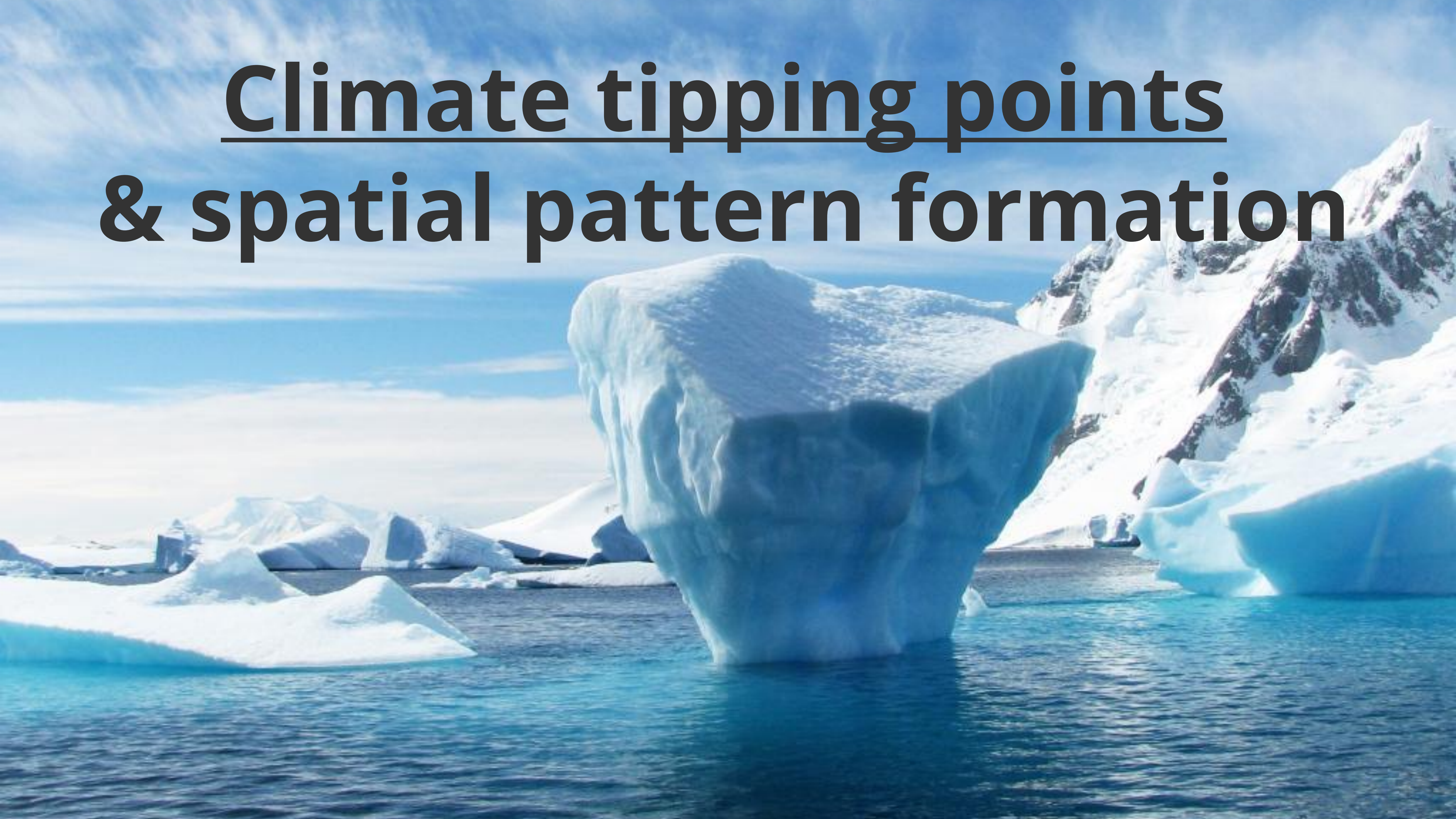
# Climate tipping points & spatial pattern formation

2025-06-04, Abacus's Mathematical Symposium "Nature's Numbers"

