Pattern formation on progressively less regular networks

2024-07-05, Yerseke
Robbin Bastiaansen
(r.bastiaansen@uu.nl)

Pattern formation on progressively less regular networks

What if systems get more connected?

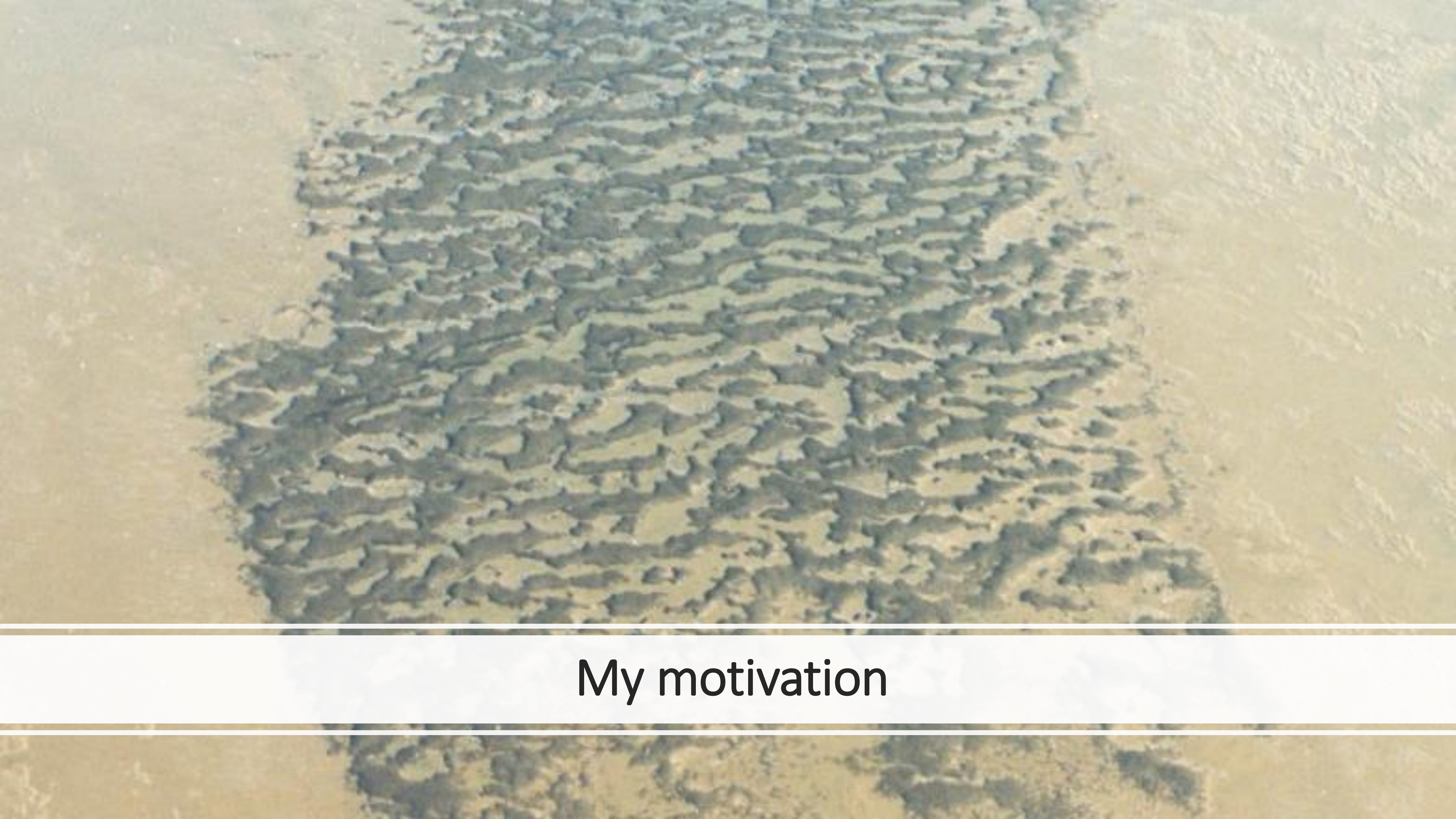
slides at bastiaansen.github.io

based on
MSc thesis by Margriet Spoorenberg
'Pattern Formation on Progressively Less Regular Networks'

2024-07-05, Yerseke

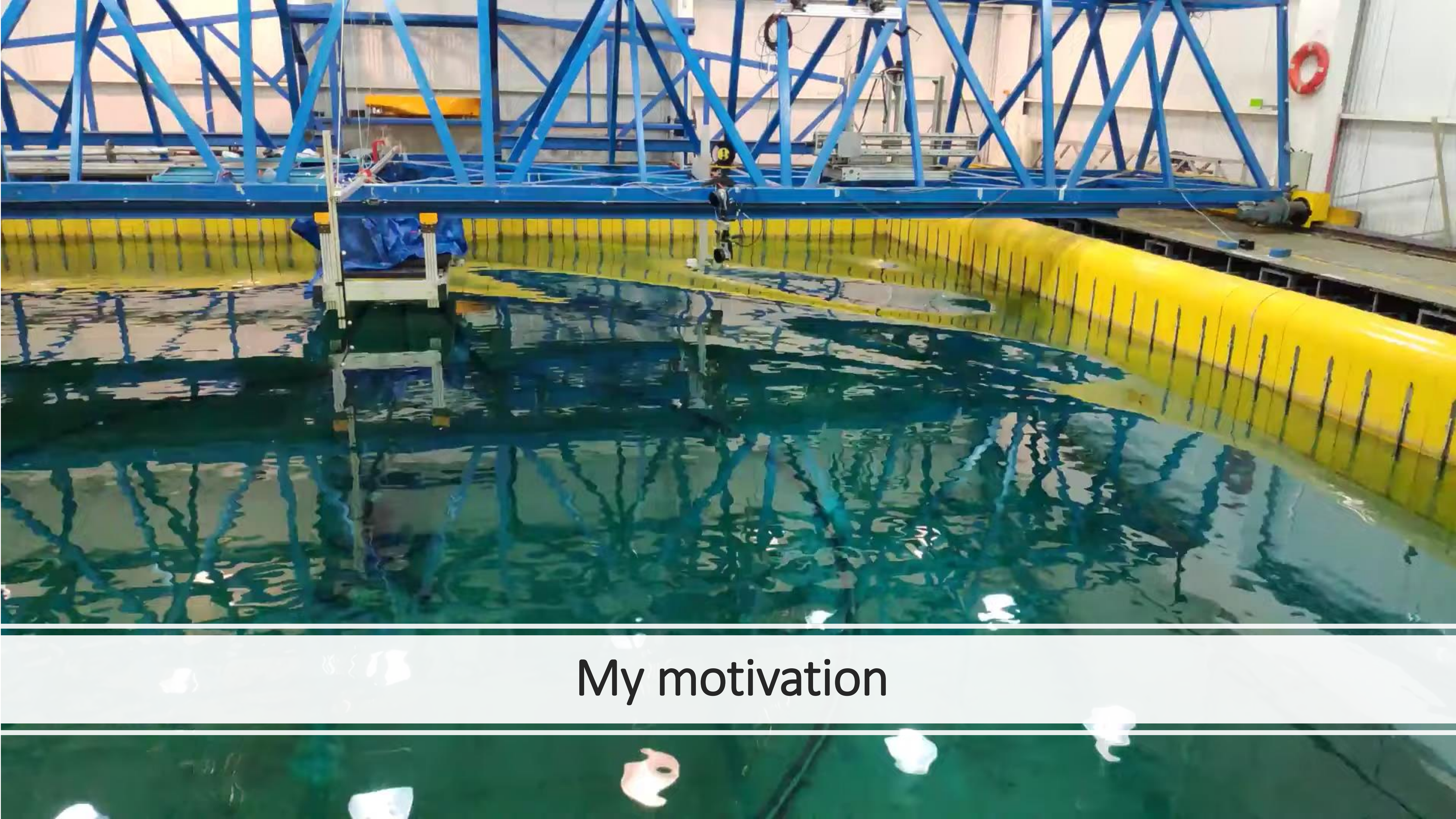
Robbin Bastiaansen
(r.bastiaansen@uu.nl)

