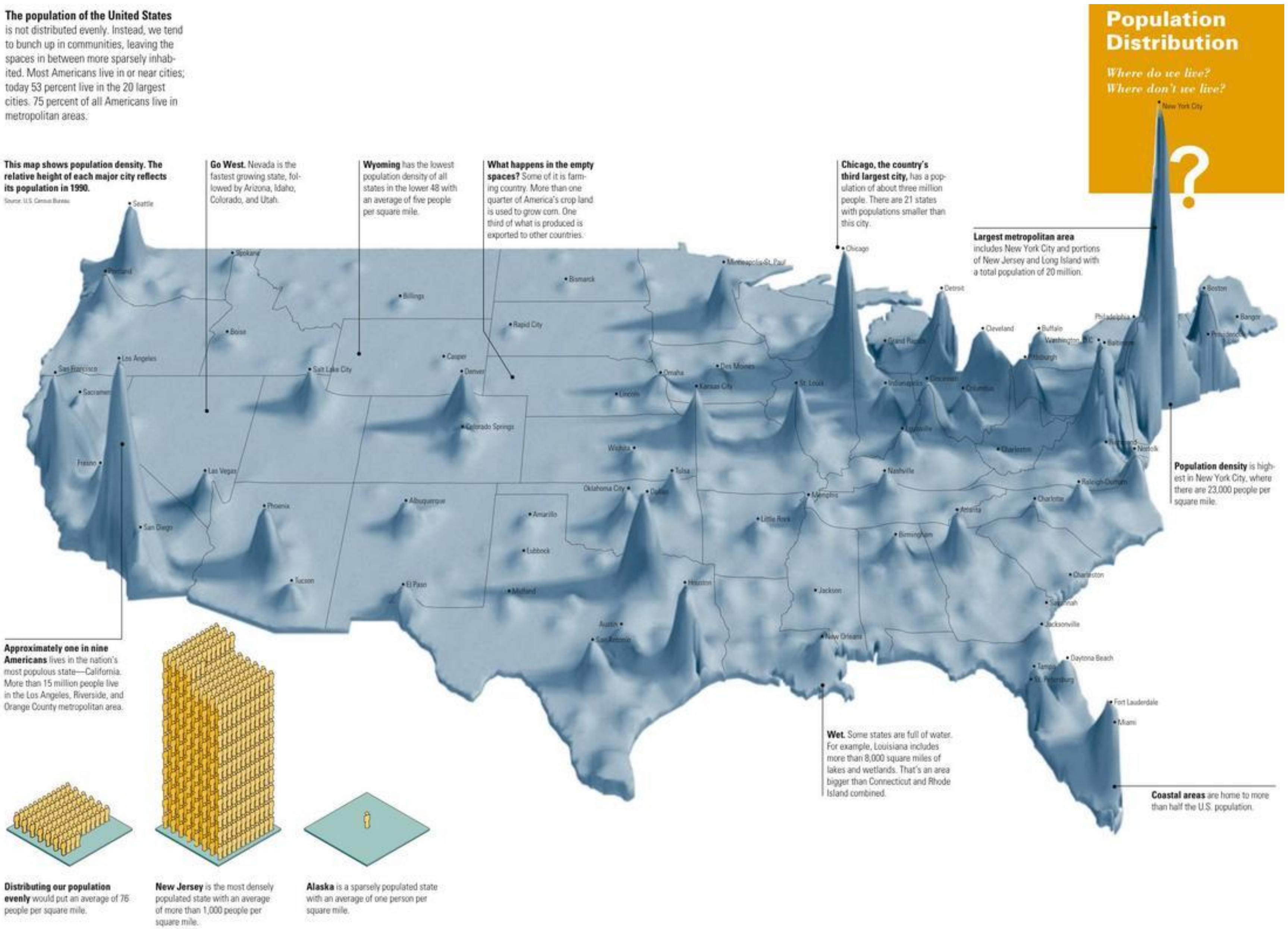
melt ponds

sand dunes



clouds

Examples of spatial patterning – sociology



Examples of spatial patterning – physics

Examples of spatial patterning – physics

Examples of spatial patterning – physics/mussels

Programme

0) Lengthy introduction with patterns in all sort of systems

1) How do spatial patterns emerge?
   - Importance for climate tipping points?

2) How do spatial patterns behave?
   - Why ice cream does not stay soft?

algae bloom
in Lake St. Clair
[NASA's Earth observatory]

A bit about myself

Since 2022: Assistant professor @ Utrecht University

Joint appointment:

- Mathematical Institute, Mathematical Department
- Institute for Marine and Atmospheric Research Utrecht (IMAU), Physics Department

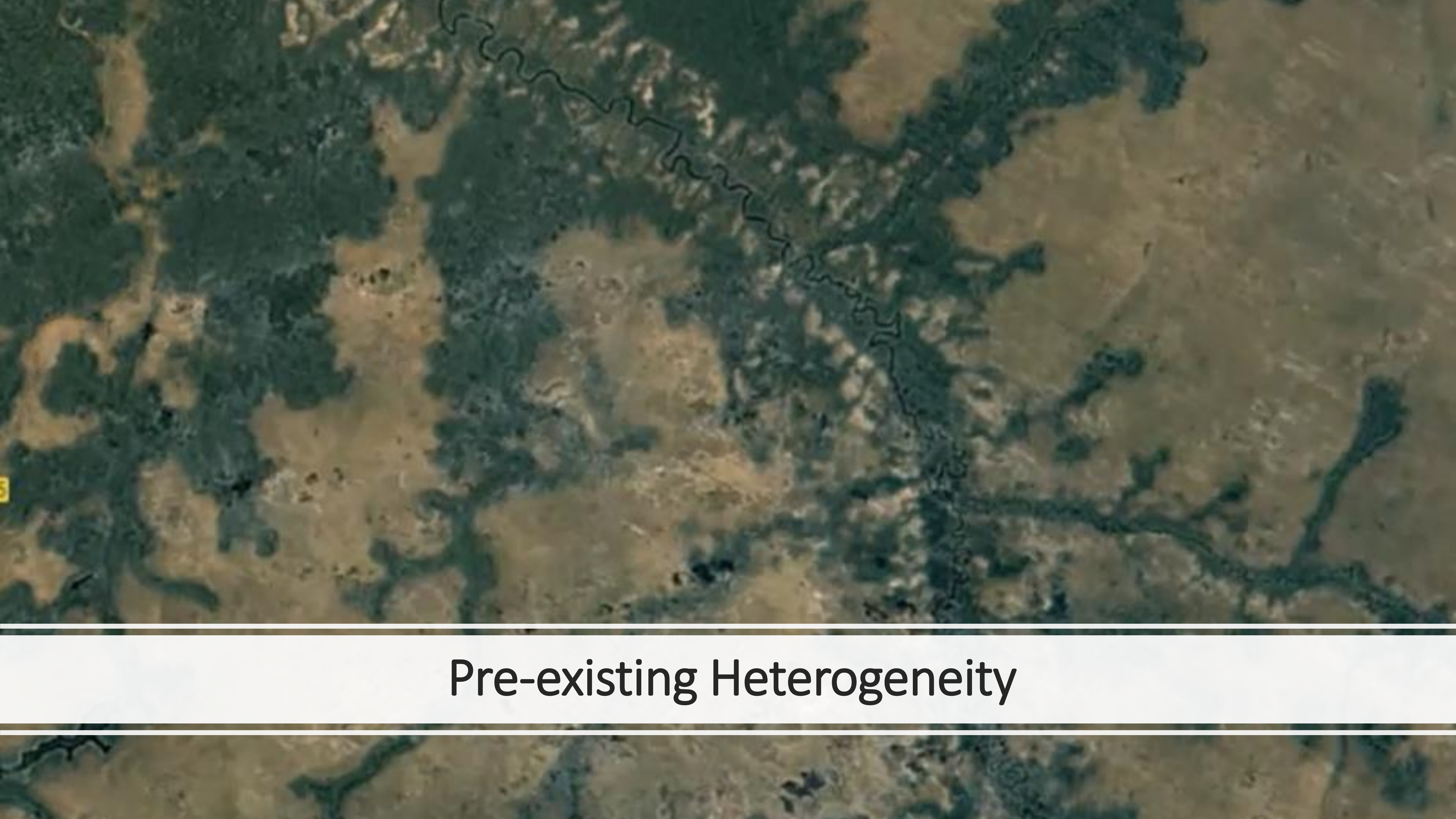
Research focusses on climate and ecosystem responses in context of climatic changes using techniques from dynamical systems theory.

<u>EARLIER</u>:
2020-2022: Postdoc @ IMAU on climate response and tipping points
2015-2019: PhD @ Leiden University on vegetation patterns and desertification

Part 1: Emergence of Spatial Patterns

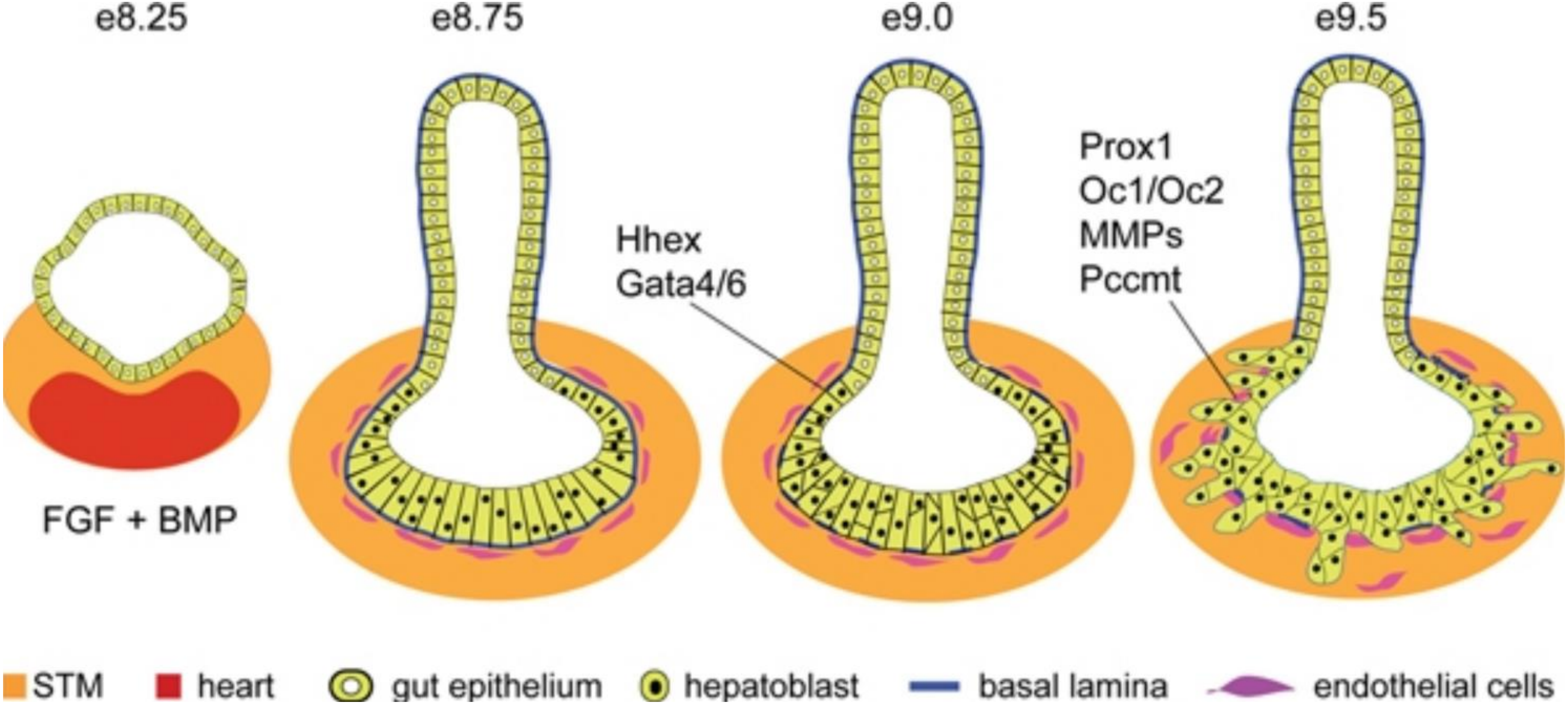
image courtesy: National Science Foundation

Active Creation

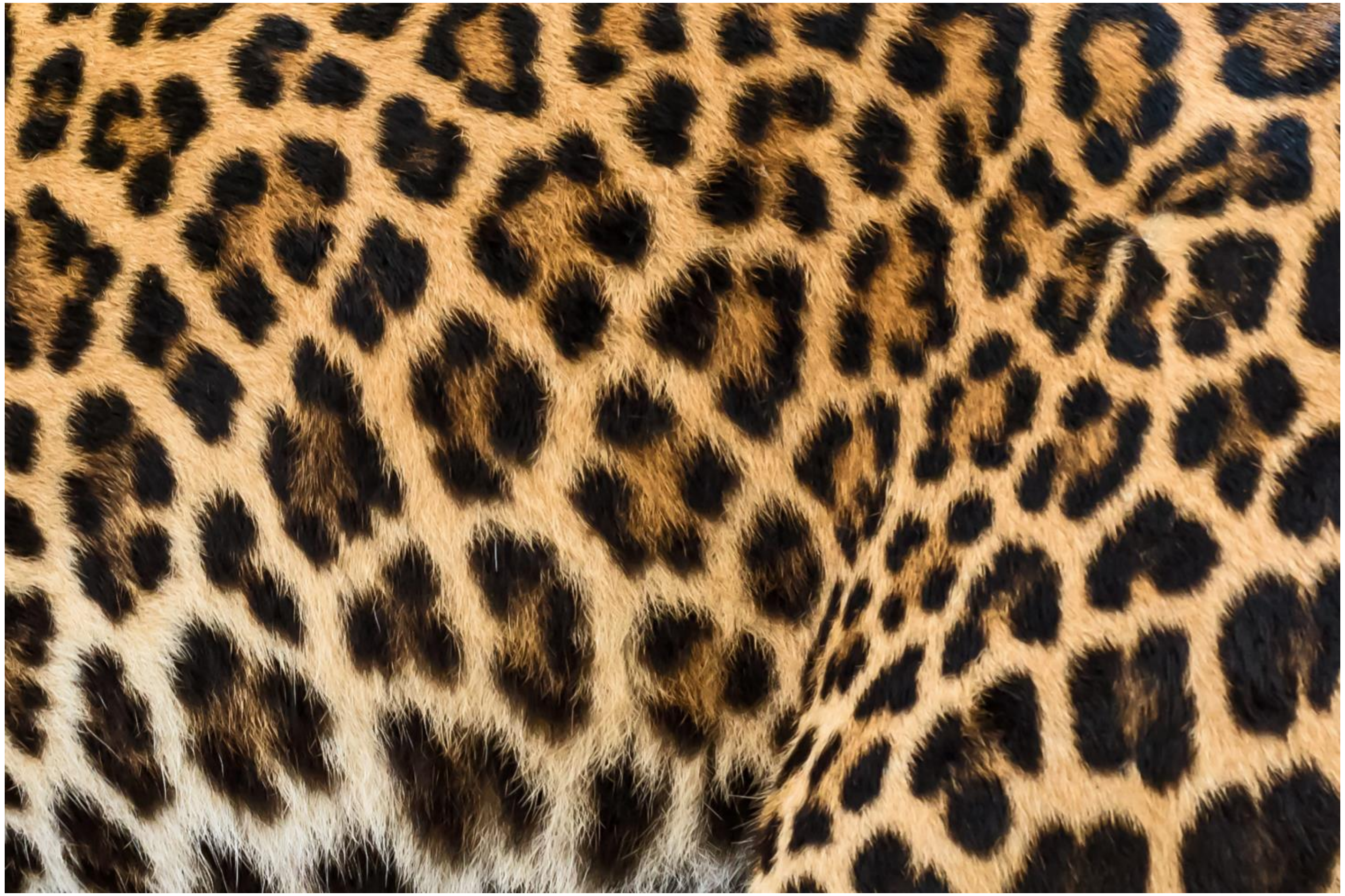


Pre-existing Heterogeneity

Self-Organisation



Turing Patterns

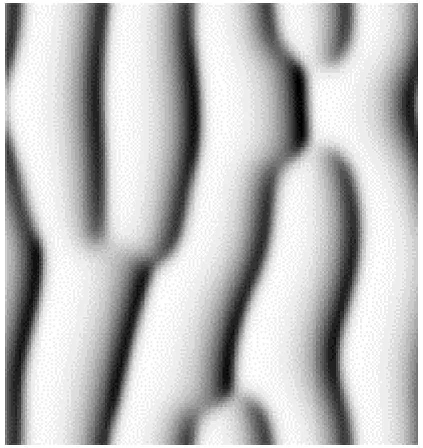


[wikipedia]