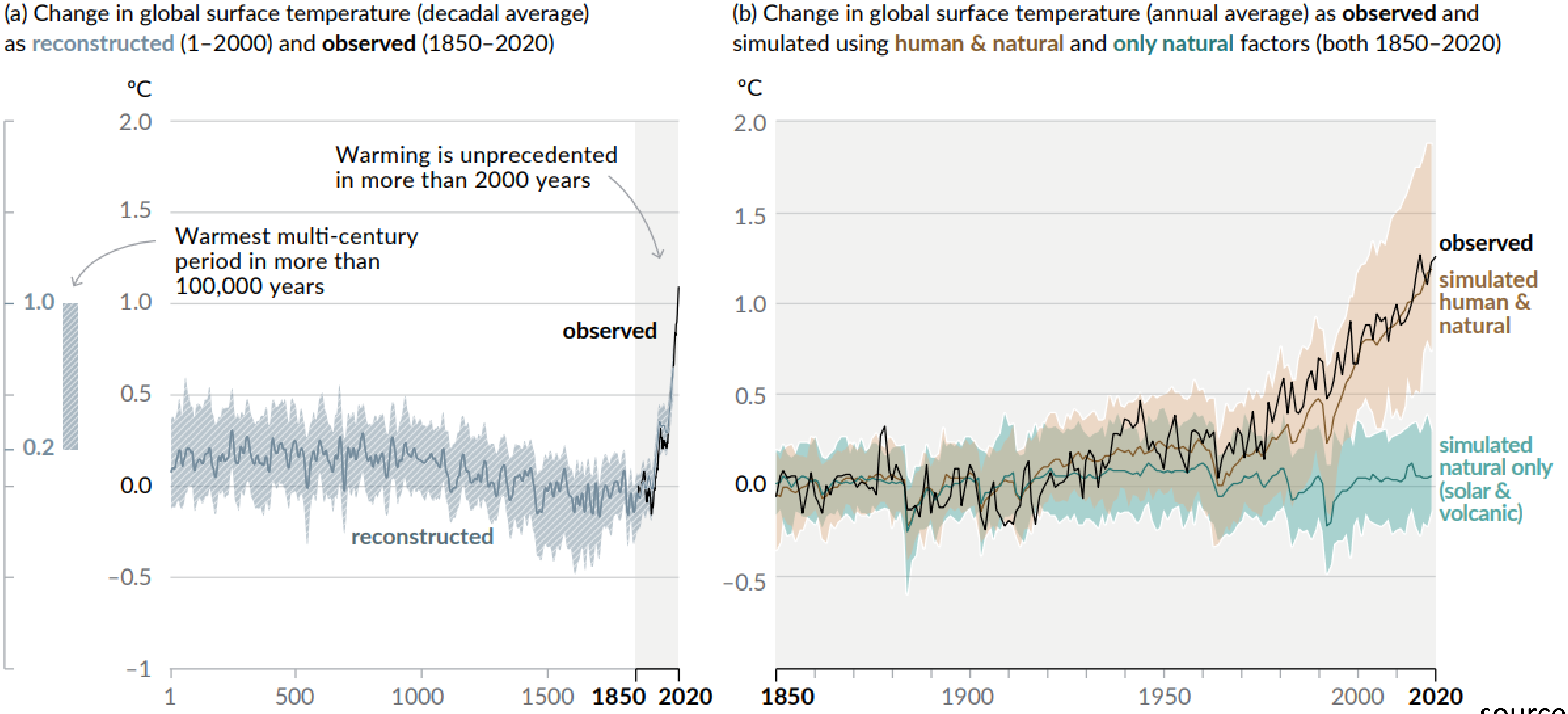
Robbin Bastiaansen (r.bastiaansen@uu.nl)