My motivation



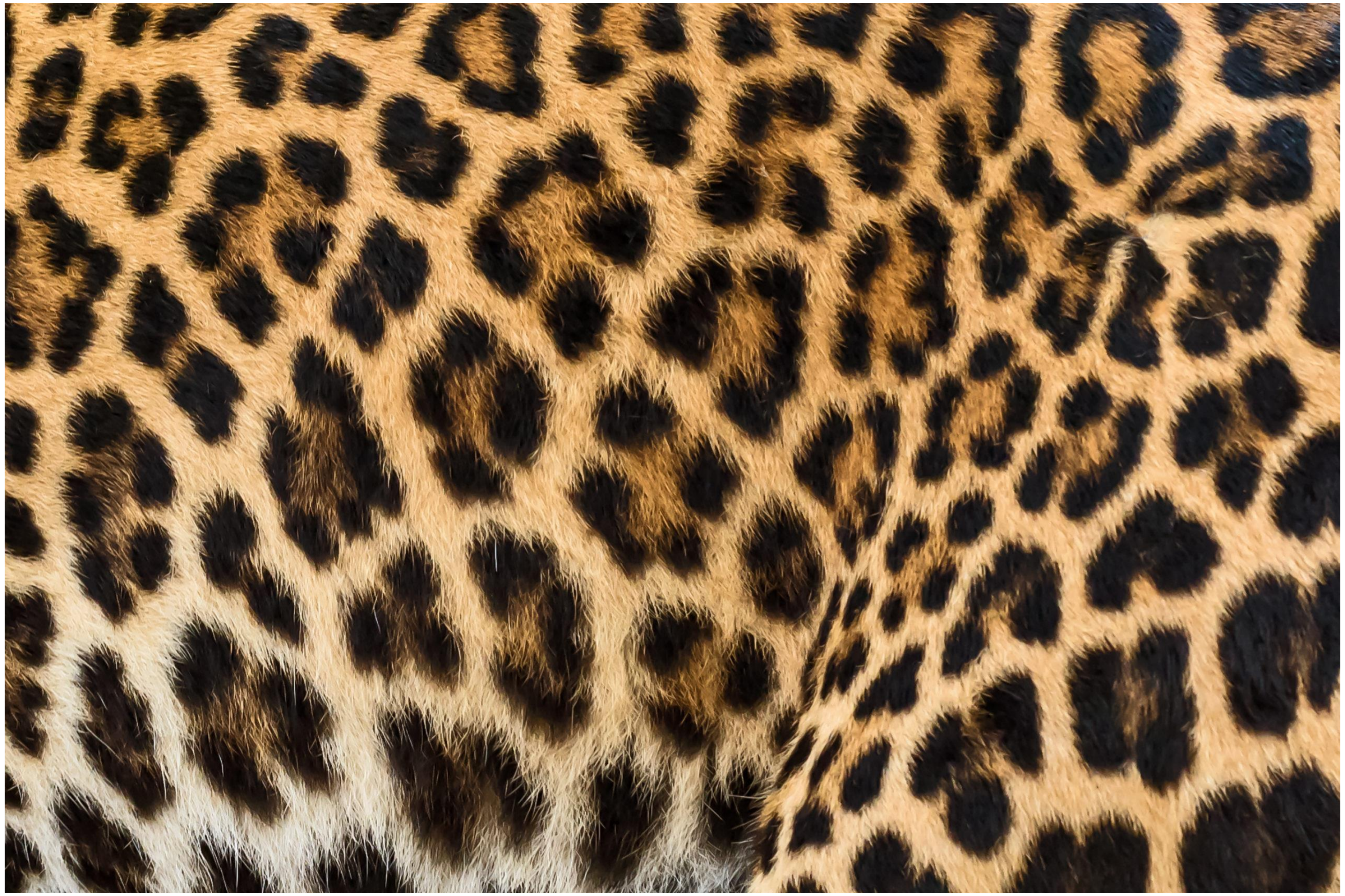
My motivation

My motivation



My motivation

Turing Patterns

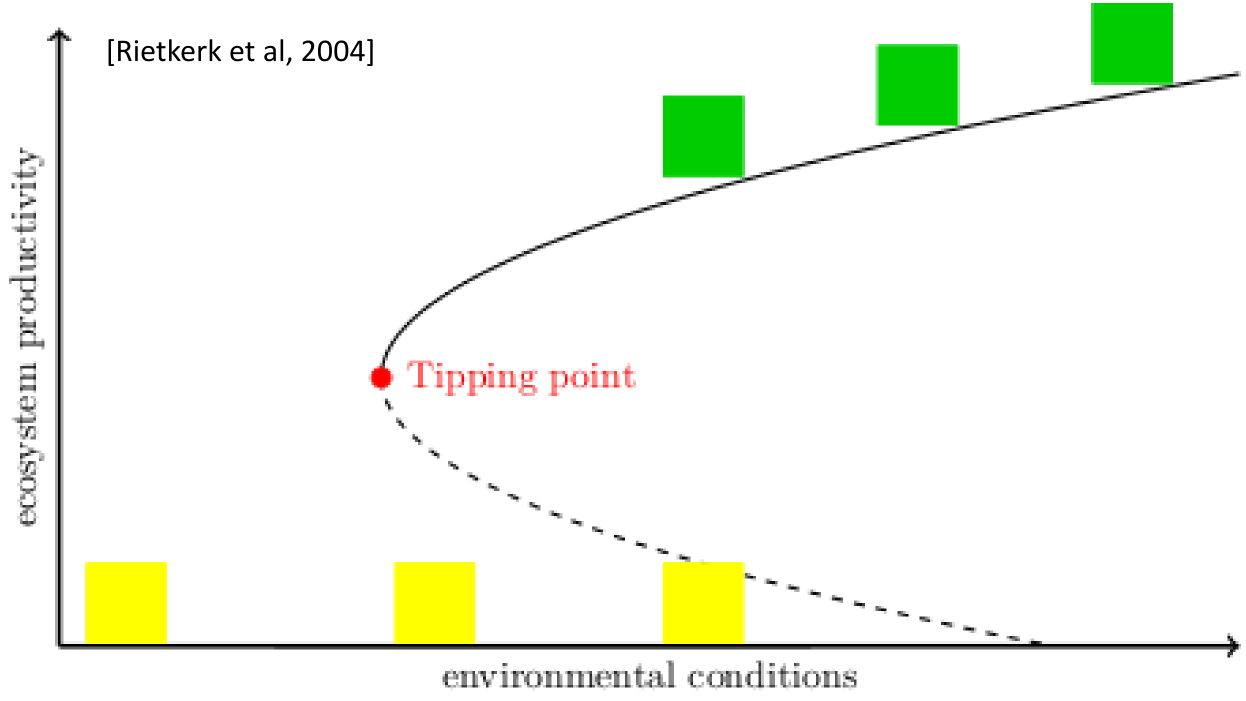


[wikipedia]

Seminal paper in 1952: “The chemical basis of morphogenesis”

Reaction-Diffusion Equation for Dryland Ecosystems

$$
\begin{aligned}
w_t &= w_{xx} - w + a - wv^2 \\
v_t &= D^2 v_{xx} - mv + wv^2
\end{aligned}
$$



Turing patterns

$$\begin{cases} \dfrac{du}{dt} = f(u,v) + D_u \, \Delta u \\ \dfrac{dv}{dt} = g(u,v) + D_v \, \Delta v \end{cases}$$

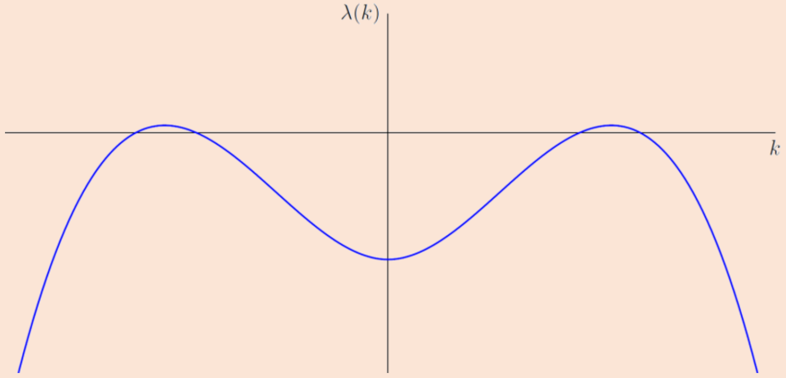
Turing bifurcation

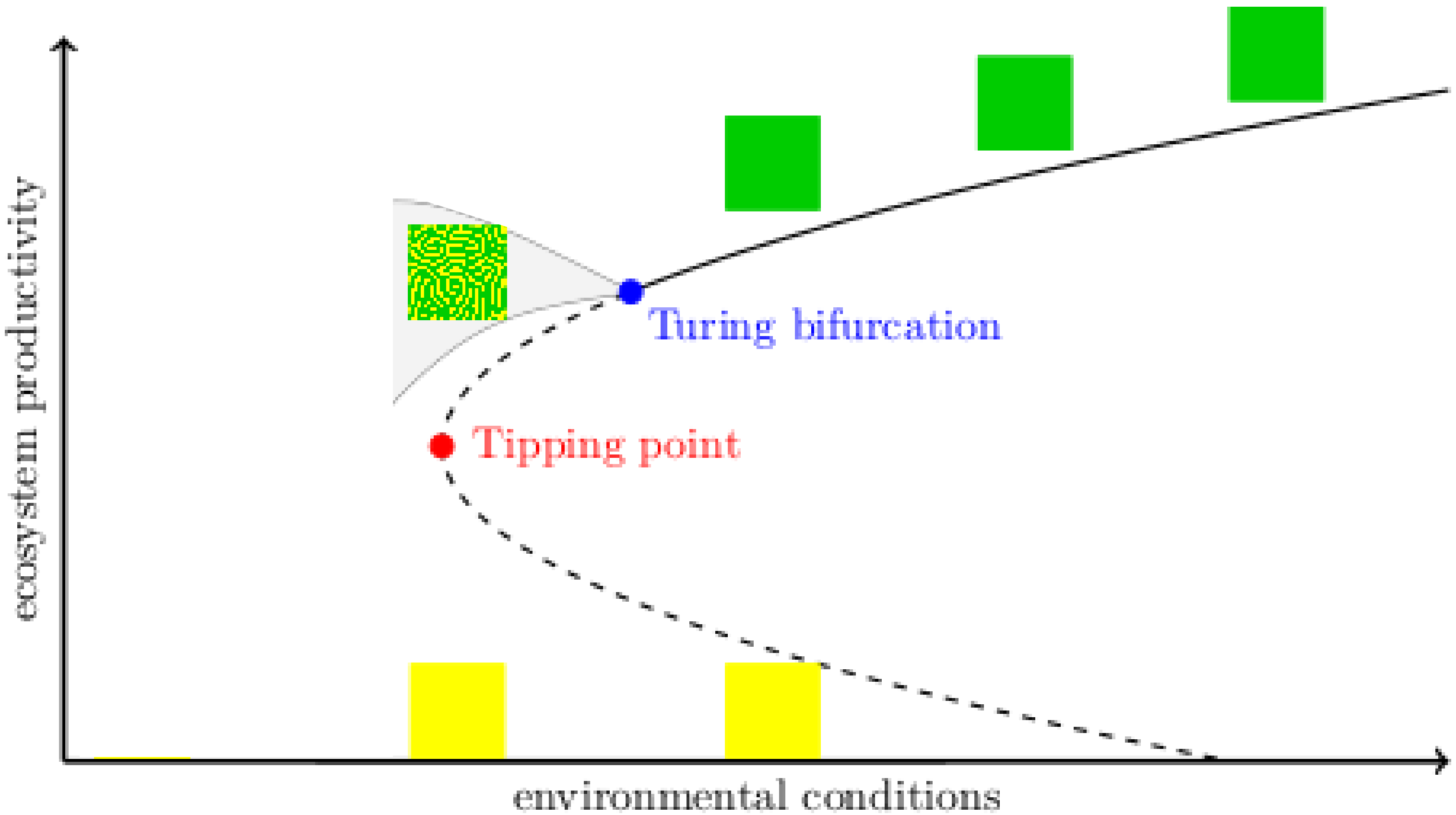
Instability to non-uniform perturbations

$$\binom{u}{v} = \binom{u_*}{v_*} + e^{\lambda t} e^{ikx} \binom{\bar{u}}{\bar{v}}$$