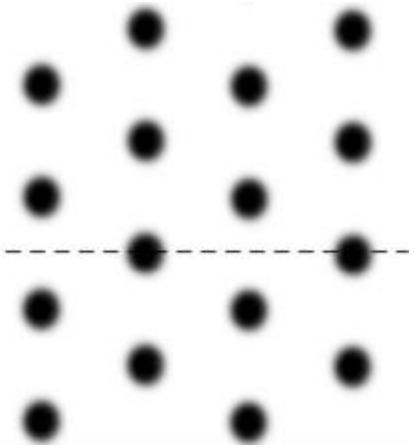
Seminal paper in 1952: “The chemical basis of morphogenesis”

Reaction-Diffusion Equations

$$\begin{cases} \dfrac{du}{dt} = f(u,v) + D_u\, \Delta u \\ \dfrac{dv}{dt} = g(u,v) + D_v\, \Delta v \end{cases}$$

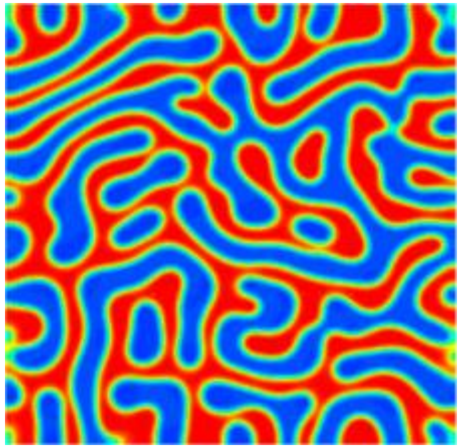


[Klausmeier, 1999]



[Gilad et al, 2004]

[Rietkerk et al, 2002]



[Liu et al, 2013]

Reaction-Diffusion Equation for Dryland Ecosystems

$$w_t = w_{xx} - w + a - wv^2$$
$$v_t = D^2 v_{xx} - mv + wv^2$$

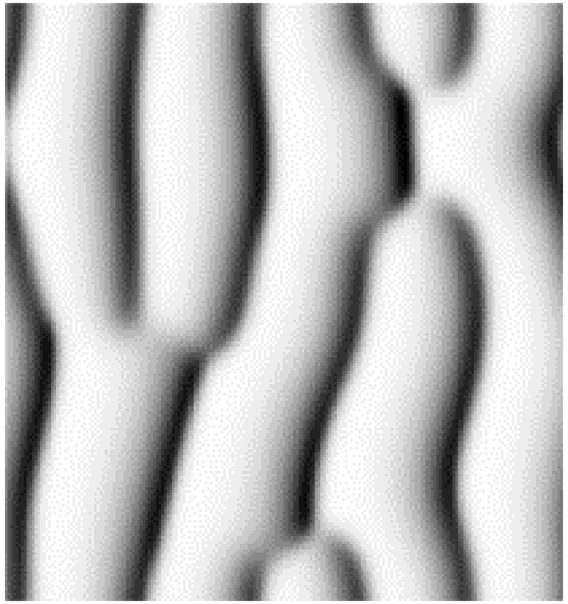
w : water

v : vegetation

D : ratio of diffusion

a : rainfall

m : mortality



[Klausmeier, 1999]

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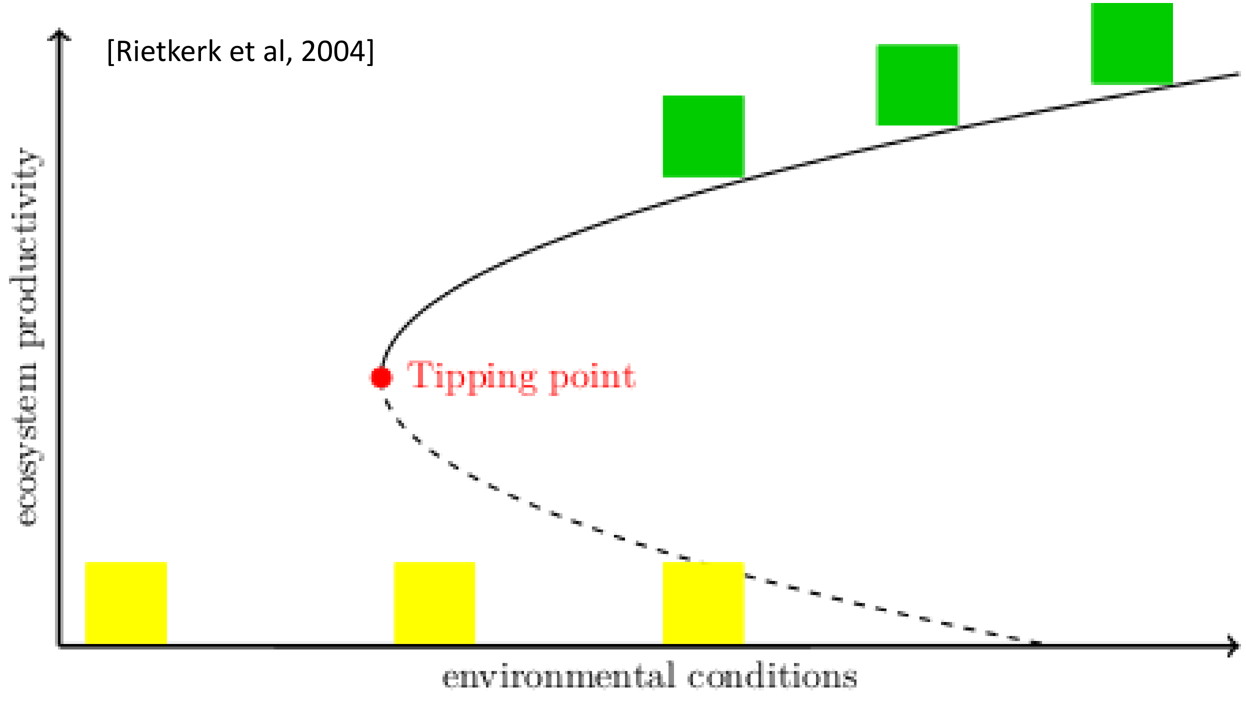
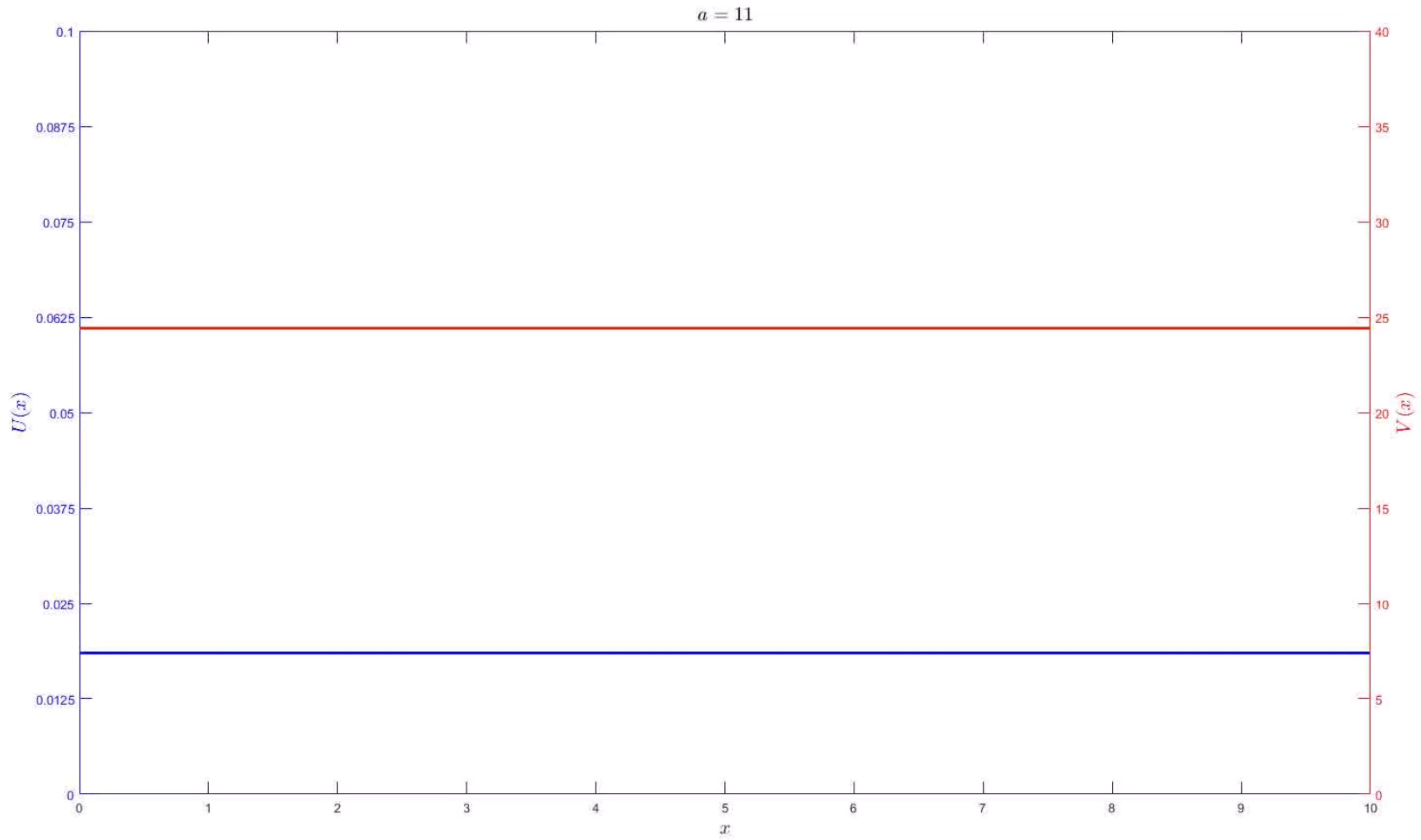


Spontaneous Pattern Formation



Turing patterns

$$\begin{cases} \frac{du}{dt} = f(u,v) + D_u\,\Delta u \\ \frac{dv}{dt} = g(u,v) + D_v\,\Delta v \end{cases}$$

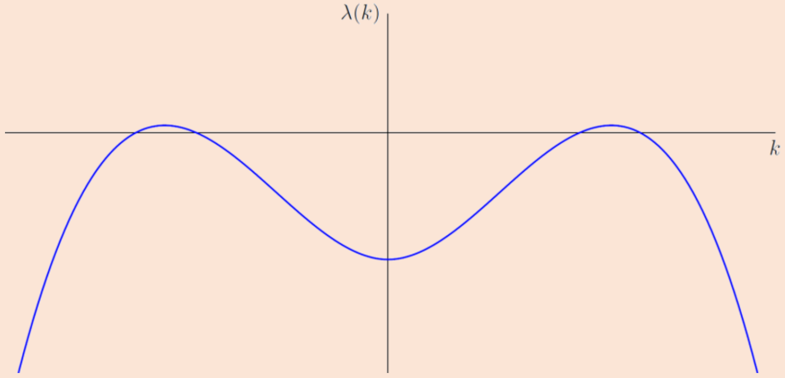
Turing bifurcation

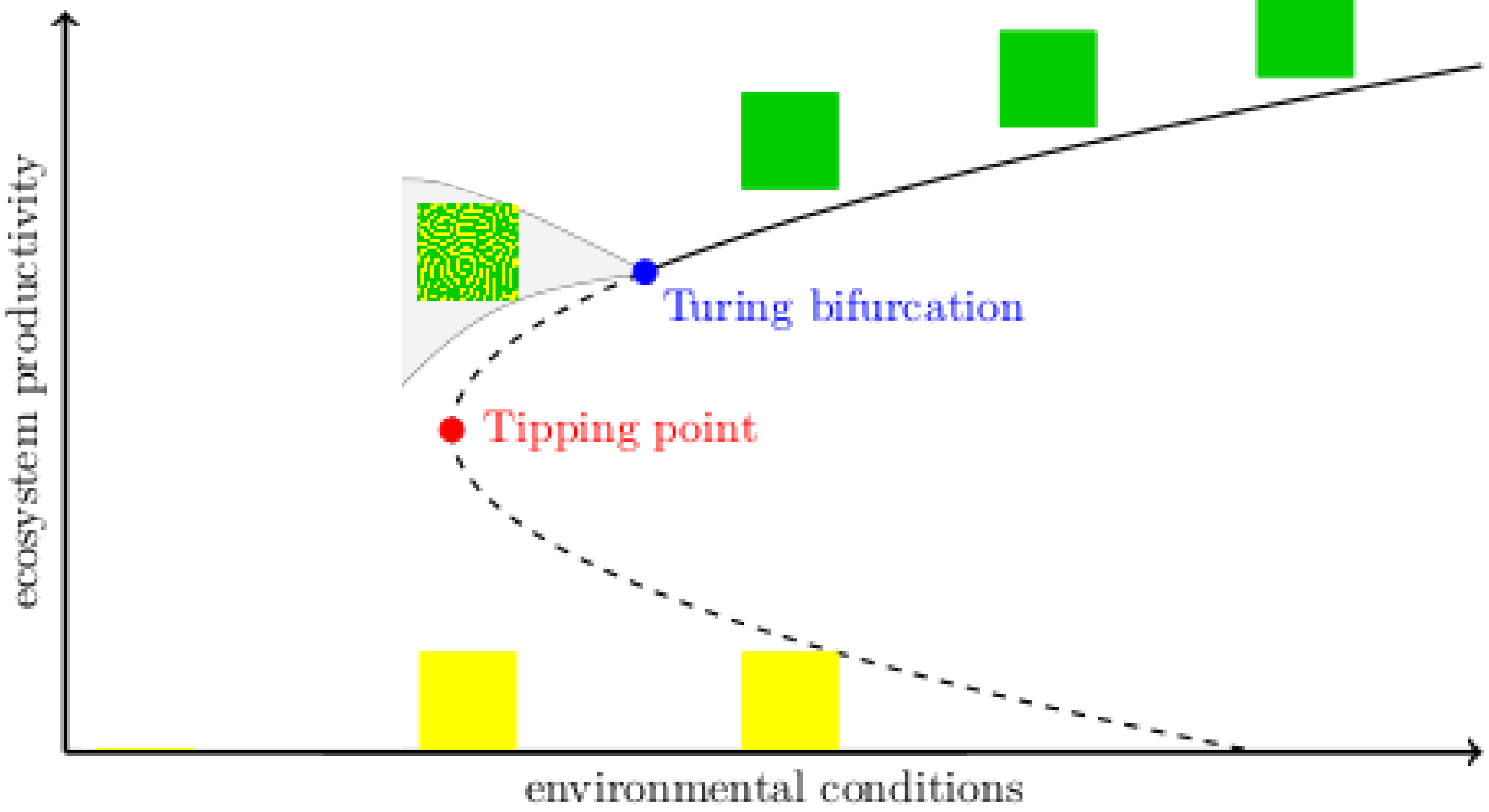
Instability to non-uniform perturbations

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u_* \\ v_* \end{pmatrix} + e^{\lambda t} e^{ikx} \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix}$$

→ Dispersion relation

$$\lambda(k) = \cdots$$





Weakly non-linear analysis

Ginzburg-Landau equation / Amplitude Equation

& Eckhaus/Benjamin-Feir-Newel criterion

[Eckhaus, 1965; Benjamin & Feir, 1967; Newell, 1974]

Busse balloon

$$\begin{cases} \dfrac{du}{dt} = f(u,v) + D_u\,\Delta u \\ \dfrac{dv}{dt} = g(u,v) + D_v\,\Delta v \end{cases}$$