# Climate tipping points & spatial pattern formation



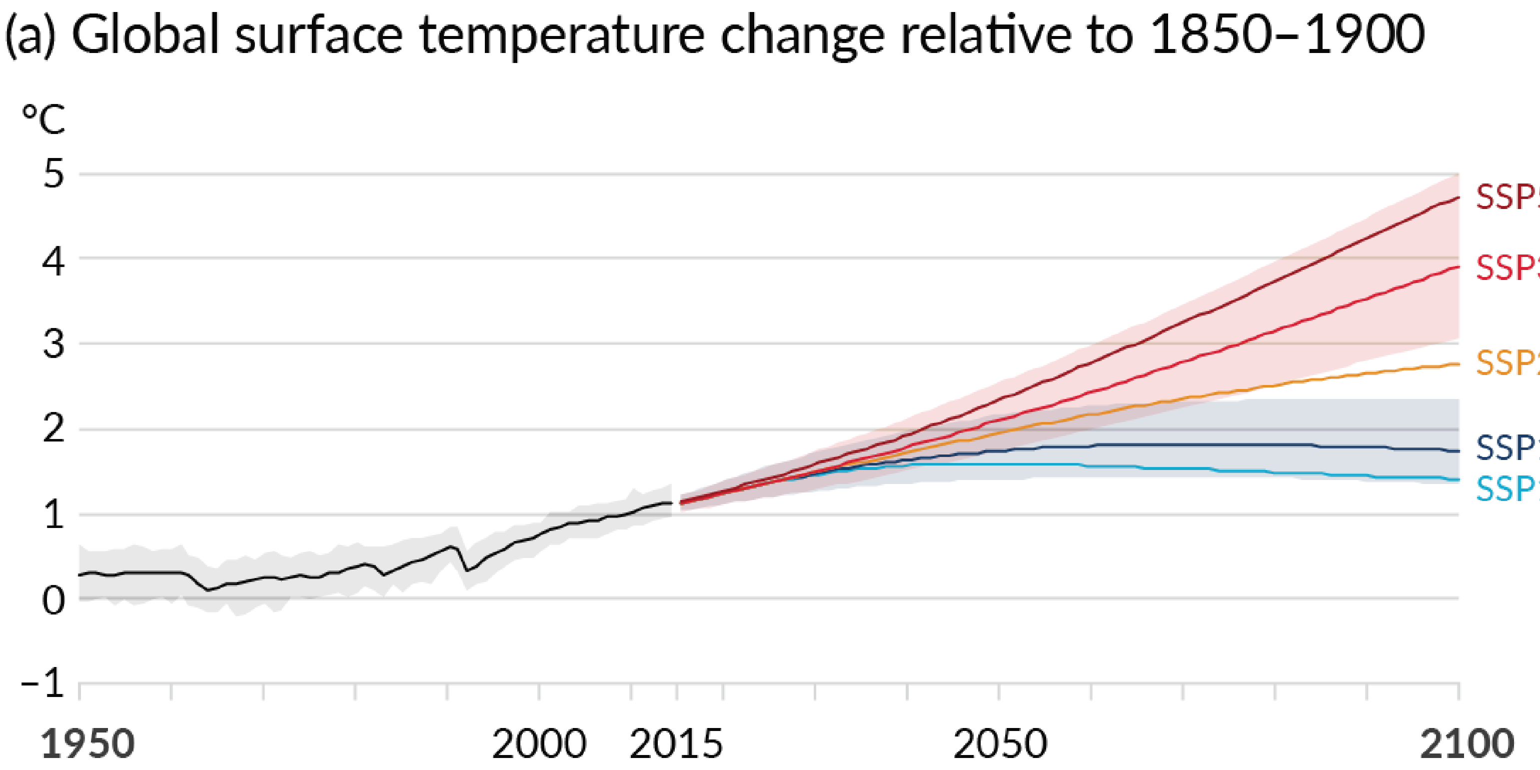
# Human influence has warmed the climate at a rate that is unprecedented in at least the last 2000 years