→ Dispersion relation

$$\lambda(k) = \cdots$$





Weakly non-linear analysis

Ginzburg-Landau equation / Amplitude Equation

& Eckhaus/Benjamin-Feir-Newel criterion

[Eckhaus, 1965; Benjamin & Feir, 1967; Newell, 1974]

Dispersion Relations

$$\begin{cases} \dfrac{du}{dt} = f(u,v) + \quad \Delta u \\ \dfrac{dv}{dt} = g(u,v) + D\Delta v \end{cases}$$

Set $\binom{u}{v} = \begin{pmatrix} u_* \\ v_* \end{pmatrix} + e^{\lambda t}\, e^{ikx} \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix}$

Temporal Evolution

Spatial Structure

Component Structure

Meaningfull choice since:

$$\Delta\, e^{ikx} = -k^2\, e^{ikx}$$

eigenvalue

eigenfunction

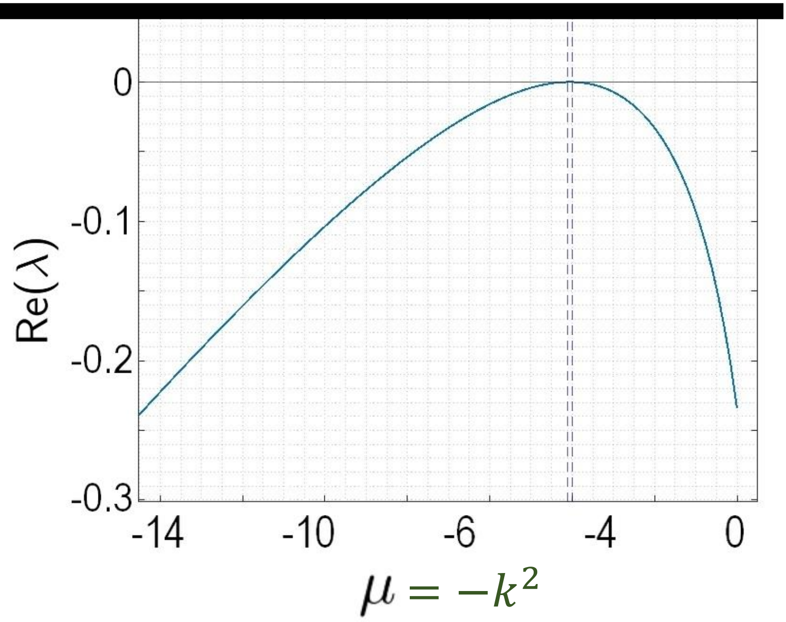
Dispersion Relations

$$\begin{cases} \dfrac{du}{dt} = f(u,v) + \quad \Delta u \\ \dfrac{dv}{dt} = g(u,v) + D\Delta v \end{cases}$$

Set $\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u_* \\ v_* \end{pmatrix} + e^{\lambda t}\, e^{ikx} \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix}$

To obtain per wavenumber k:

$$\lambda(k) \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} = -k^2 \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} + \begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix} \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix}$$



This yields the dispersion relation $\lambda_1(k) = \cdots, \lambda_2(k) = \cdots$

Perturbation Evolution on Different Domains

- **Infinite Domains:**

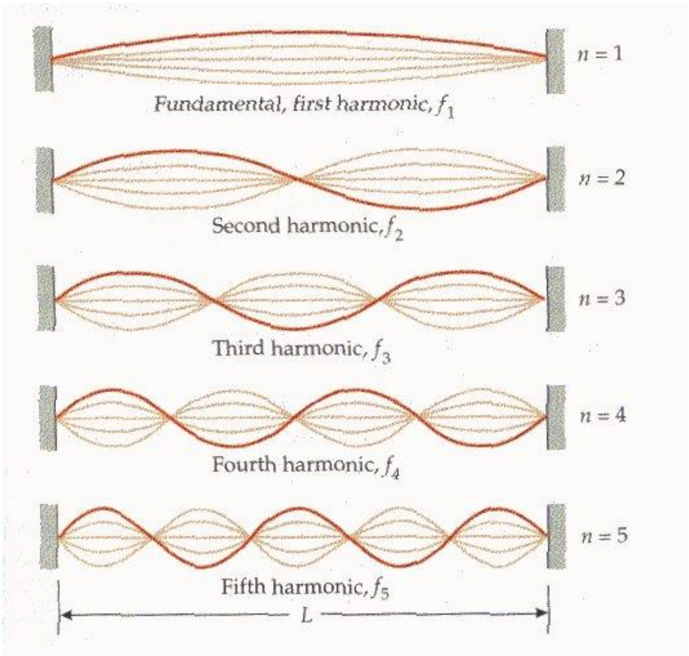
  All wavenumbers k possible

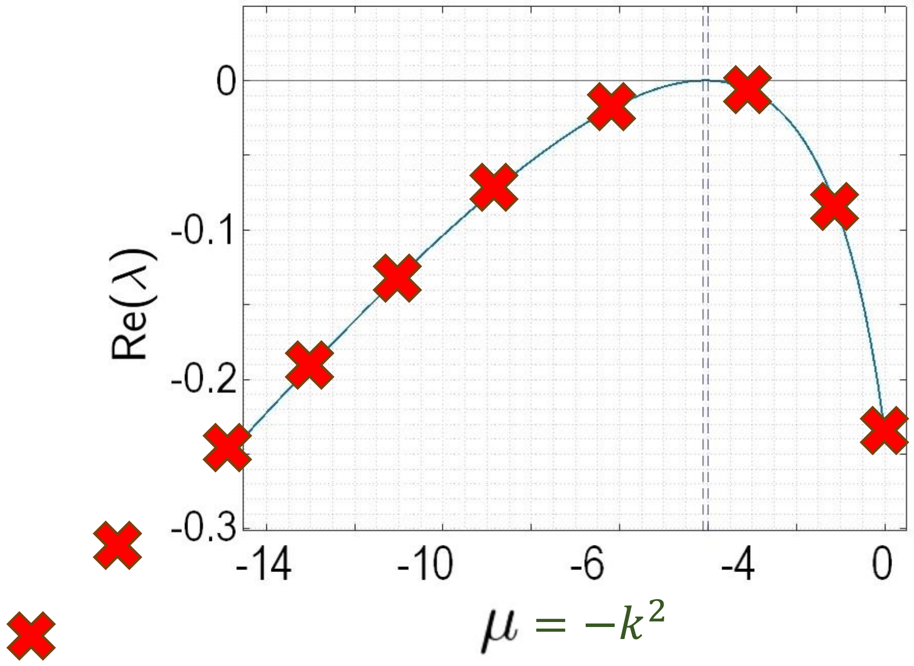
$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u_* \\ v_* \end{pmatrix} + \int_{-\infty}^{\infty} dk \sum_{i=1}^{2} c_{k,i}\, e^{\lambda_i(k)\, t}\, e^{ikx} \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix}_{k,i}$$

- **Finite Domains:**

  Only wavenumbers k possible that fit the domain

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u_* \\ v_* \end{pmatrix} + \sum_{k \in \Gamma} \sum_{i=1}^{2} c_{k,i}\, e^{\lambda_i(k)\, t}\, e^{ikx} \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix}_{k,i}$$





Perturbation Evolution on Different Domains

- **Infinite Domains:**

  All wavenumbers k possible

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u_* \\ v_* \end{pmatrix} + \int_{-\infty}^{\infty} dk \sum_{i=1}^{2} c_{k,i}\, e^{\lambda_i(k)\, t}\, e^{ikx} \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix}_{k,i}$$