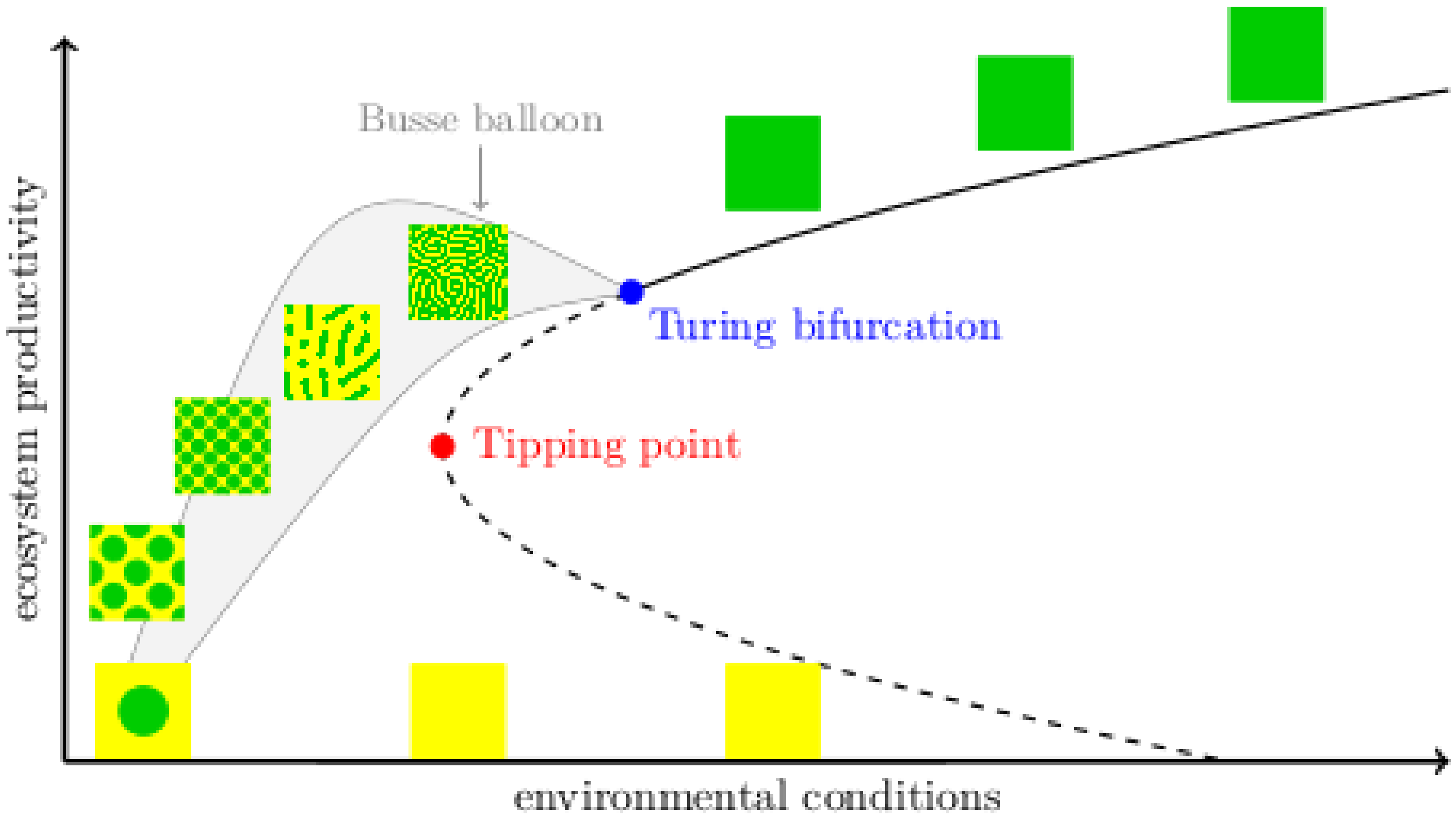
Busse balloon

A model-dependent shape in *(parameter, observable)* space that indicates all stable patterned solutions to the PDE.

Construction Busse balloon

Via numerical continuation

few general results on the shape of Busse balloon



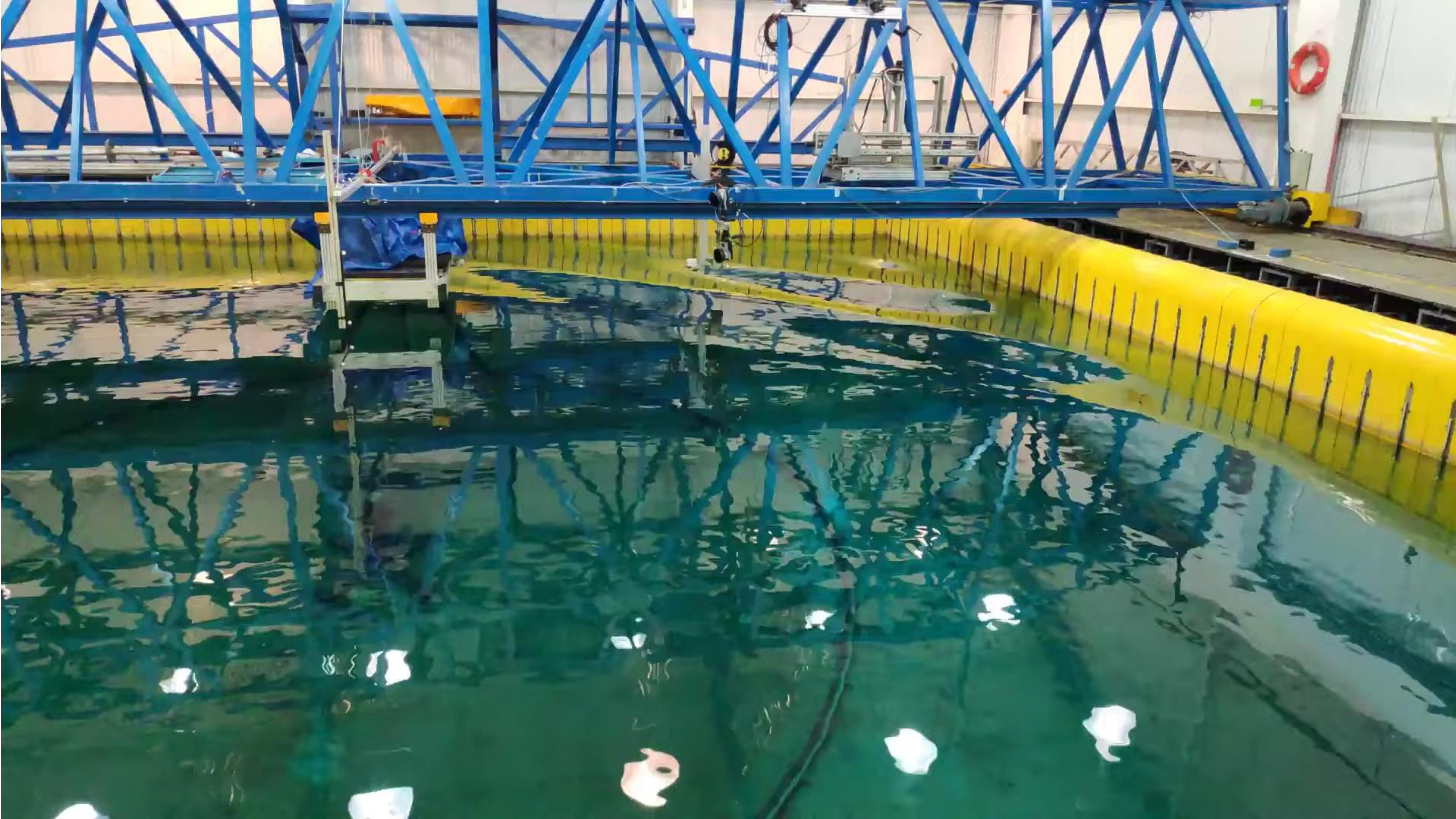
Busse balloon

Idea originates from thermal convection

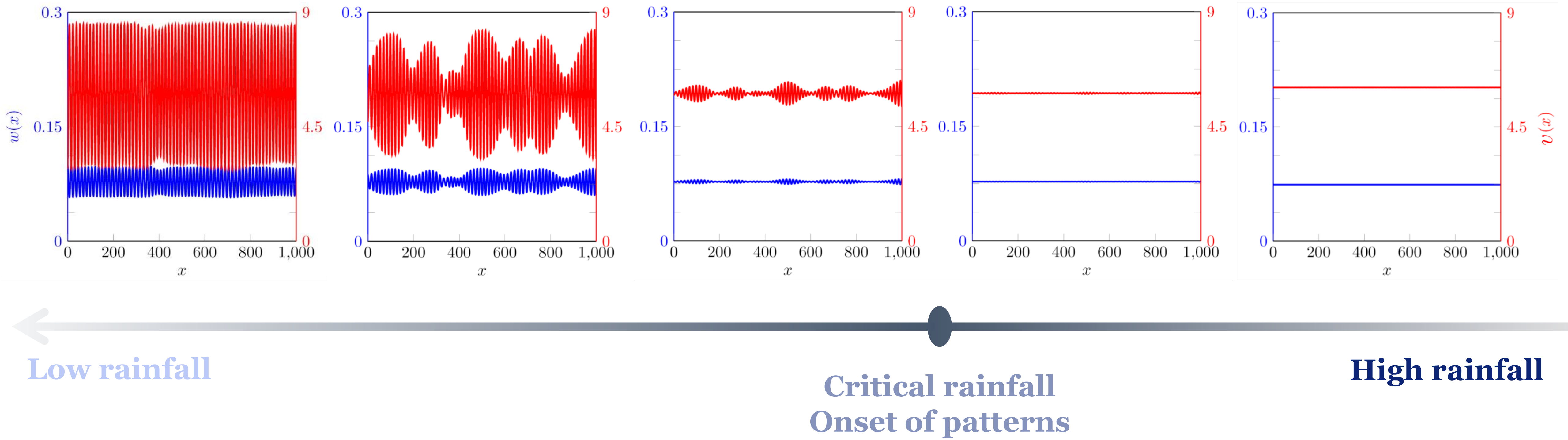
[Busse, 1978]

Rayleigh Bénard thermal convection

Video source: wikiRigaou (wikimedia commons)

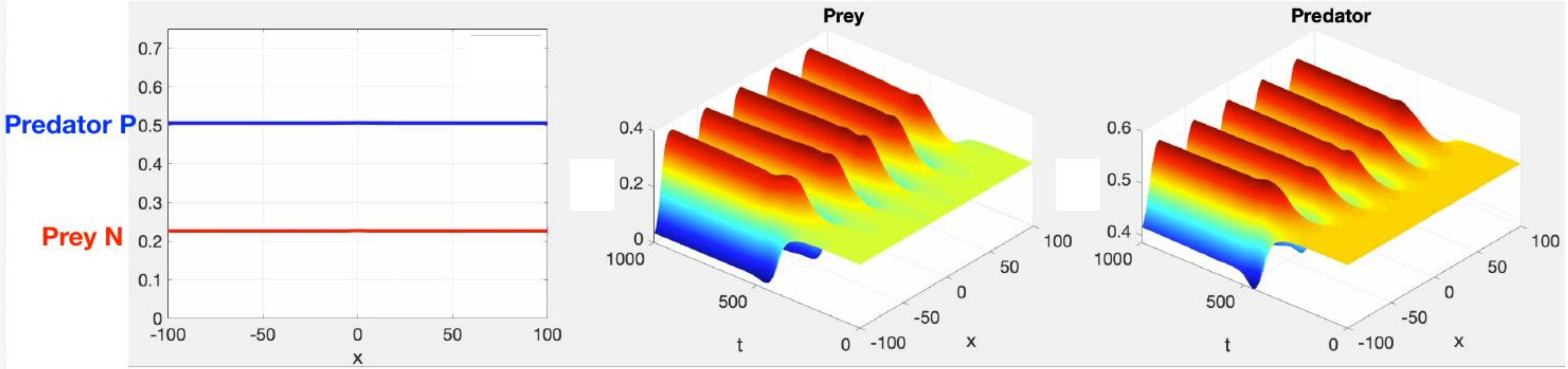


The orgin of patterns in dryland model



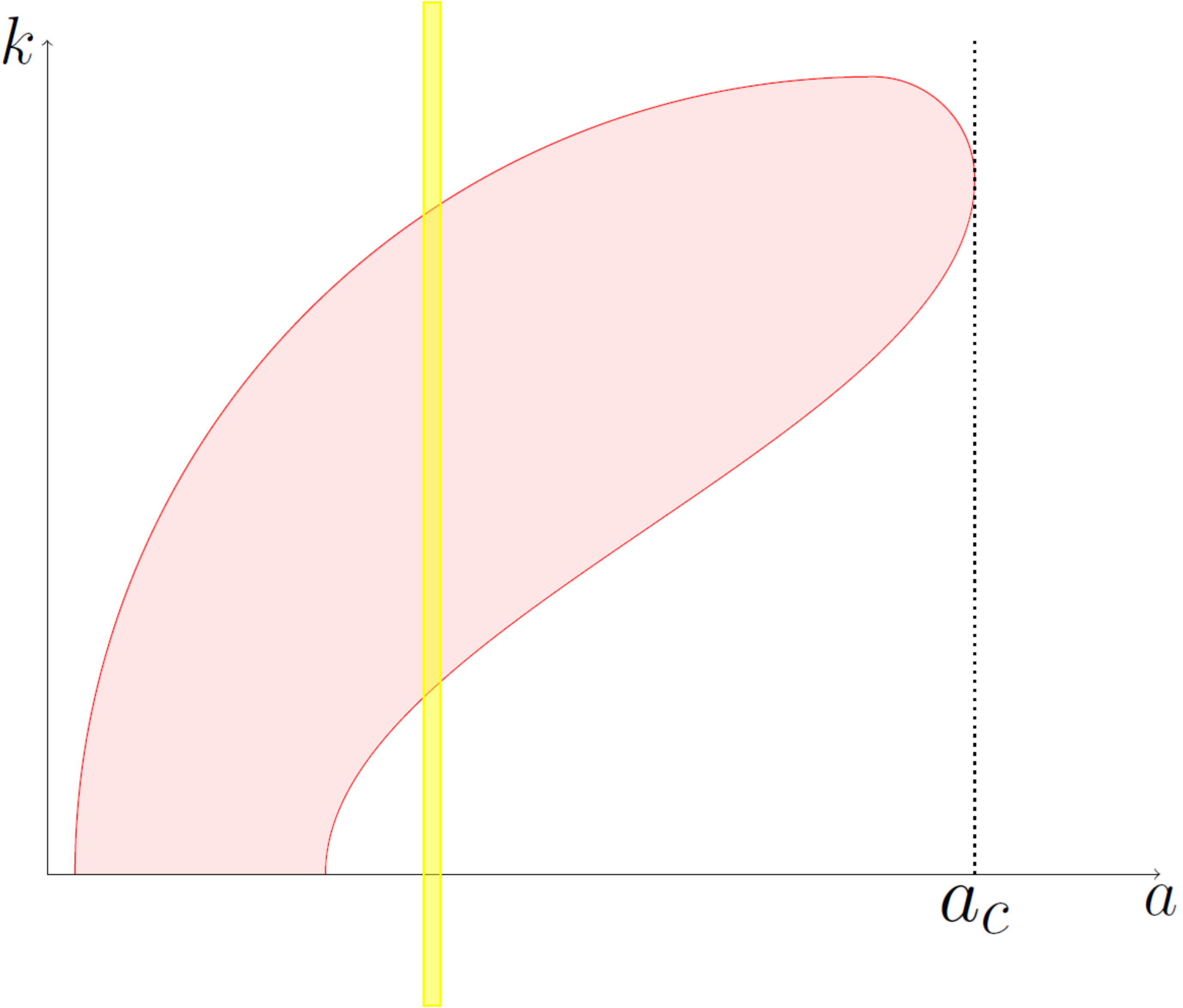
Example: Diffusive Holling–Tanner predator-prey model with an alternative food source for the predator

$$N_t = rN\left(1 - \frac{N}{K}\right) - \frac{qNP}{N+a} + D_1 N_{xx},$$
$$P_t = sP\left(1 - \frac{P}{hN+c}\right) + D_2 P_{xx}.$$