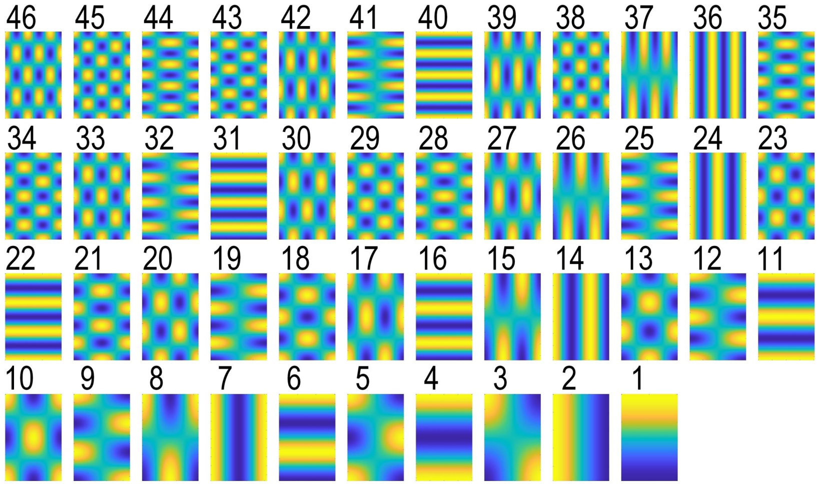
- **Finite Domains:**

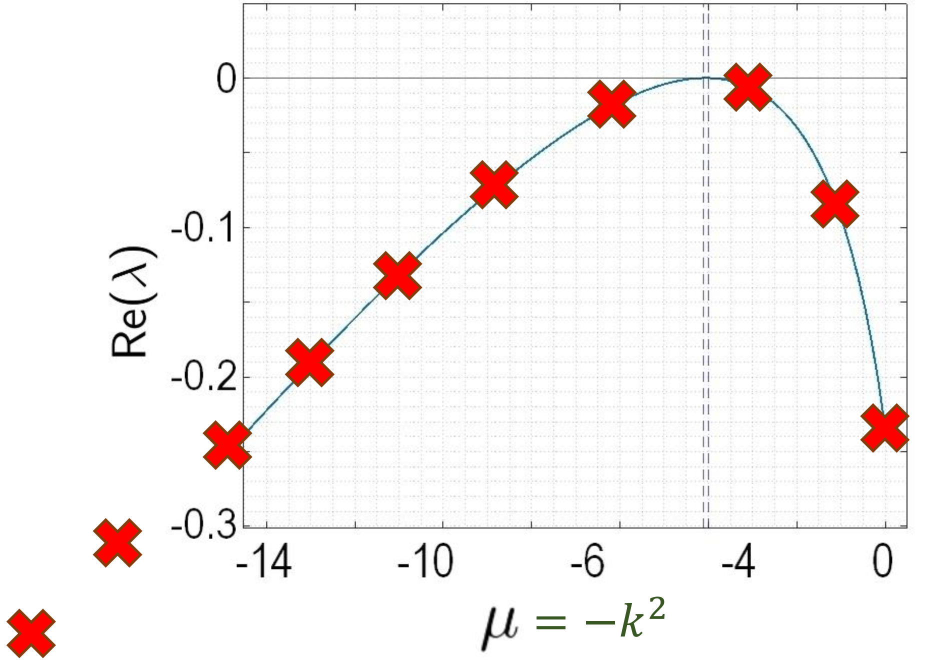
  Only wavenumbers k possible that fit the domain

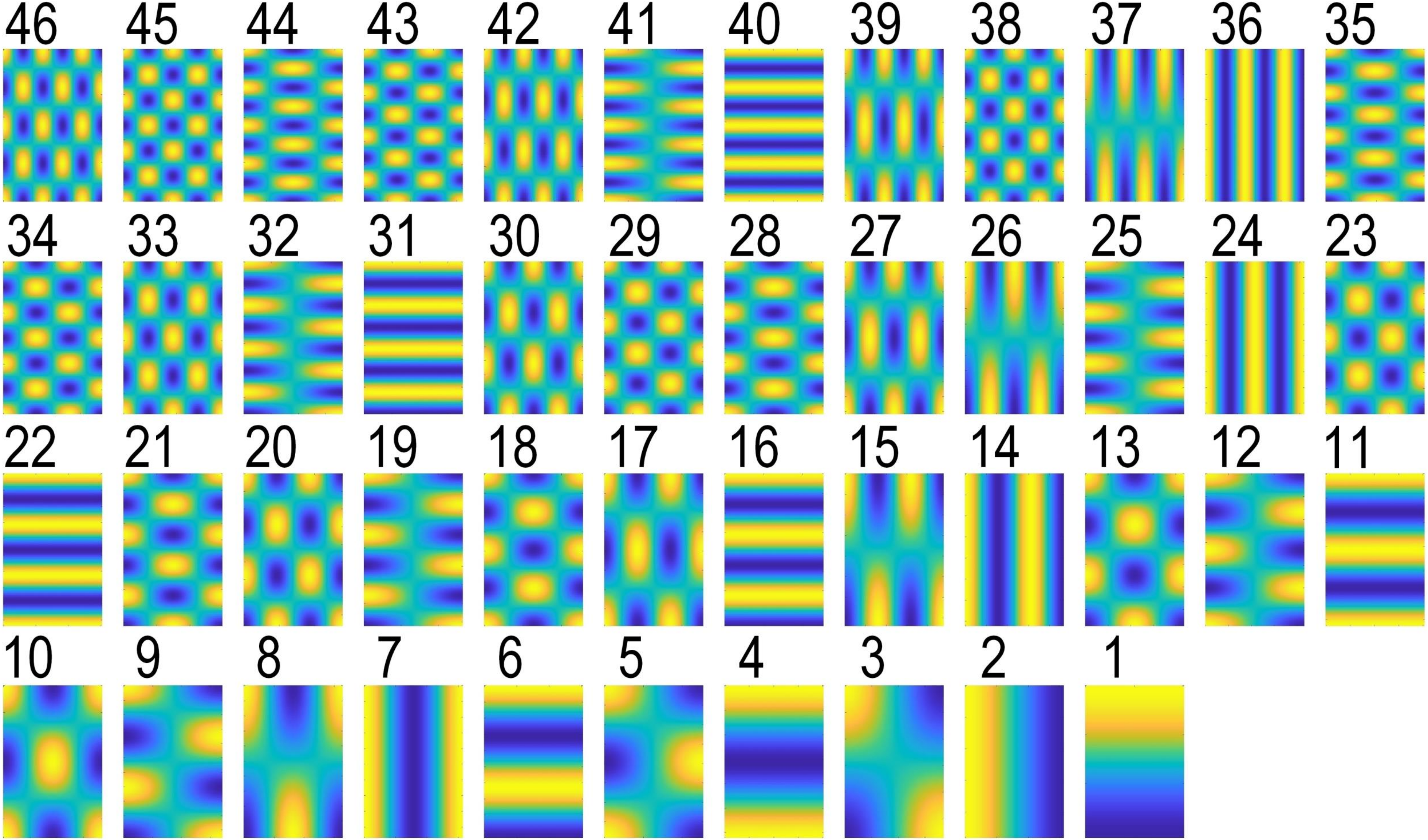
$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u_* \\ v_* \end{pmatrix} + \sum_{k \in \Gamma} \sum_{i=1}^{2} c_{k,i}\, e^{\lambda_i(k)\, t}\, e^{ikx} \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix}_{k,i}$$

- **Discretised Finite Domains:**

  Highest wavenumbers also k become impossible






46
45
44
43
42
41
40
39
38
37
36
35
34
33
32
31
30
29
28
27
26
25
24
23
22
21
20
19
18
17
16
15
14
13
12
11
10
9
8
7
6
5
4
3
2
1

Perturbation Evolution on Different Domains

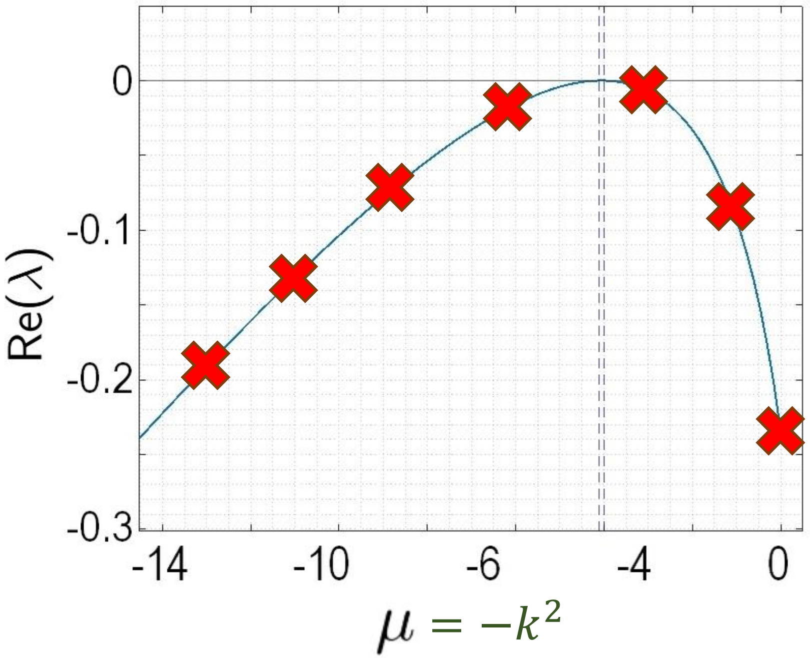
- **Infinite Domains:**
  All wavenumbers k possible

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u_* \\ v_* \end{pmatrix} + \int_{-\infty}^{\infty} dk \sum_{i=1}^{2} c_{k,i}\, e^{\lambda_i(k)\, t}\, e^{ikx} \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix}_{k,i}$$