[Arancibia-Ibarra et al., 2021]

Multistability in the Busse balloon



<u>Observation:</u>

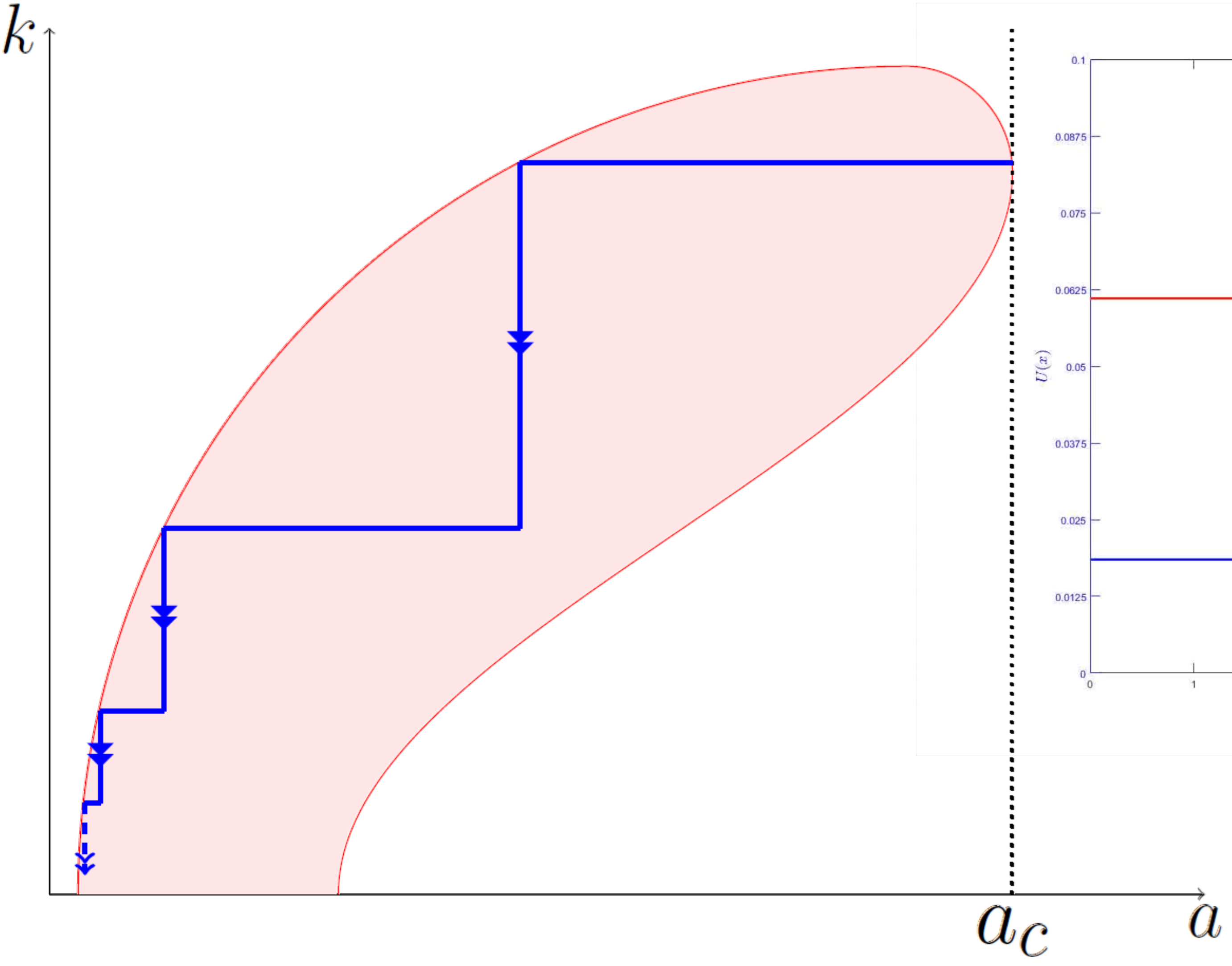
For a fixed parameter value, there is a **continuous** range of wavenumbers possible.

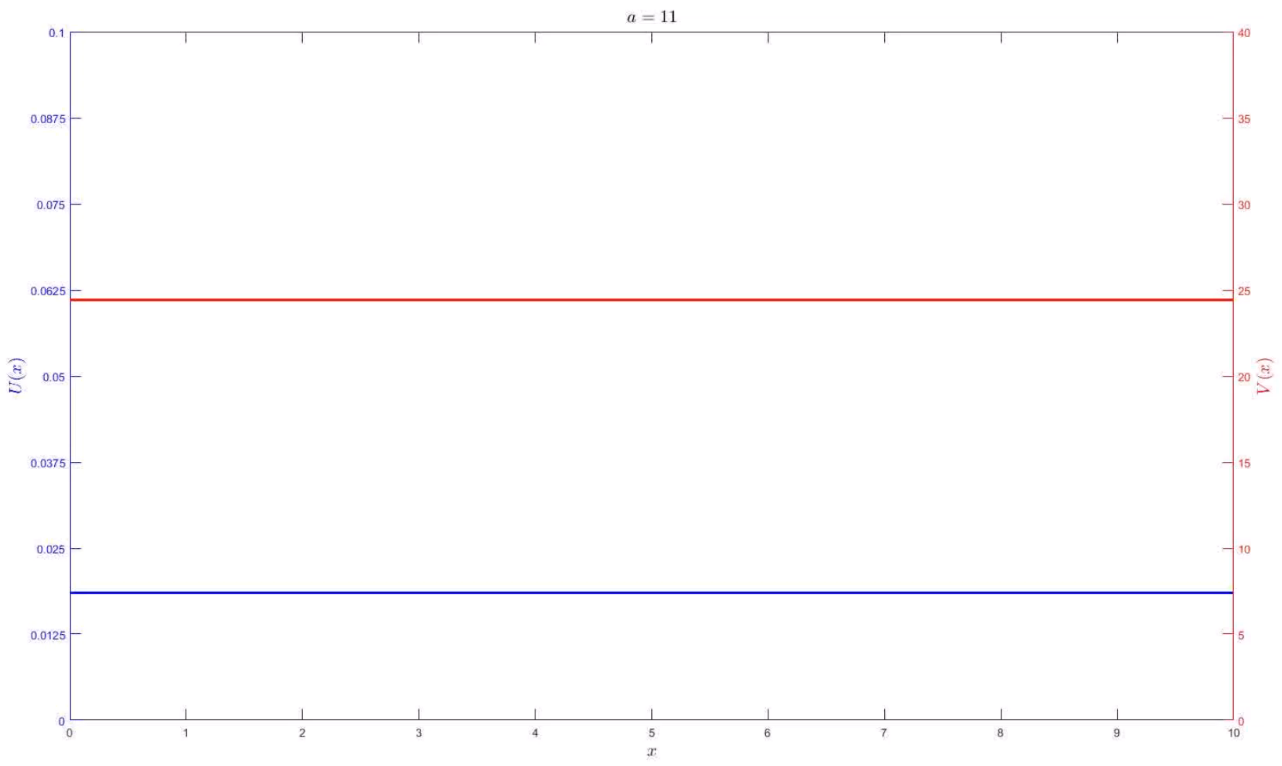
That is, there is a large **multistability** of stable pattern states to the PDE

<u>Consequence:</u>

Specifying only parameter values is ambiguous, as it does not correspond to only one patterned state.

A Walk through the Busse balloon





<u>Consequence:</u>

In patterned shifts, you see smaller transitions from one patterned states to another

<u>APPLICATION</u>:

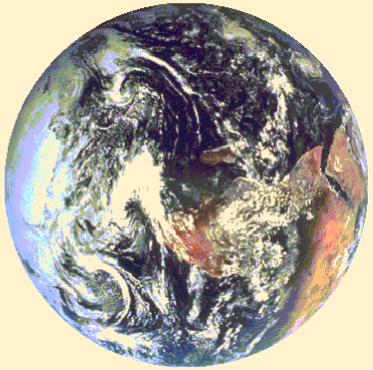
Tipping

in

Spatially Extended Systems

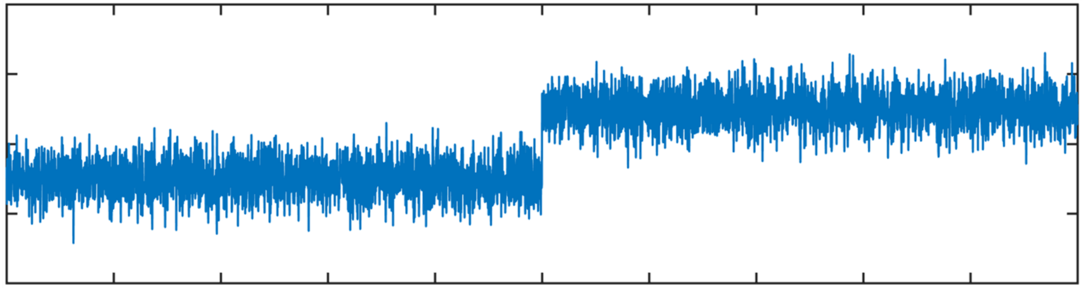
Tipping Points

IPCC AR6 (2021) : “a critical threshold beyond which a system reorganizes, often abruptly and/or irreversibly”



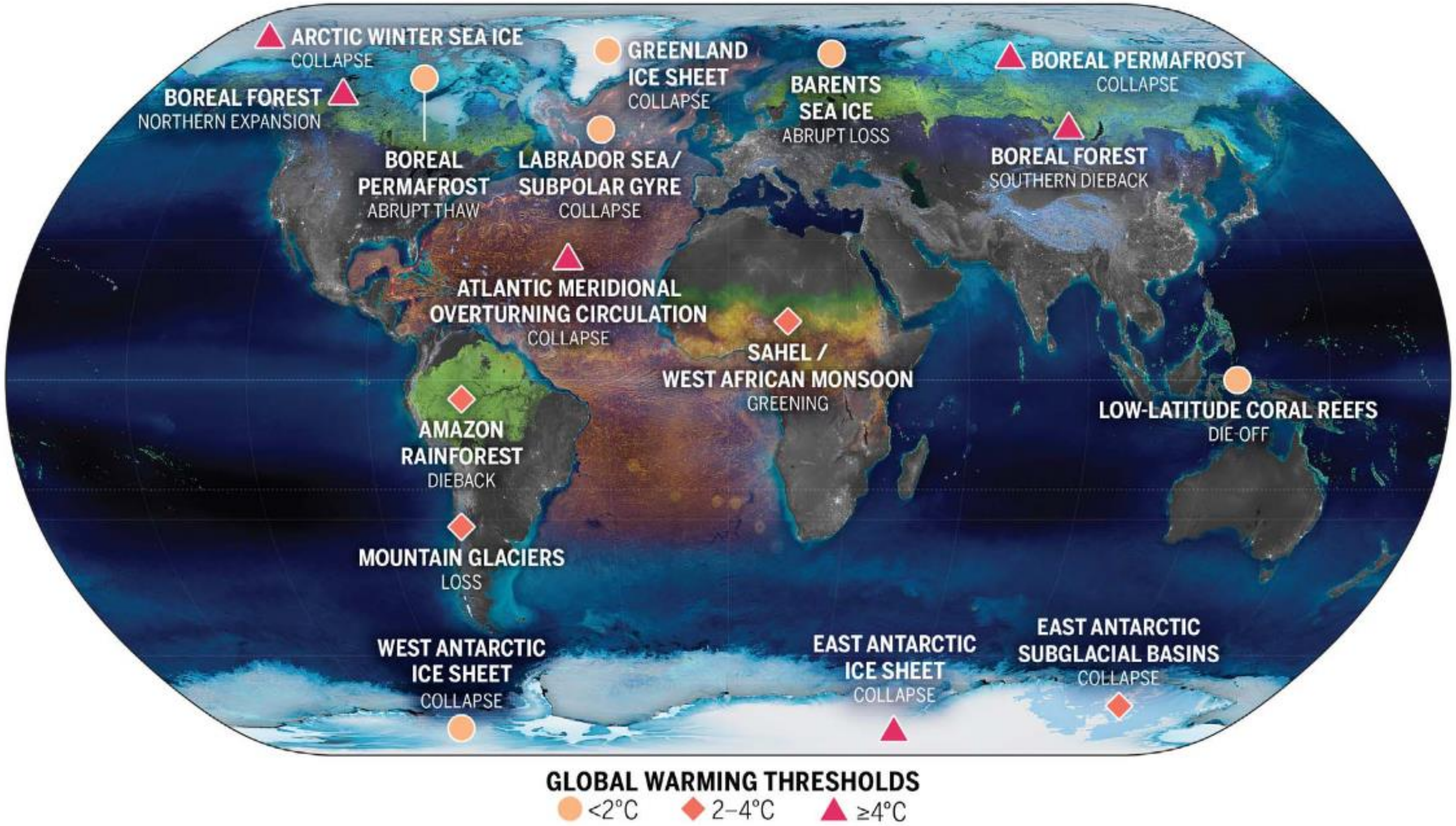
Planetary transitions

Ecosystem shifts



Tipping Points

IPCC AR6 (2021) : “a critical threshold beyond which a system reorganizes, often abruptly and/or irreversibly”