- **Finite Domains:**
  Only wavenumbers k possible that fit the domain

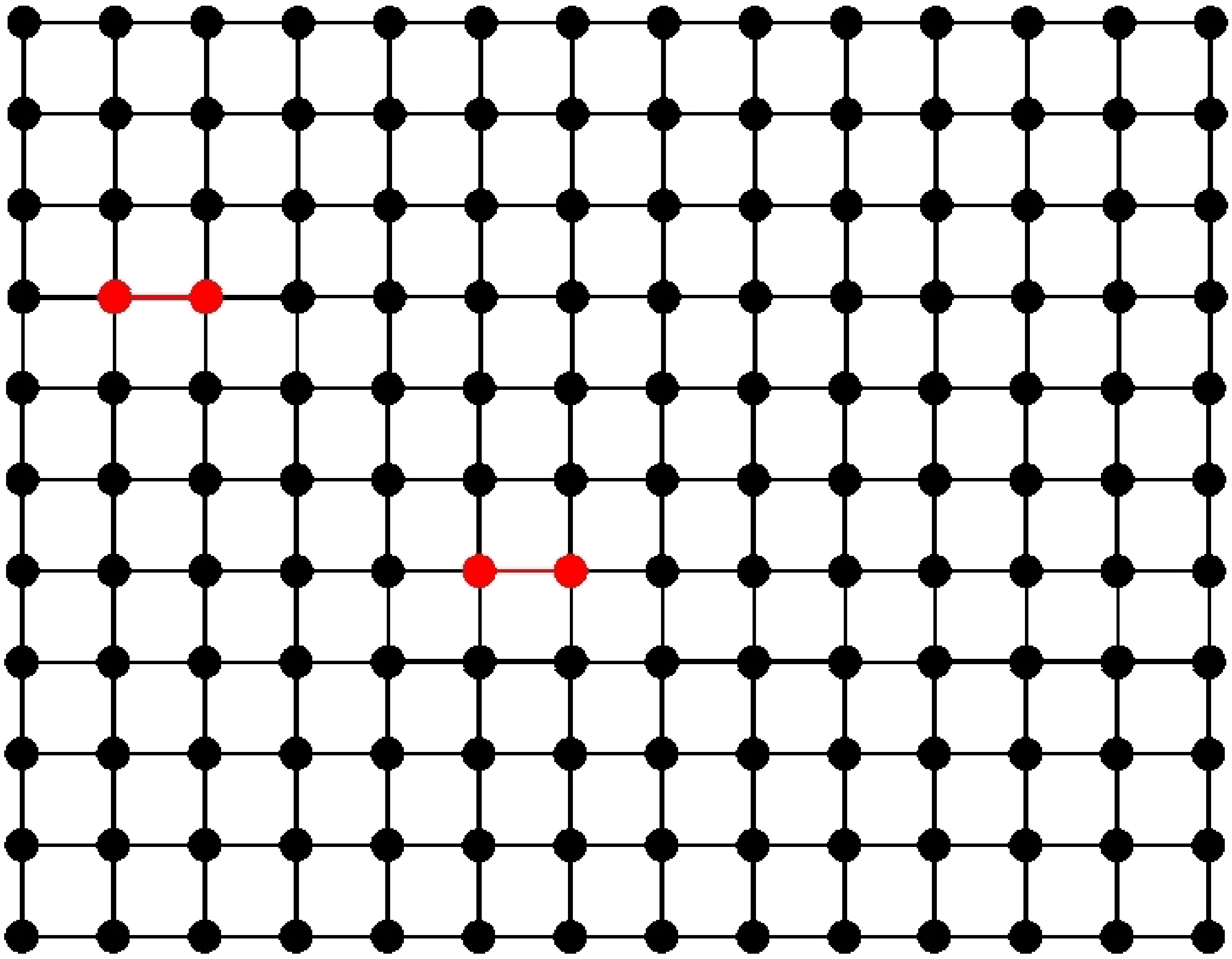
$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u_* \\ v_* \end{pmatrix} + \sum_{k\in\Gamma} \sum_{i=1}^{2} c_{k,i}\, e^{\lambda_i(k)\, t}\, e^{ikx} \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix}_{k,i}$$