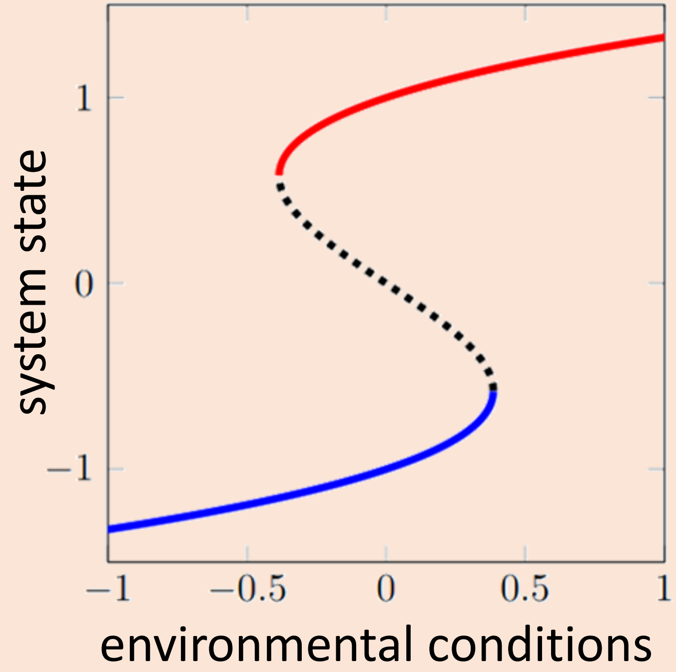


source: McKay et al, 2022

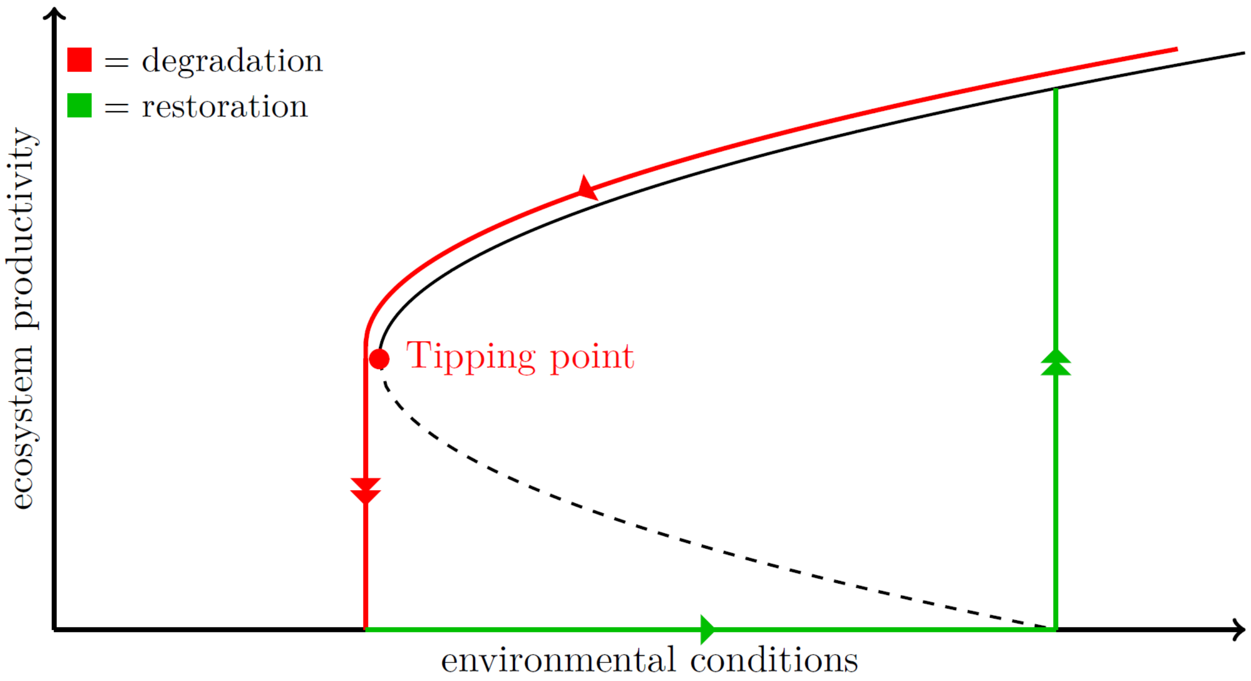
Mathematics

Tipping points ⟷ Bifurcations

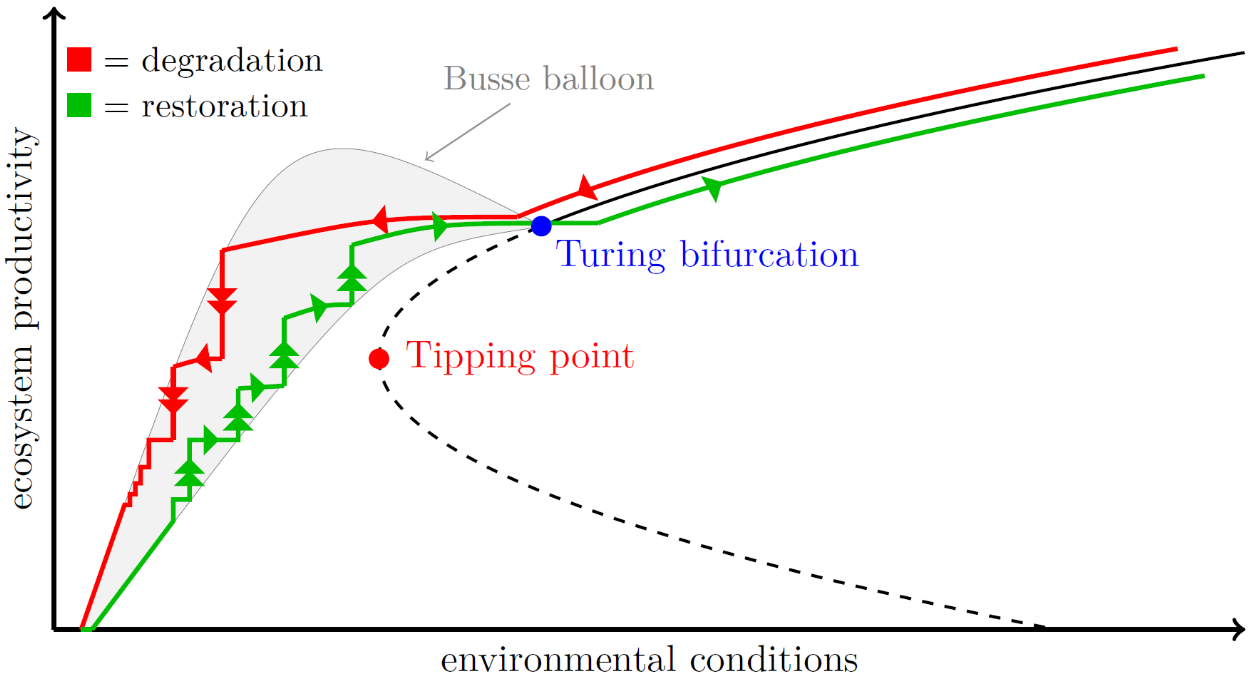
$$\frac{dy}{dt} = f(y, \mu)$$



Tipping of (Turing) patterns



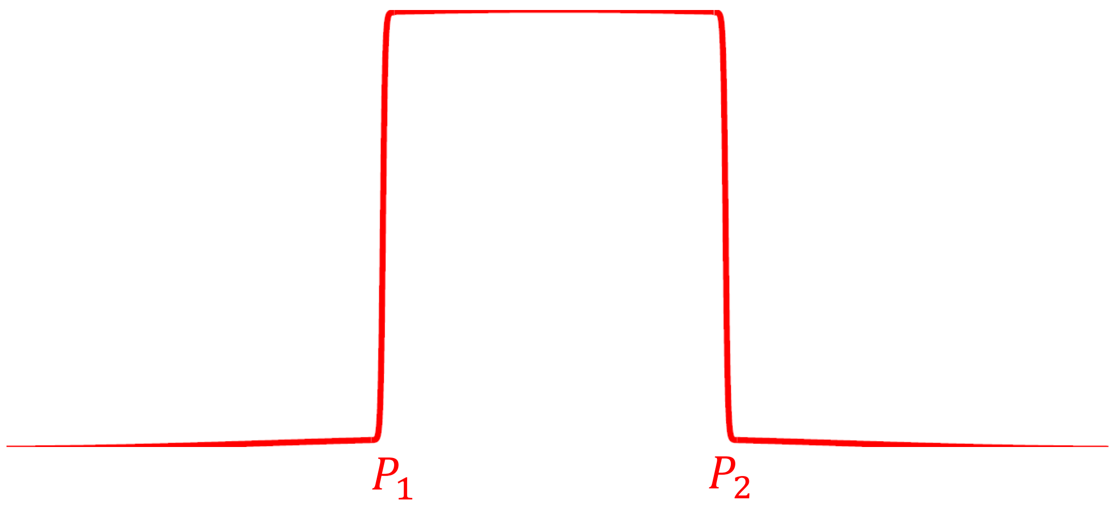
Classic tipping



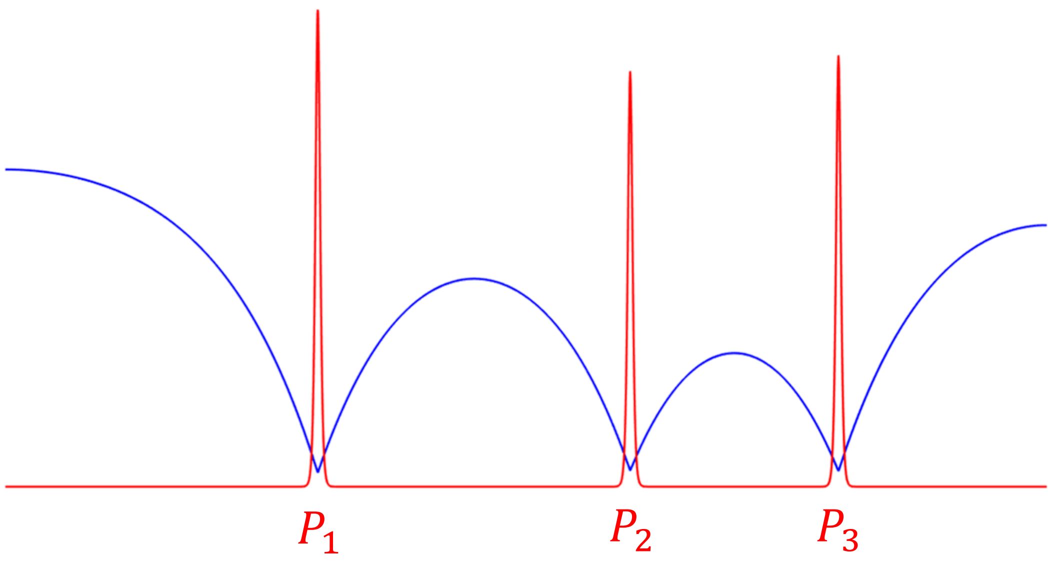
Tipping of patterns

Part 2:
Behaviour of Spatial Patterns

Dynamics of Patterned States



fronts



pulses

Mathematical Study of Localised Structures (1)

Example system:

Allen-Cahn Equation

$$\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2} + y(1 - y^2) + \mu$$

Introduce travelling wave coordinate(s):

$$\zeta \coloneqq x - ct$$

Assume state only depends on that:

$$y(x, t) = y(\zeta)$$

Gives an ordinary differential equation

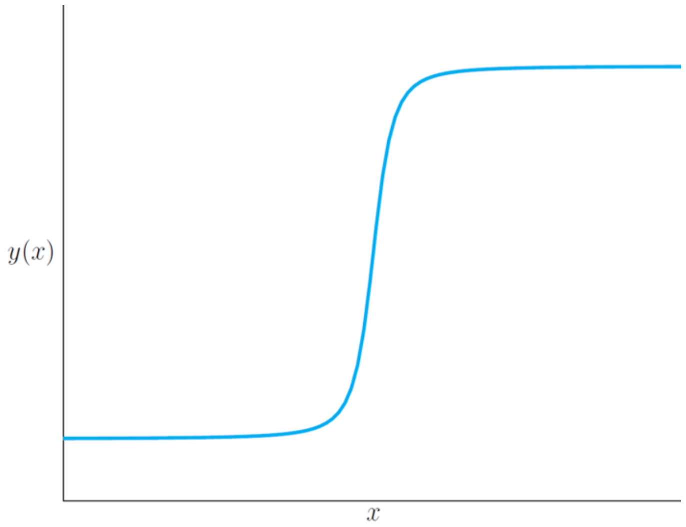
often coined the ‘spatial dynamics’:

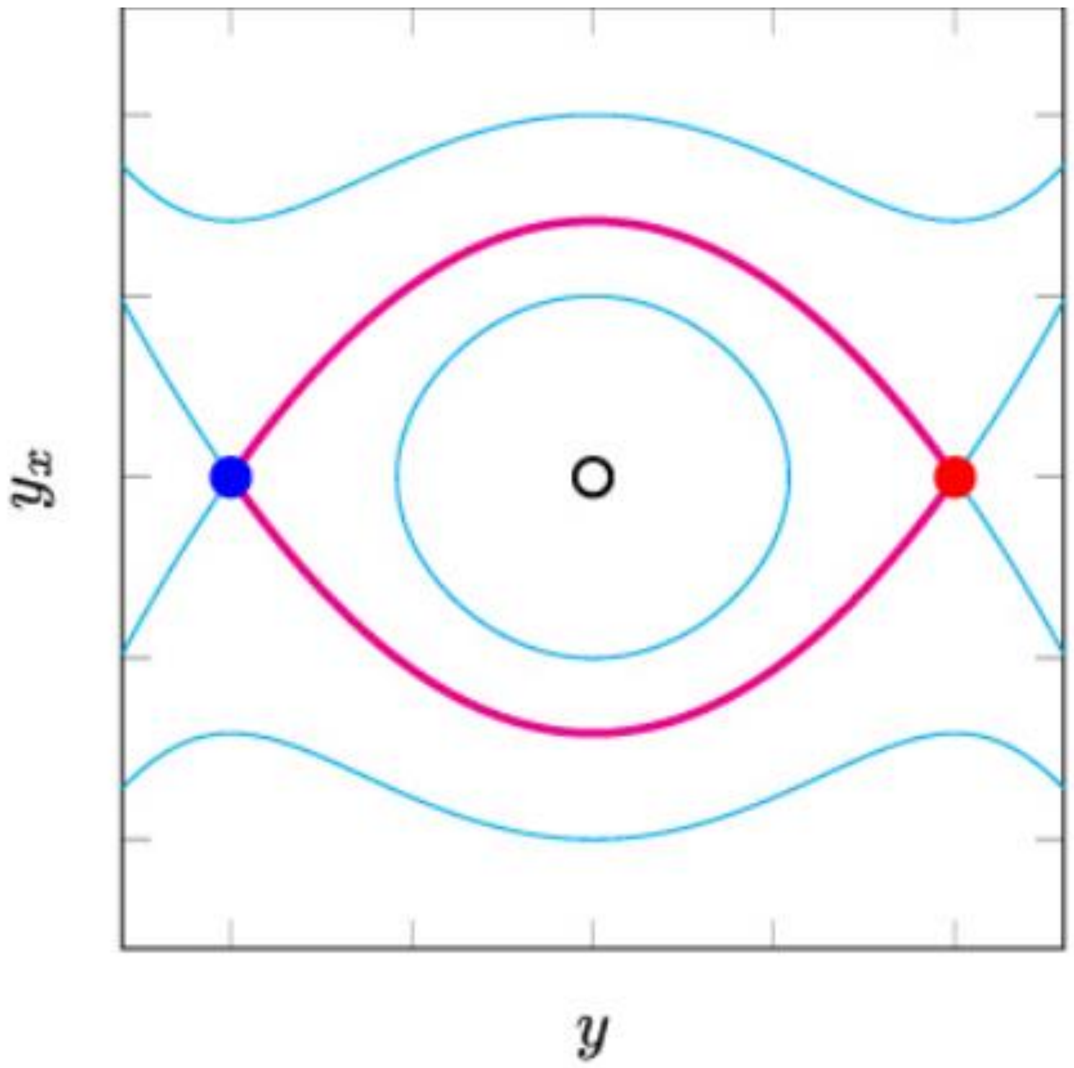
$$-cy' = y'' + y(1 - y^2) + \mu$$

Mathematical Study of Localised Structures (2)

For $c = 0$ and $\mu = 0$
system can be written as

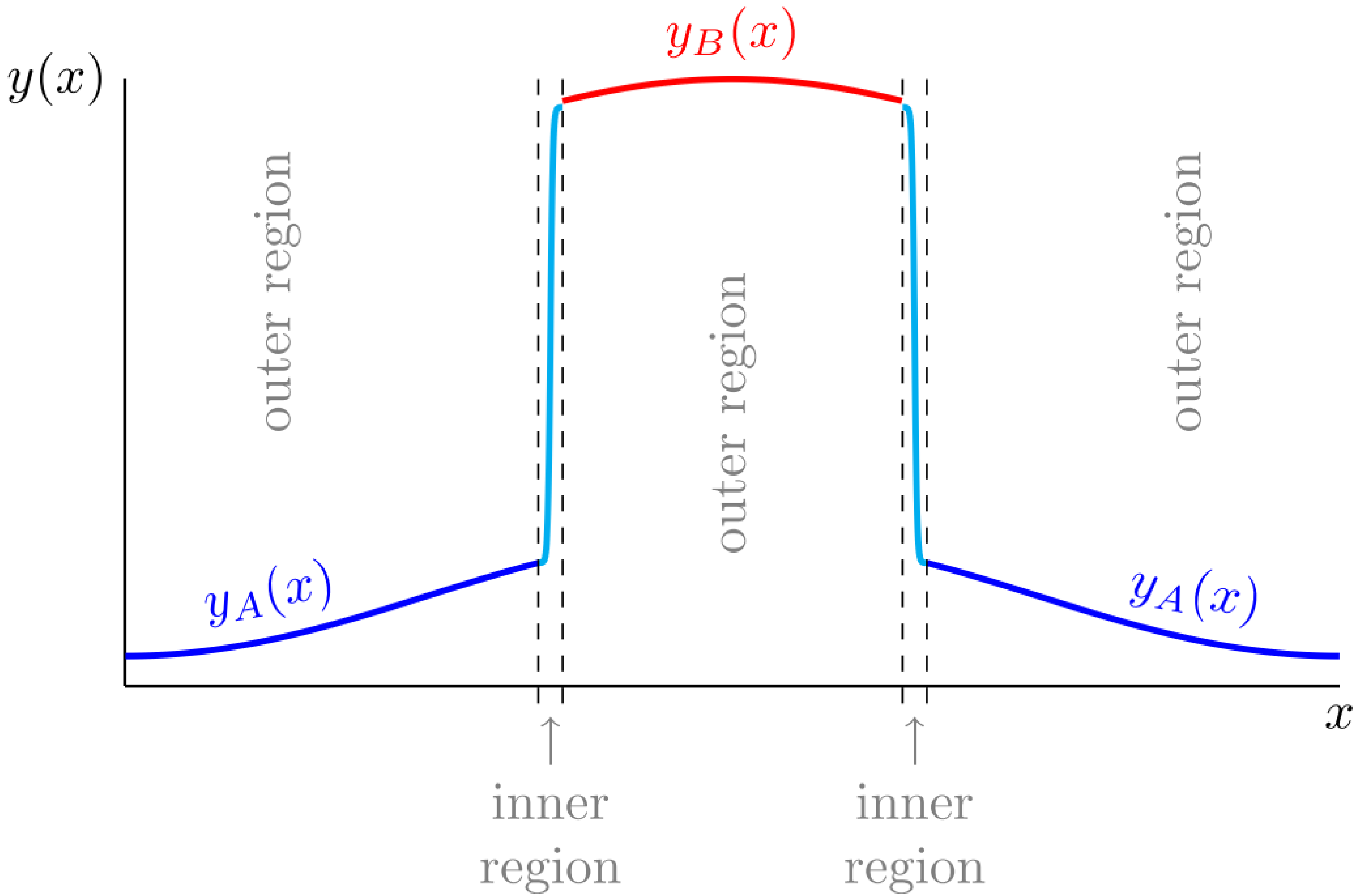
$$y' = p$$
$$p' = -y(1 - y^2)$$



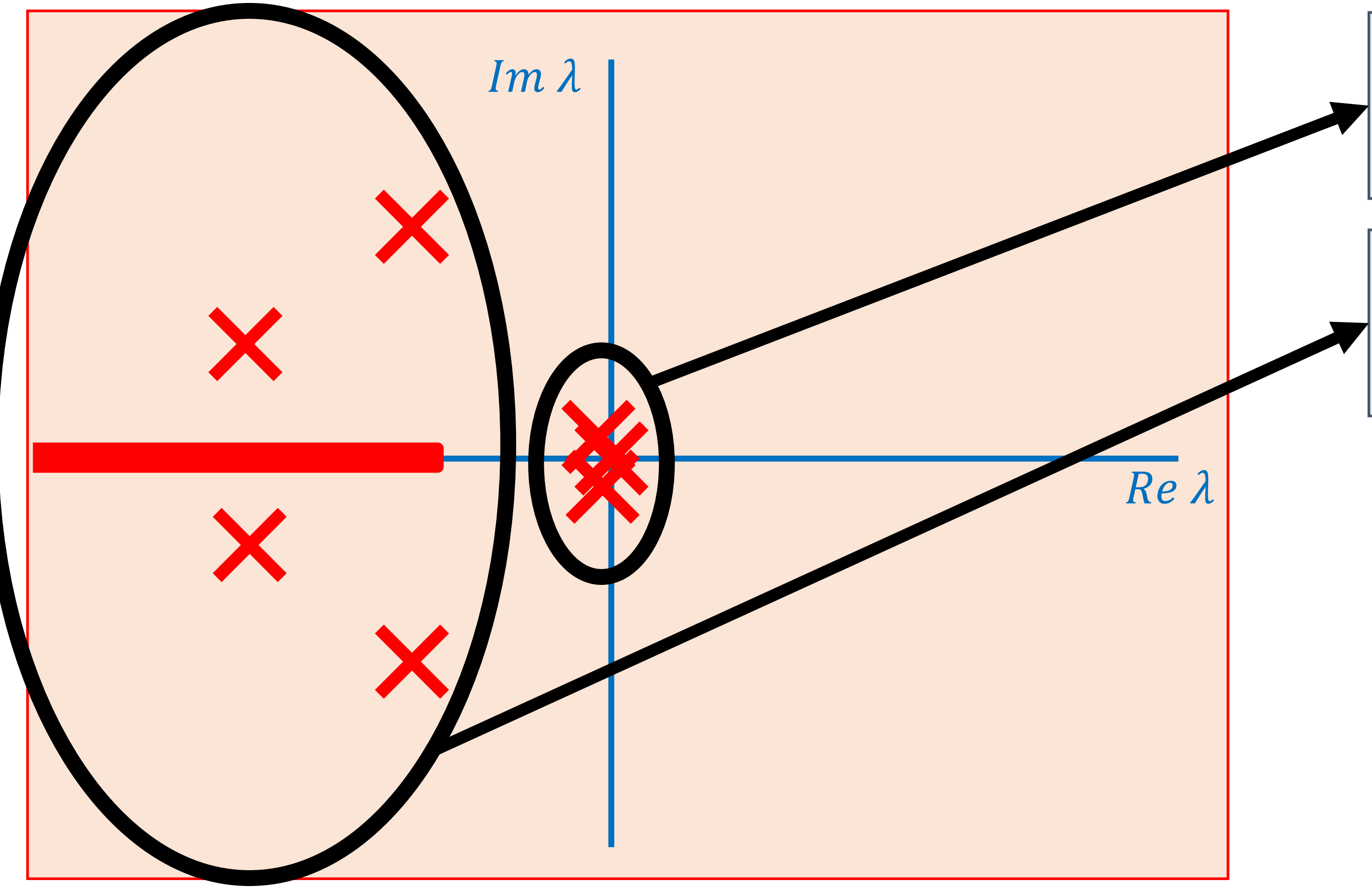


Heteroclinic connections in the spatial dynamics correspond to front solutions to the PDE

Mathematical Construction of localised structures

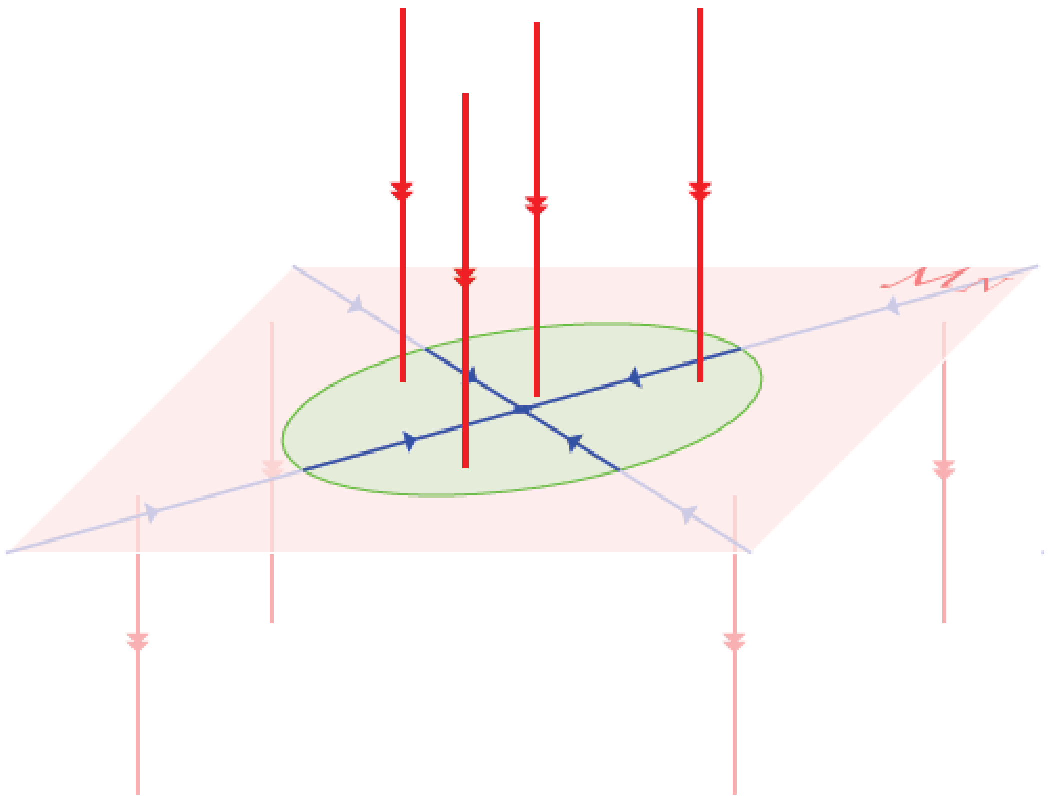


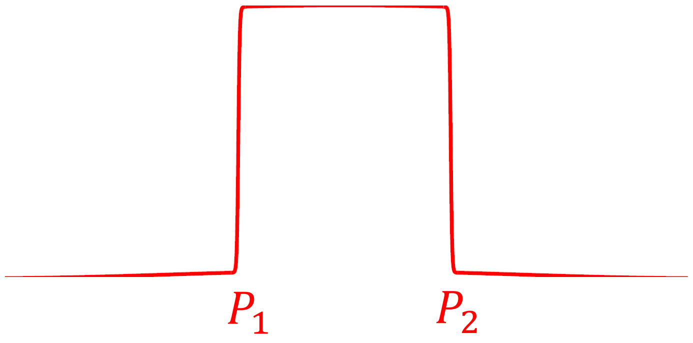
Dynamics of Patterned States

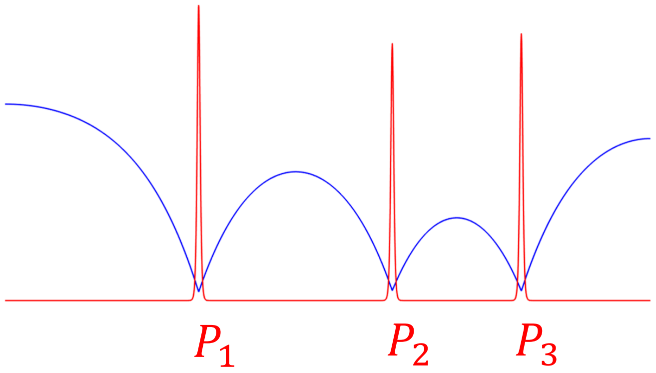


1. SLOW Pattern Adaptation

2. FAST Pattern Degradation





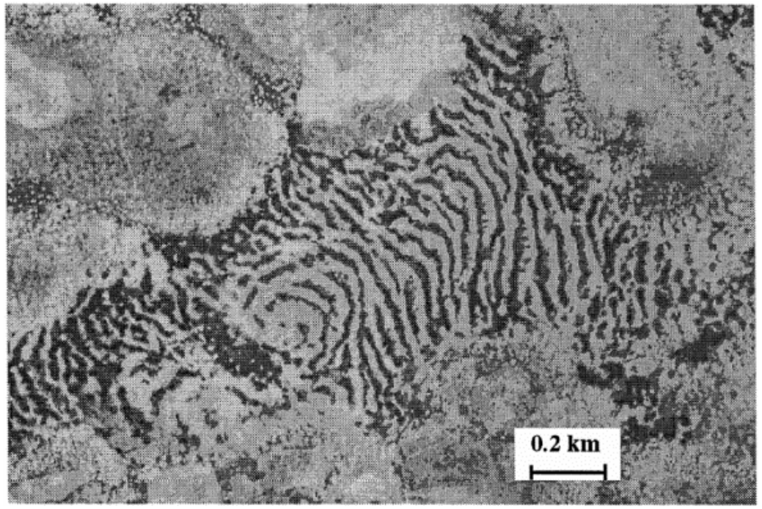


1. SLOW pattern adaptation

Somaliland, 1948 [Macfadyen, 1950]

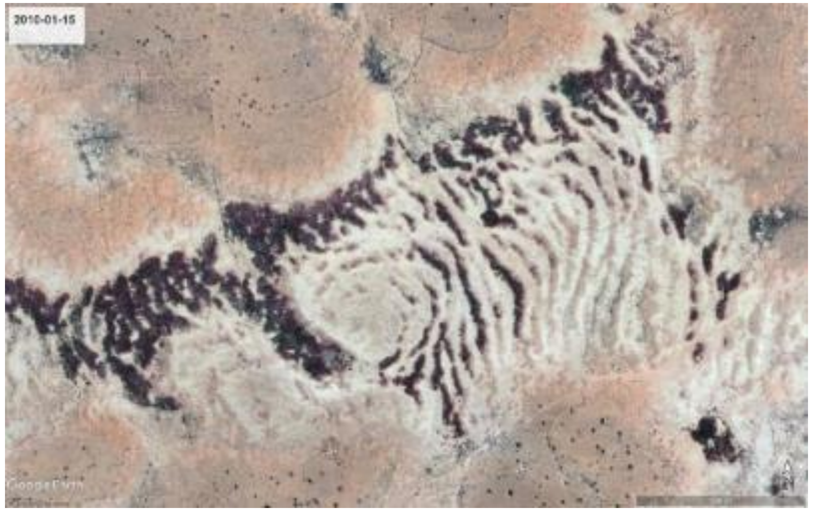
Somaliland, 2008

2. FAST Pattern Degradation

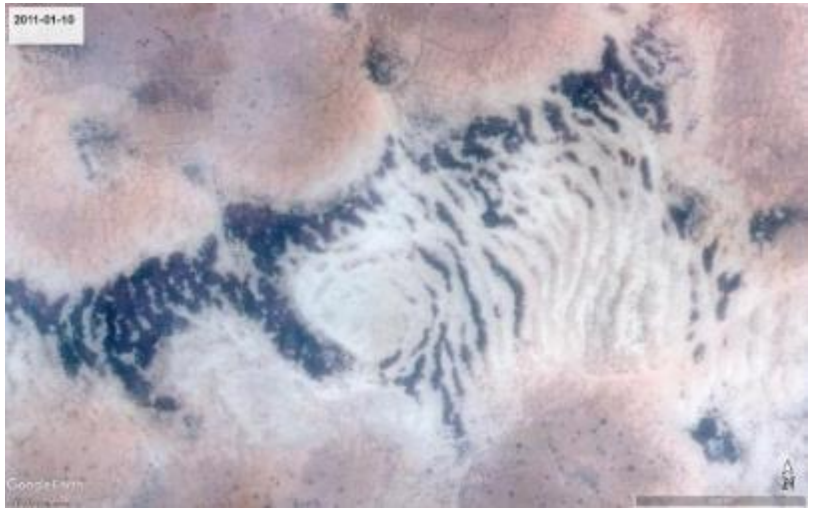


Niger, 1950 [Valentin, 1999]