- **Discretised Finite Domains:**
  Highest wavenumbers also k become impossible

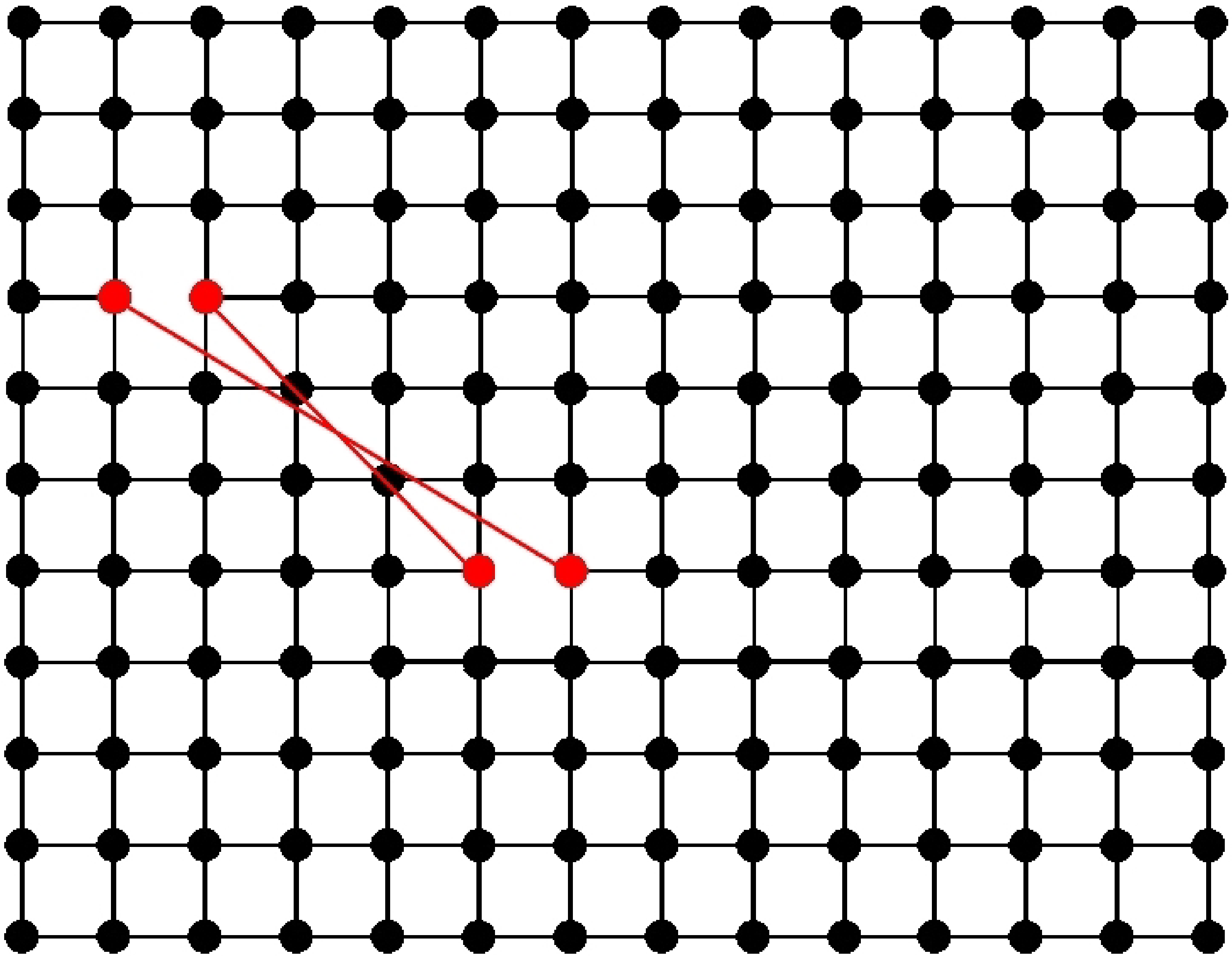


- **“Less regular Networks”:**
  See rest of this presentation

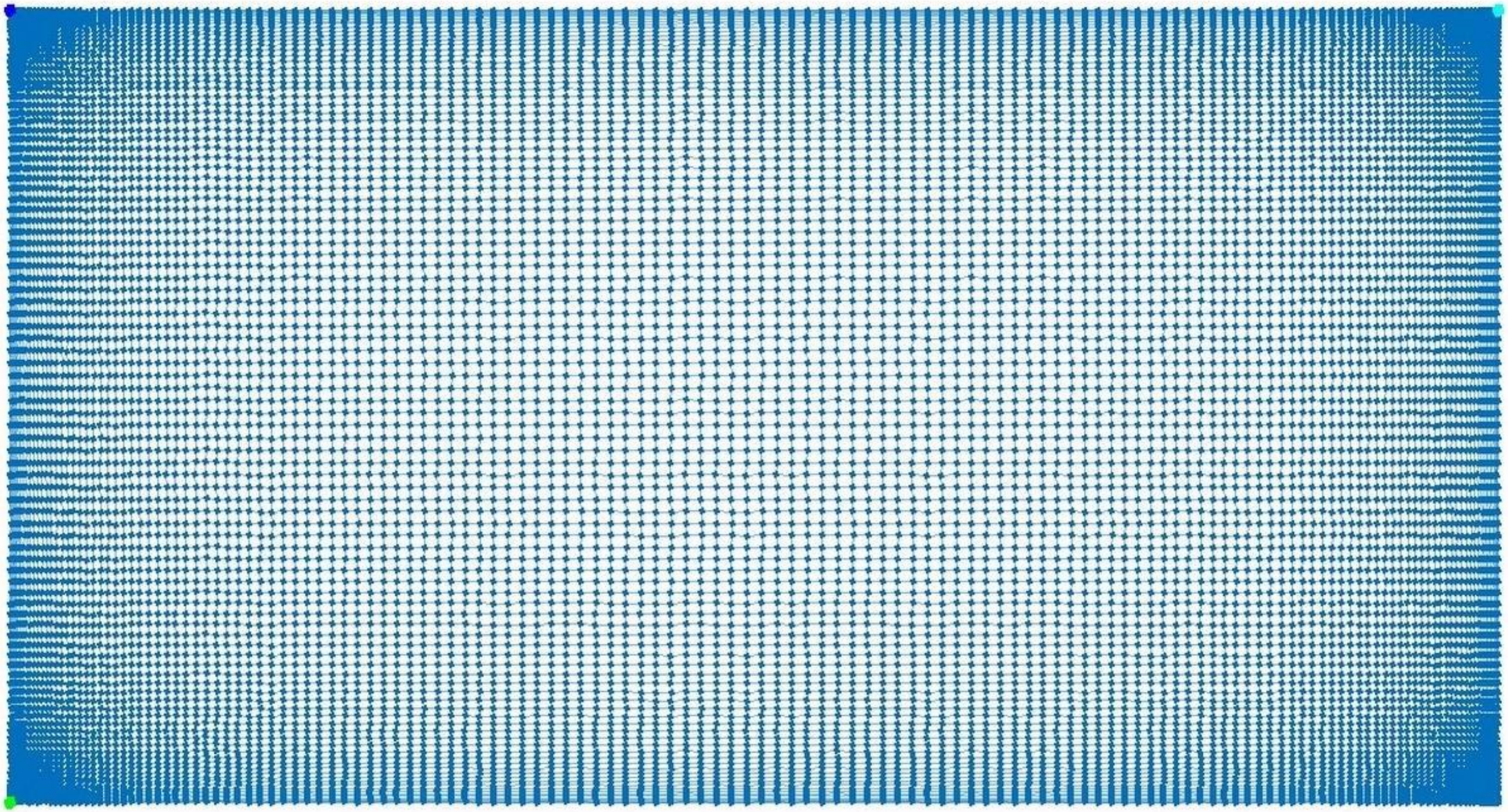
Progressively Less Regular Networks



Progressively Less Regular Networks



Progressively Less Regular Networks

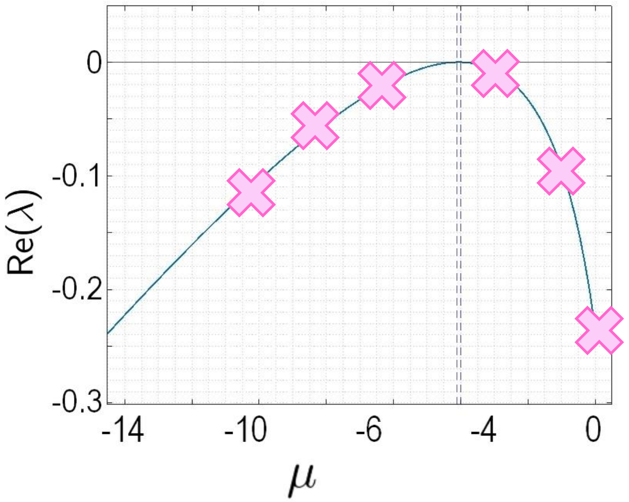


Dispersion Relations on Network

$$\begin{cases} \dfrac{du}{dt} = f(u,v) + \Delta u \\ \dfrac{dv}{dt} = g(u,v) + D\Delta v \end{cases} \quad \Rightarrow \quad \begin{cases} \dfrac{du}{dt} = f(u,v) + Lu \\ \dfrac{dv}{dt} = g(u,v) + DLv \end{cases}$$

Graph Laplacian
(almost)

Set $\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u_* \\ v_* \end{pmatrix} + e^{\lambda t}\, e^{ikx} \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u_* \\ v_* \end{pmatrix} + e^{\lambda t}\, \vec{\Psi} \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix}$



Temporal Evolution

Spatial Structure

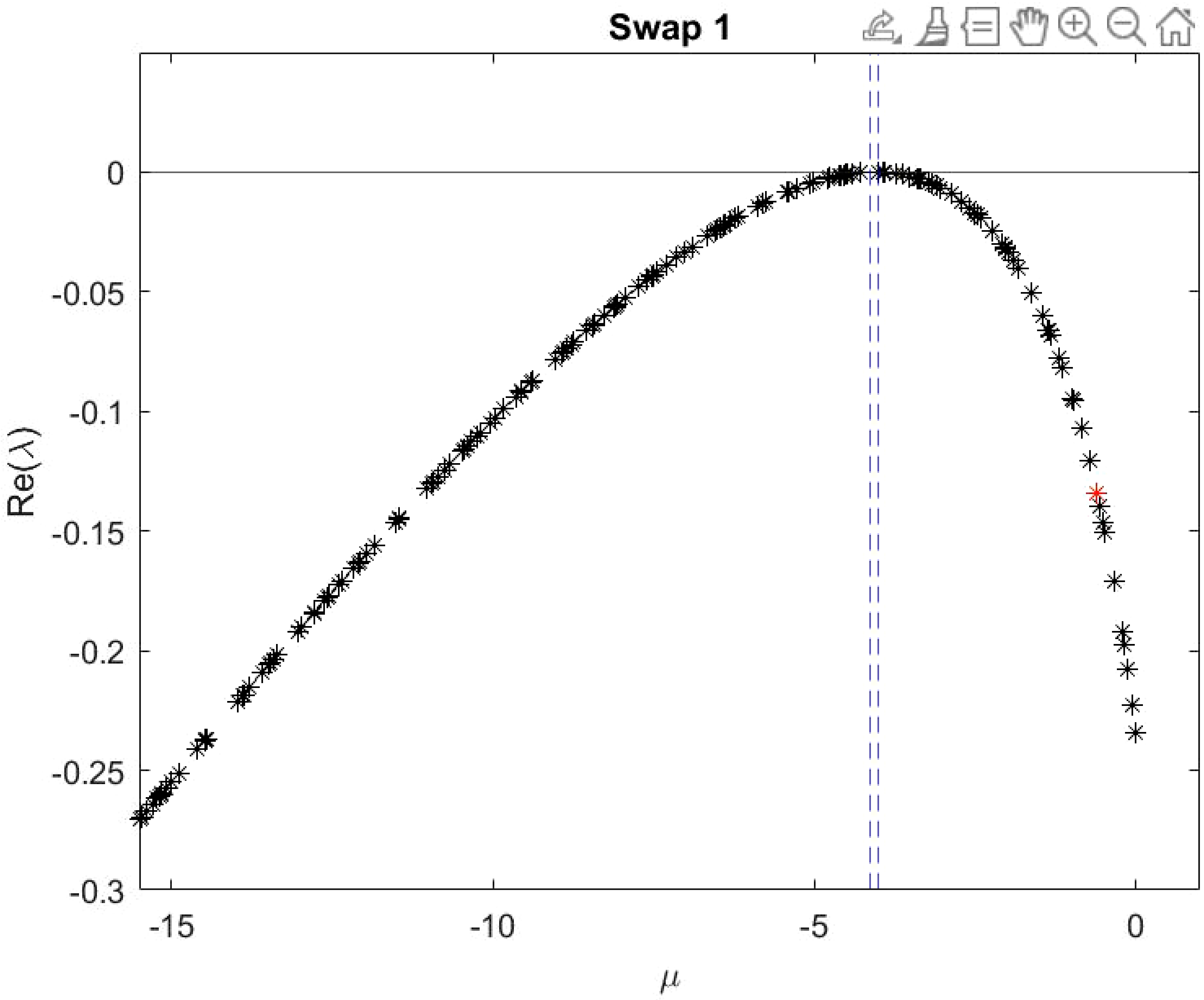
Component Structure

Spatial Structure

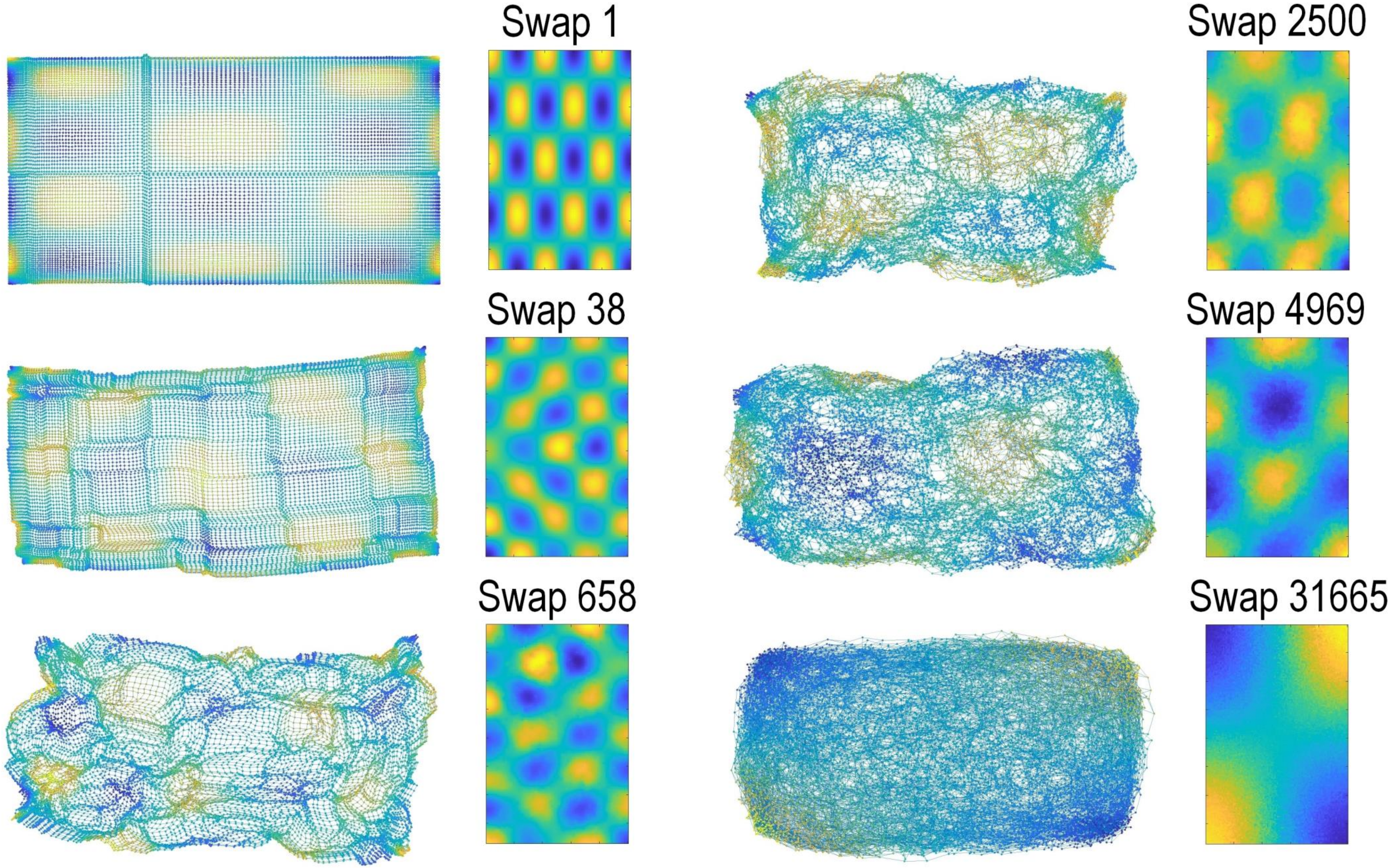
Meaningfull choice since:
$\Delta\, e^{ikx} = -k^2\, e^{ikx}$
eigenvalue
eigenfunction

Meaningfull choice IF:
$L\vec{\Psi} = \mu\vec{\Psi}$

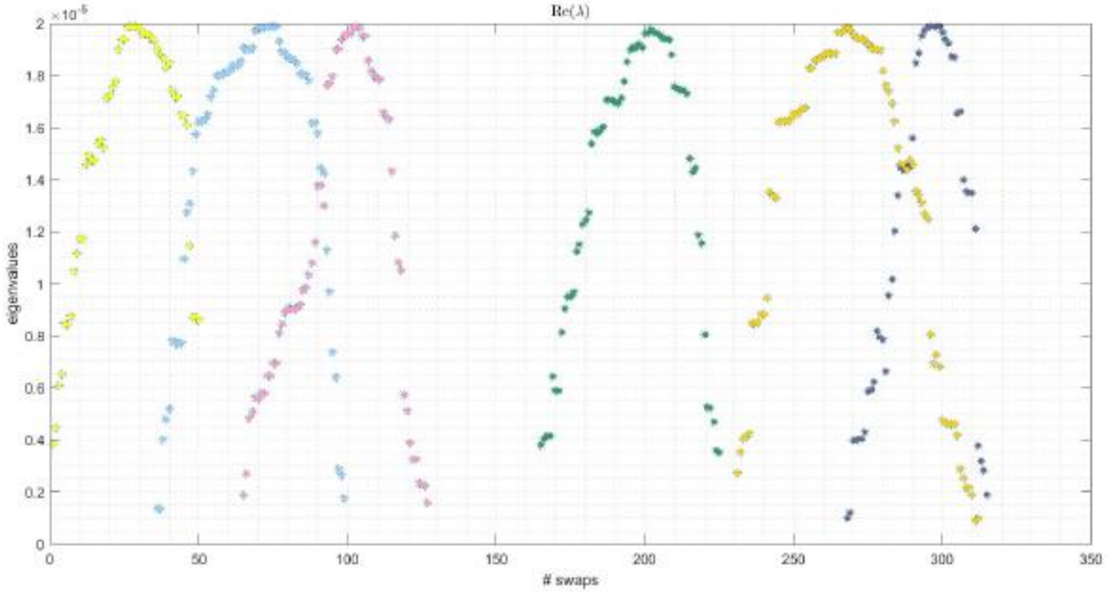
Dispersion Relation for Progressively Less Regular Networks



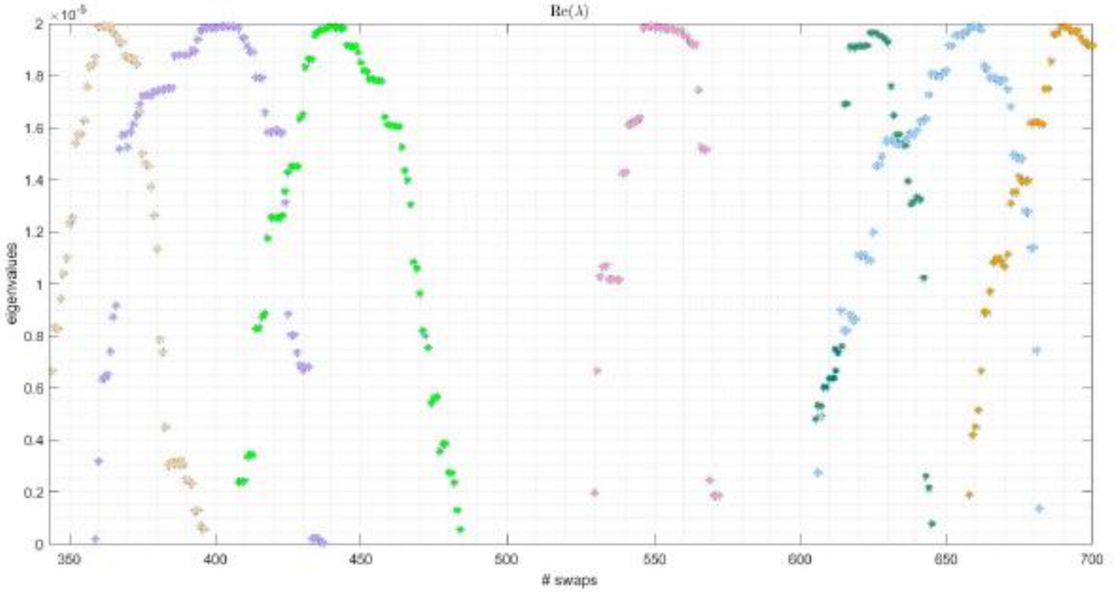
Eigenfunctions on Network



Tracking eigenfunctions



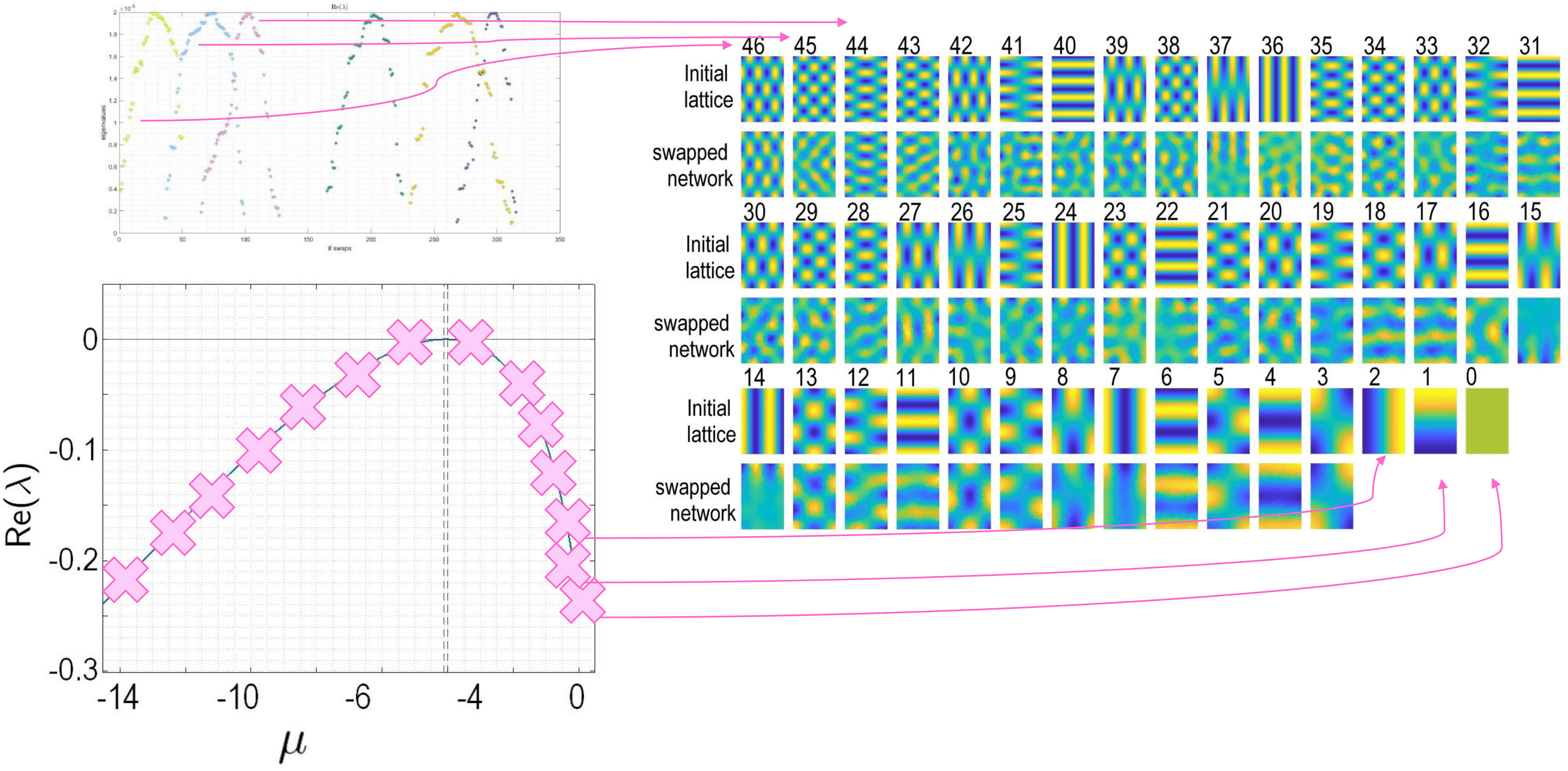
(a) Eigenvalues followed through swaps 1 to 343.