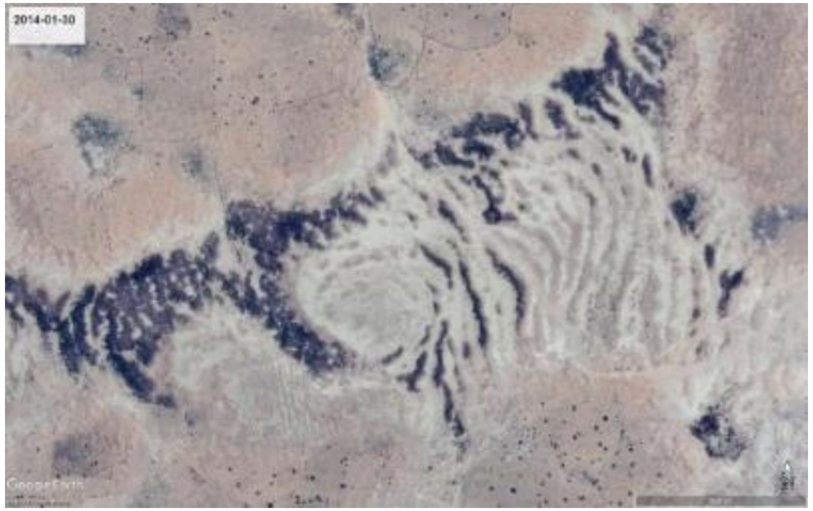
Niger, 2008



Niger, 2010



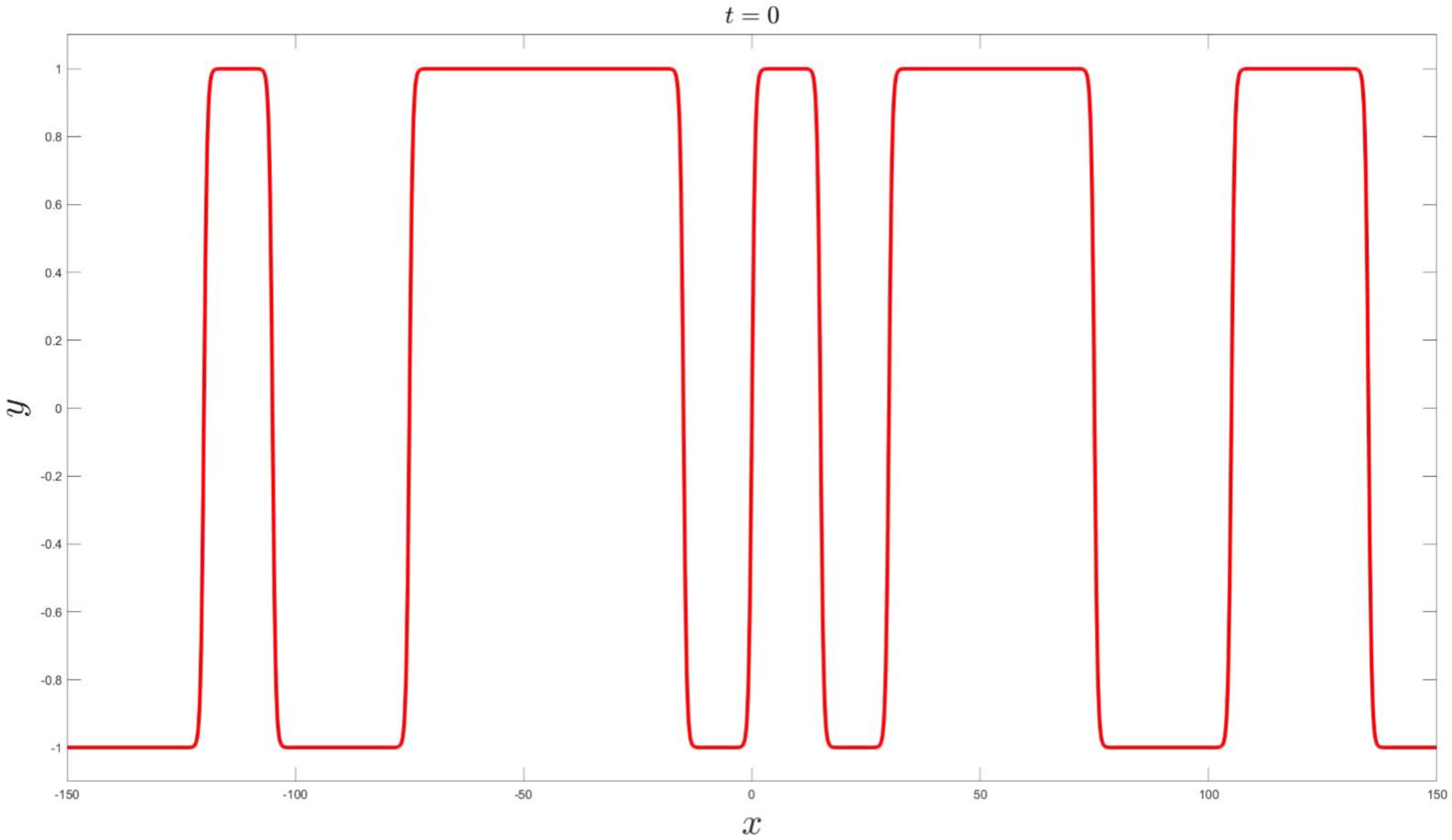
Niger, 2011



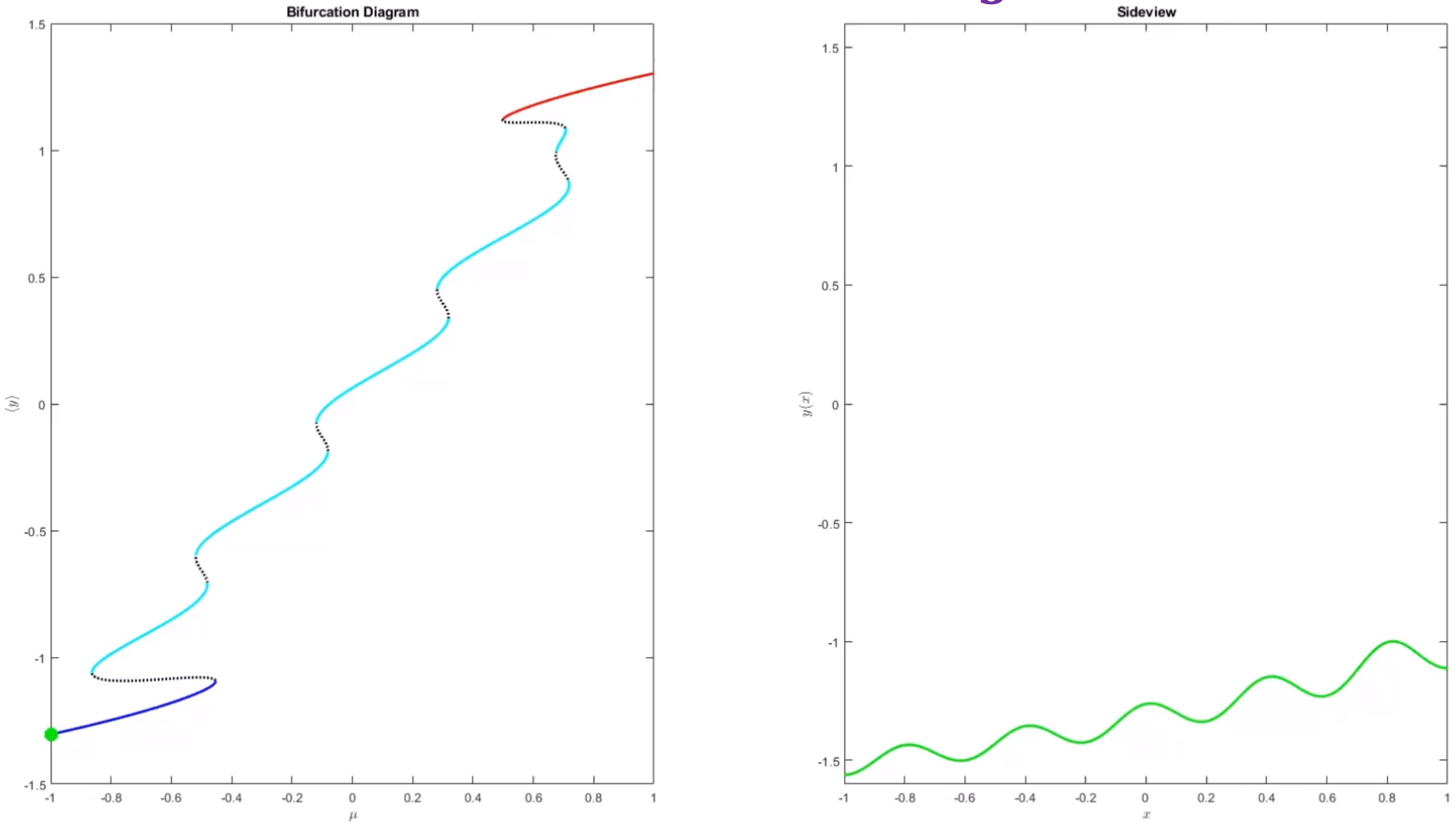
Niger, 2014

Niger, 2016

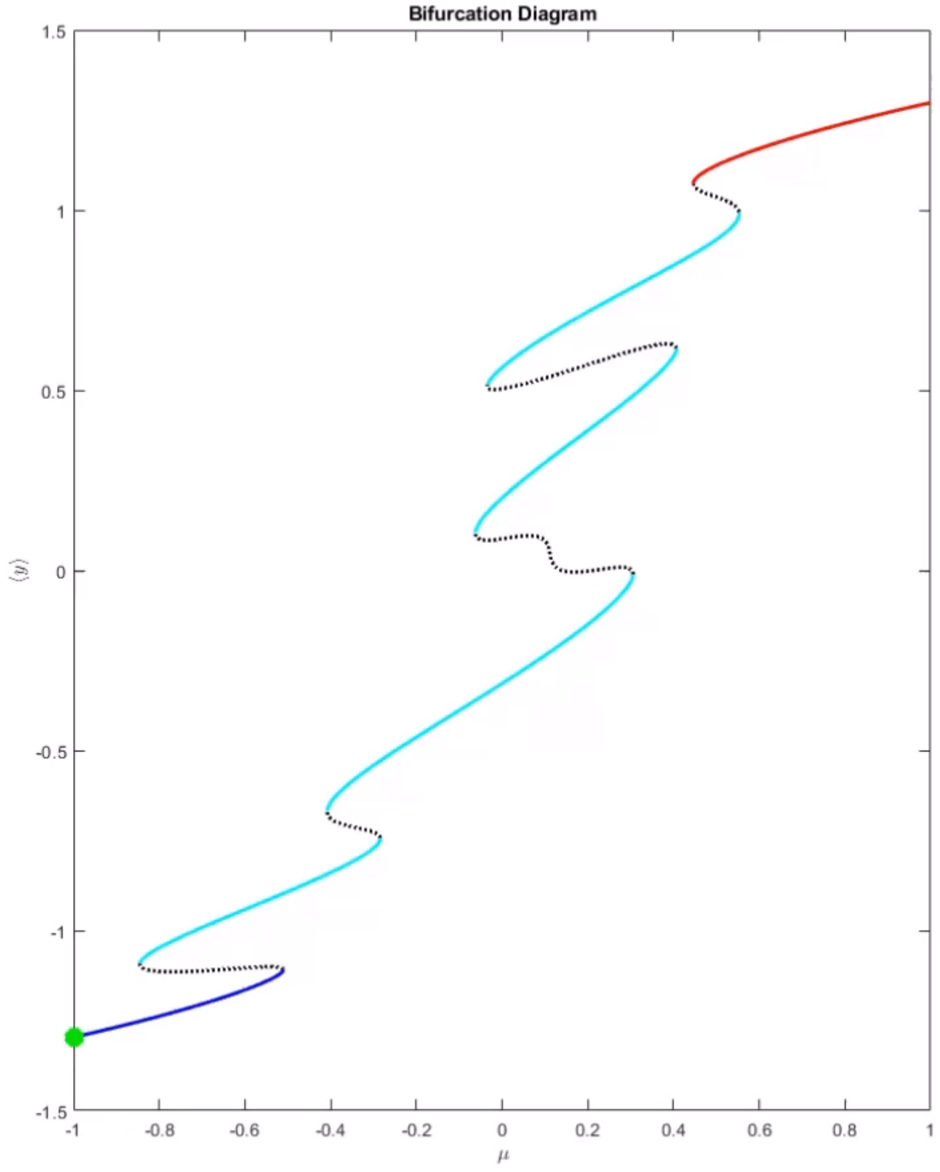
Dynamics of $\frac{\partial y}{\partial t} = D\ \frac{\partial^2 y}{\partial x^2} + y(1 - y^2) + \mu$