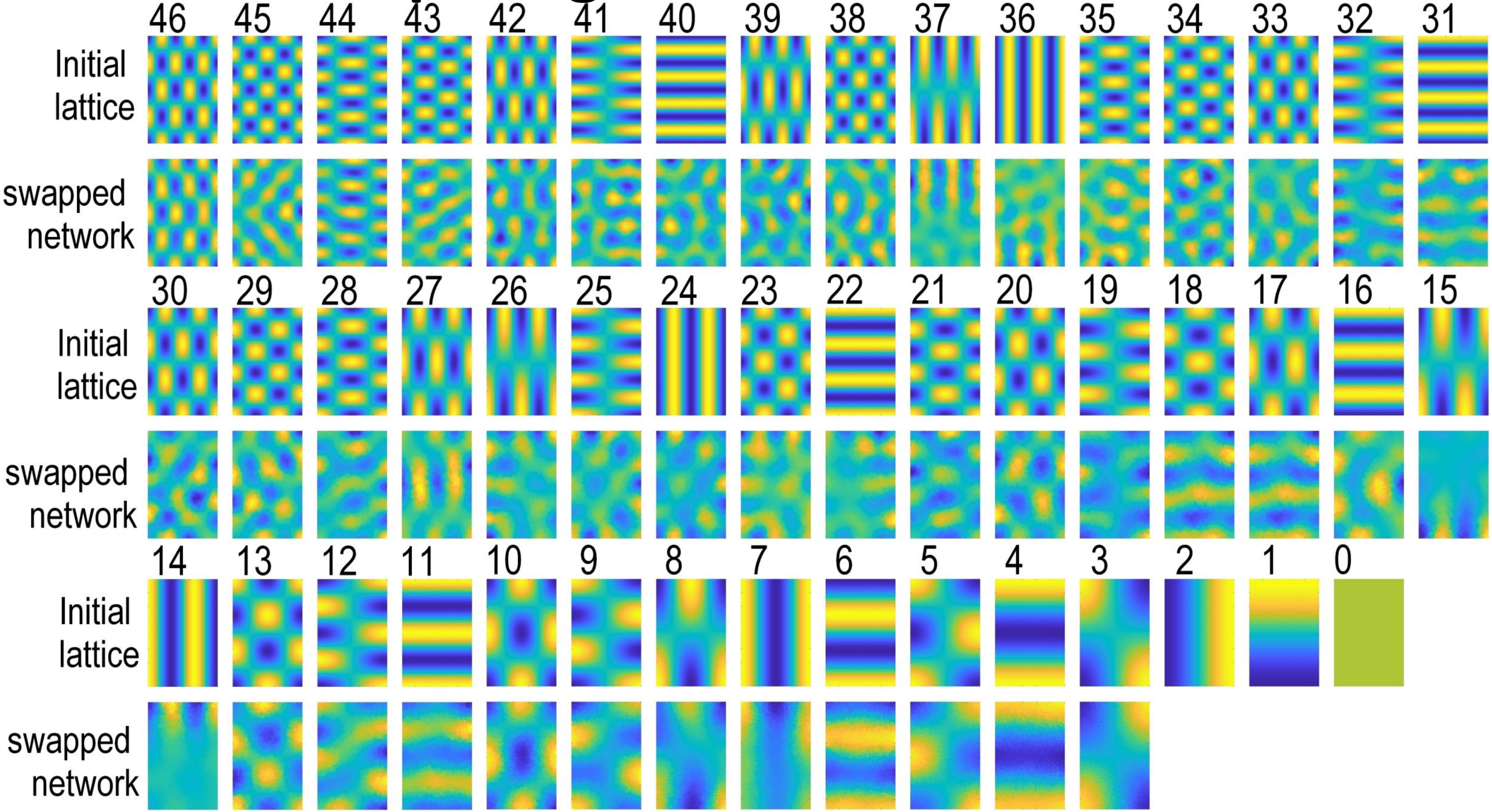
(b) Eigenvalues followed through swaps 344 to 700.

Figure 24: First thirteen patterns for which the real part of the eigenvalue, Re(λ), is larger that zero, followed through the first 700 swaps.

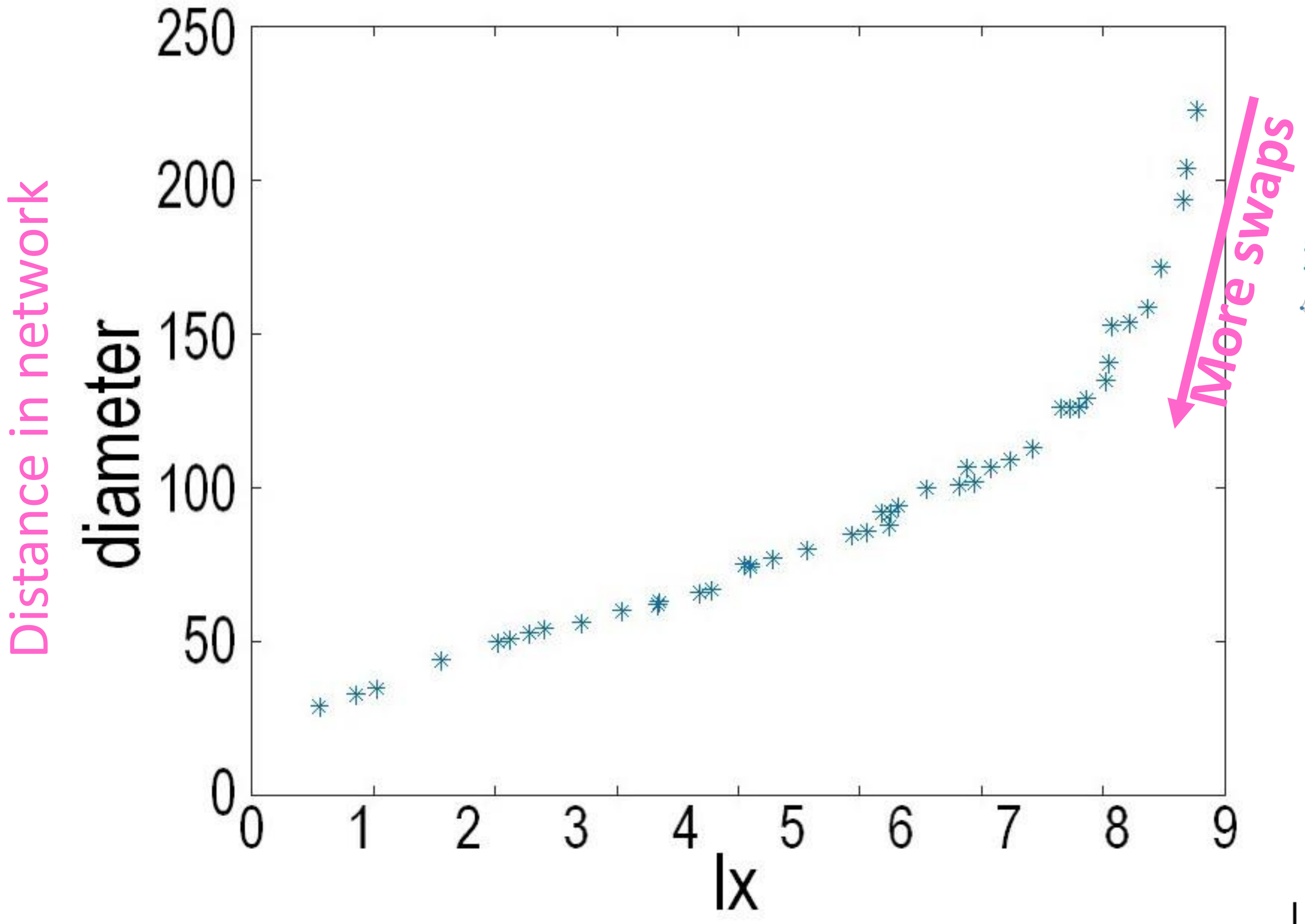
Comparing to bounded domain



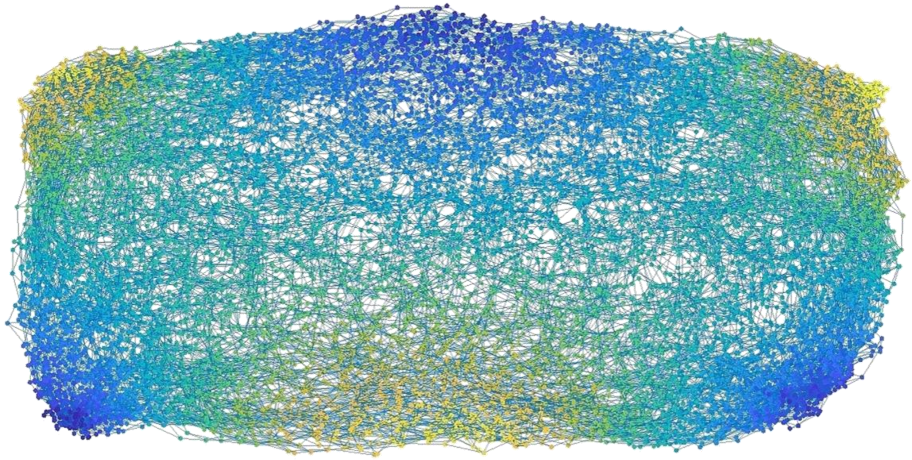
Comparing to bounded domain



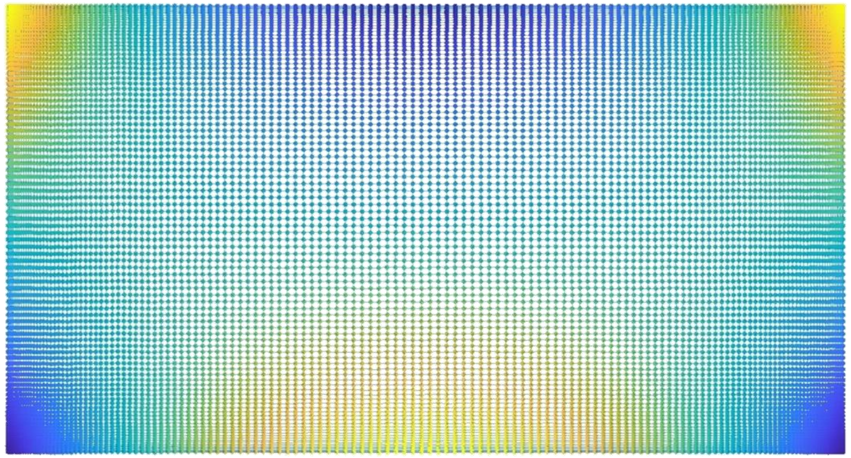
Comparable to smaller and smaller domains?



Length of domain



Network after 17170 swaps



Lattice size 28.9% of the initial domain size

Summary

The more interconnected a network is,

the more it behaves like a system with a smaller spatial domain

→ So also less resilient?

Caveat / Future Work

Now only linear stability analysis with one specific setup,

- Is there limit behaviour for infinite number of swaps?
- Do things change for other network constructions?
- What about nonlinear effects and Turing patterns on network?

MSc thesis by Margriet Spoorenberg
'Pattern Formation on Progressively Less Regular Networks'

slides at bastiaansen.github.io