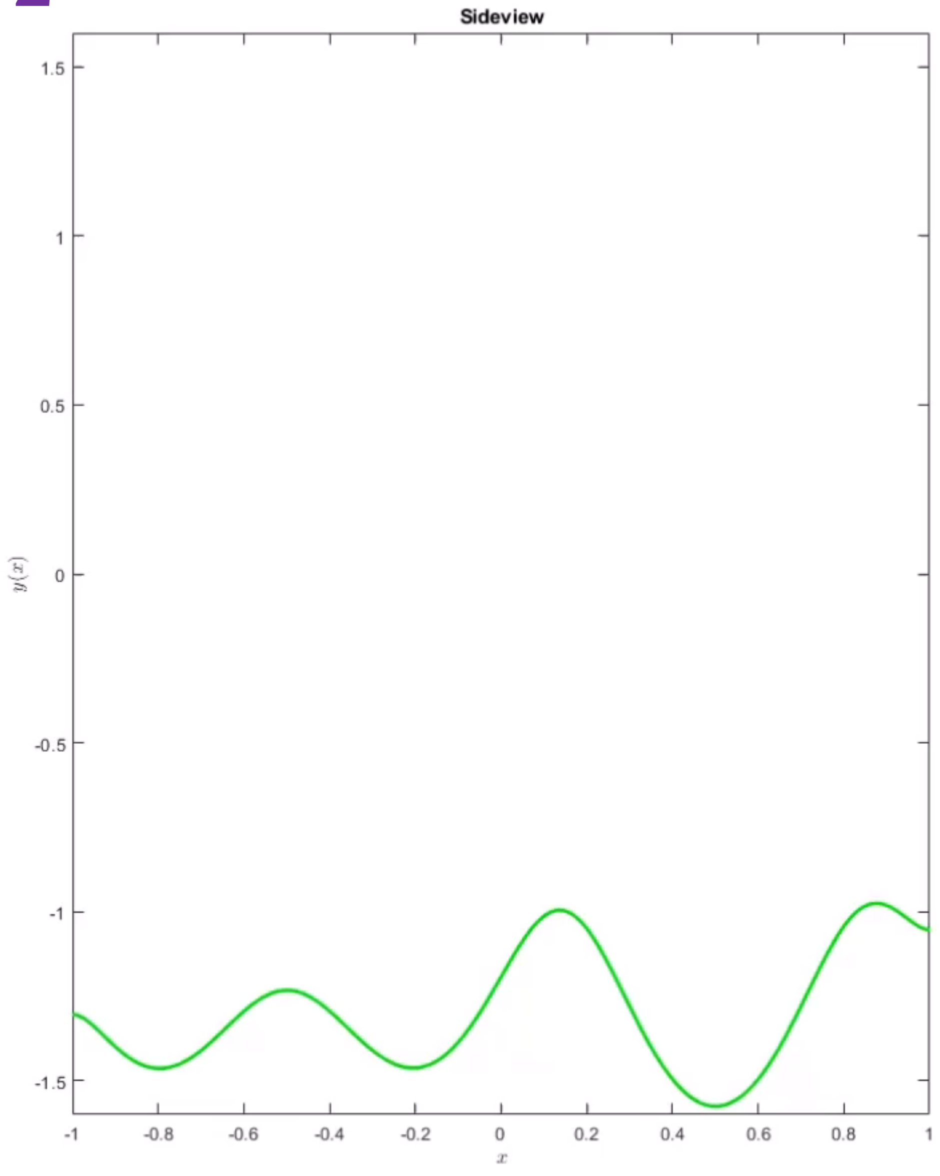


$$y_t = D\, y_{xx} + y\left(1 - y^2\right) + \mu + x + \frac{2}{5}\cos(5\pi x)$$

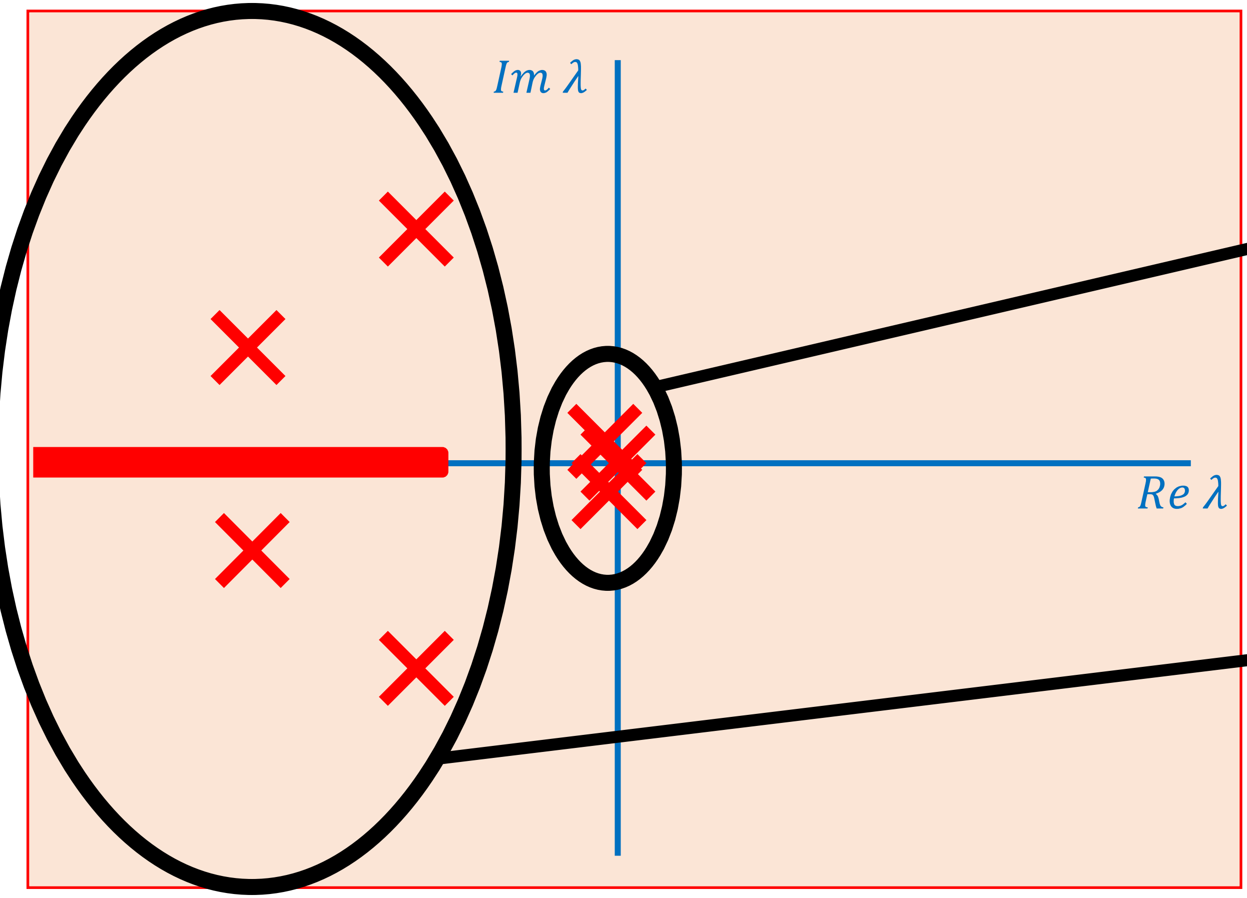


$$y_t = D\,y_{xx} + y(1 - y^2) + \mu + \frac{1}{2}\cos(2\pi x) + \sin(3\pi x)$$





Bifurcations



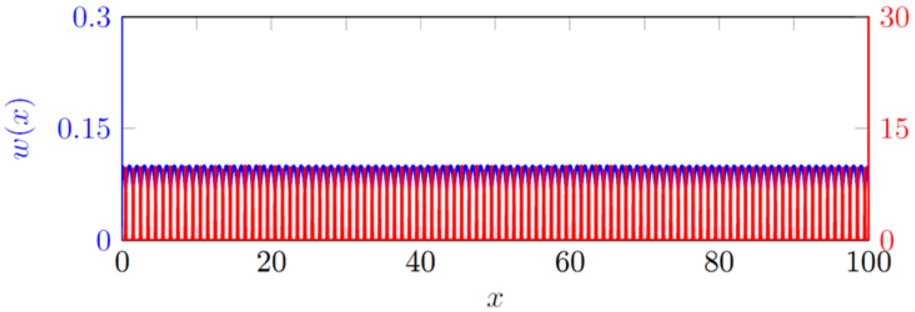
What happens at bifurcation?

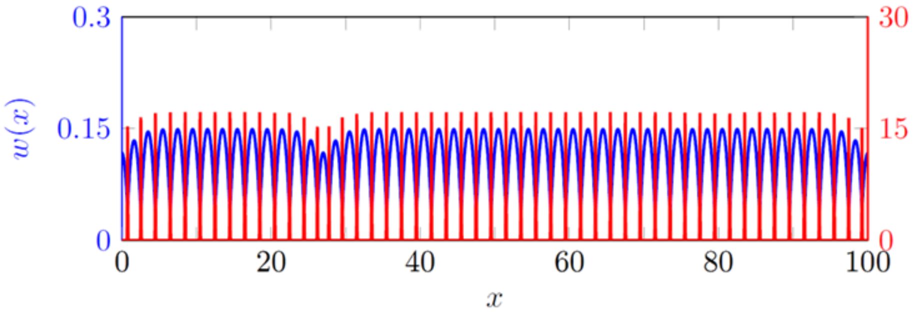
1. SLOW Pattern Adaptation

At bifurcation:
→ Location of structure changes

2. FAST Pattern Degradation

At bifurcation:
→ Structures created or destroyed



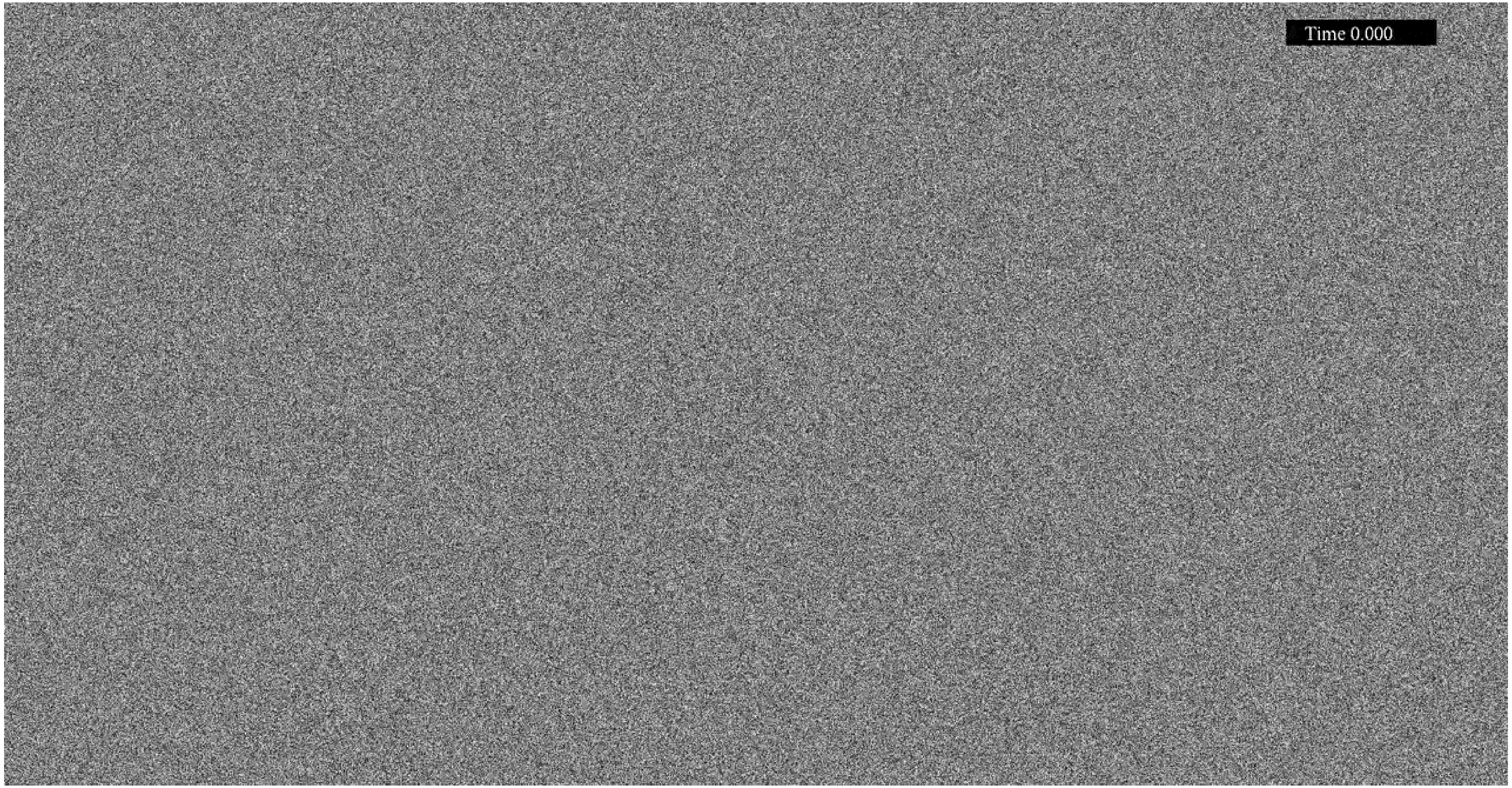



And what about that ice cream?

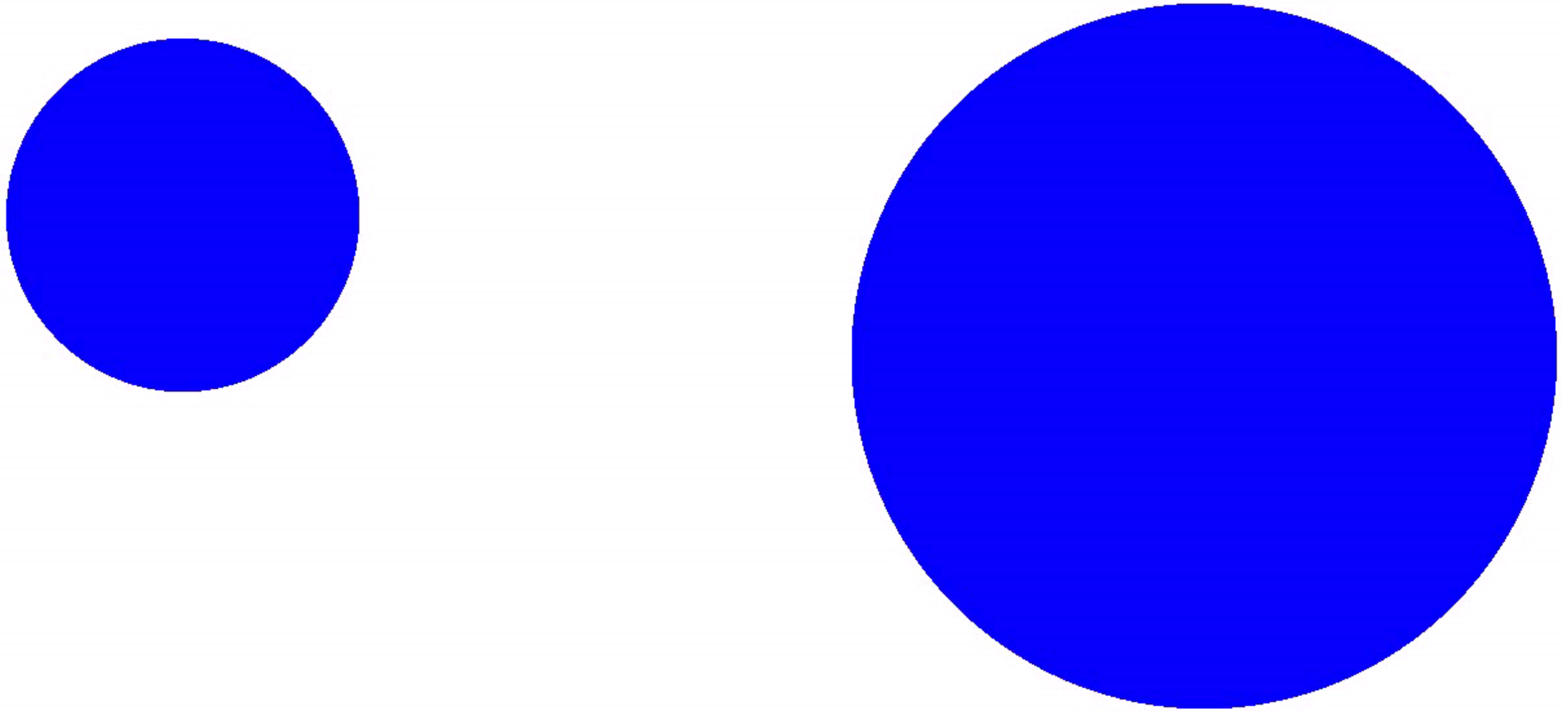
Phase Separation

Cahn-Hilliard Equation

$$\frac{\partial m}{\partial t} = \nabla[f(m)\nabla m \; - \kappa\nabla(\Delta m)]$$



Ostwald Ripening



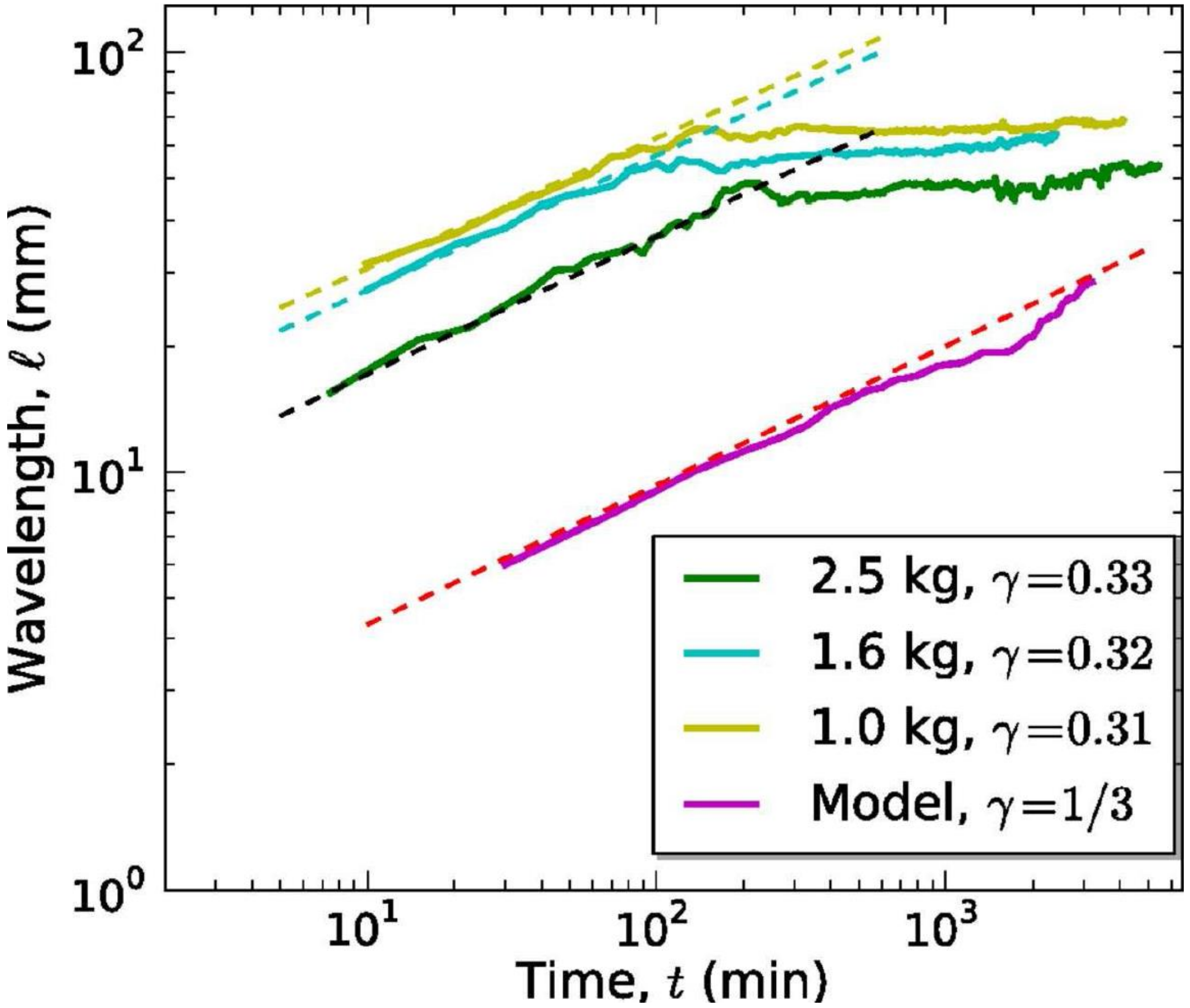
Ostwald Ripening in mussels

Lifschitz-Slyozov law

Evolution of growth

$$\langle Radius(t)\rangle \sim t^{1/3}$$

Lifschitz-Slyozov law breaks down after few hours.



Quan-Xing Liu, *et al,* PNAS (2013)

Behaviour of Spatial Patterns

Summary

Patterns in many systems

Emergence of patterns: Turing instability

Dynamics of patterns:

SLOW pattern adaptation

FAST pattern degradation

Slides available at bastiaansen.github.io