## Changes in global surface temperature relative to 1850–1900




source: IPCC AR6

# Future climate projections




source: IPCC AR6

**Experiments in Global Climate Models**

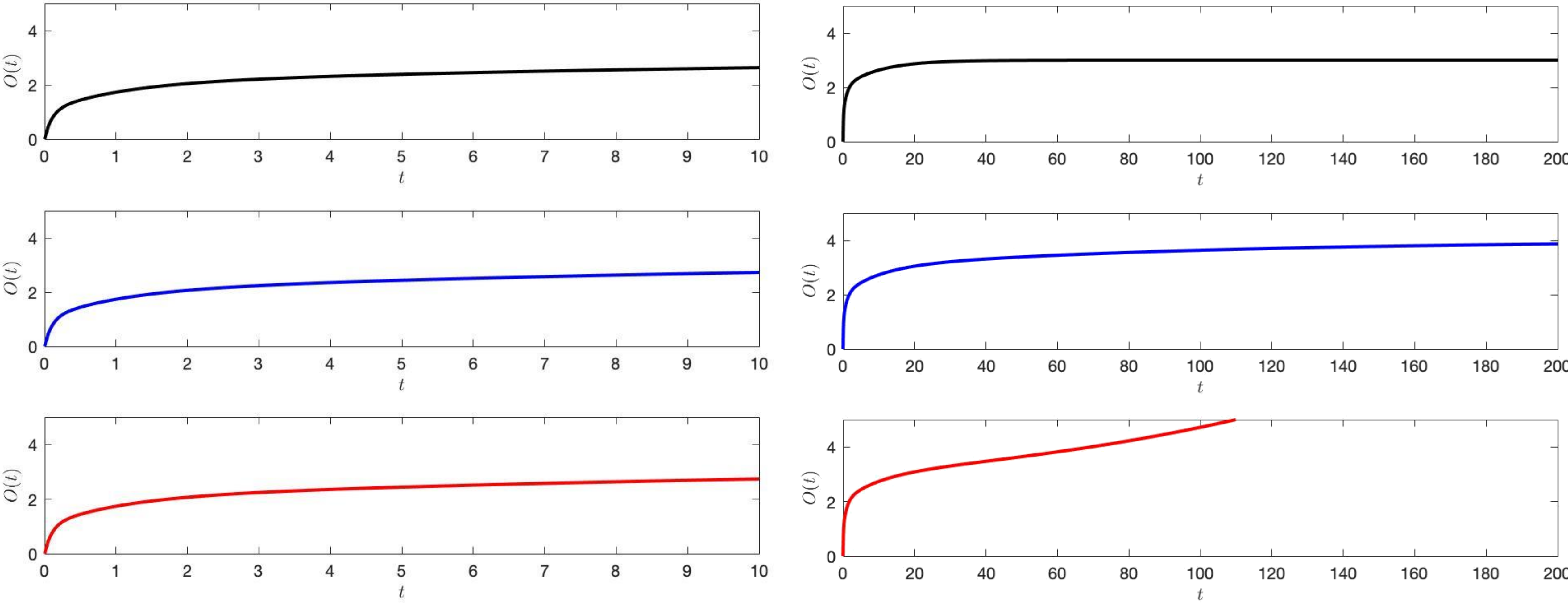
- Much computing power
- Not long-term response
- Not many scenarios

**Often, all other observables are assumed to be linearly related to the global mean surface temperature**

## Mathematical Context

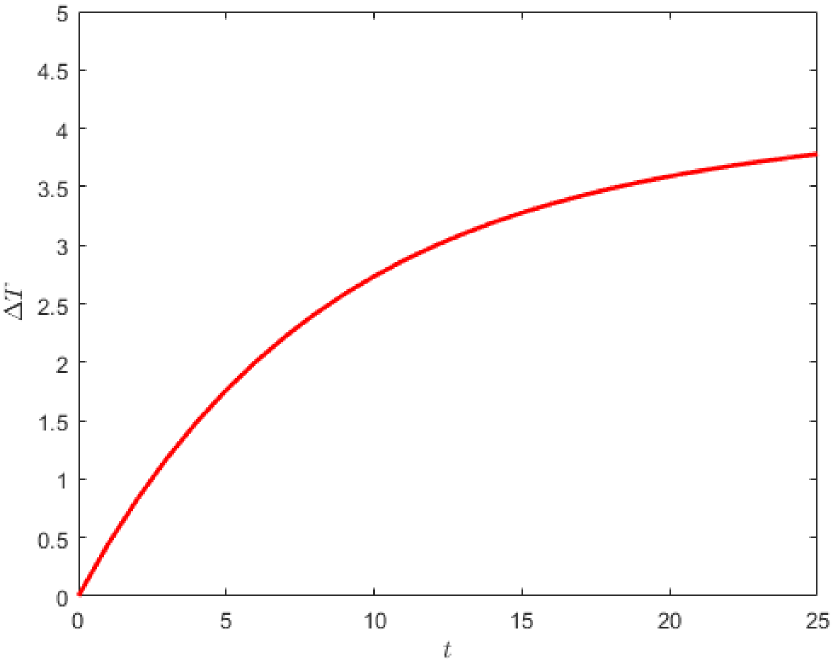
$$\frac{dy}{dt} = f(\,y\,;\,\mu(t)\,)$$

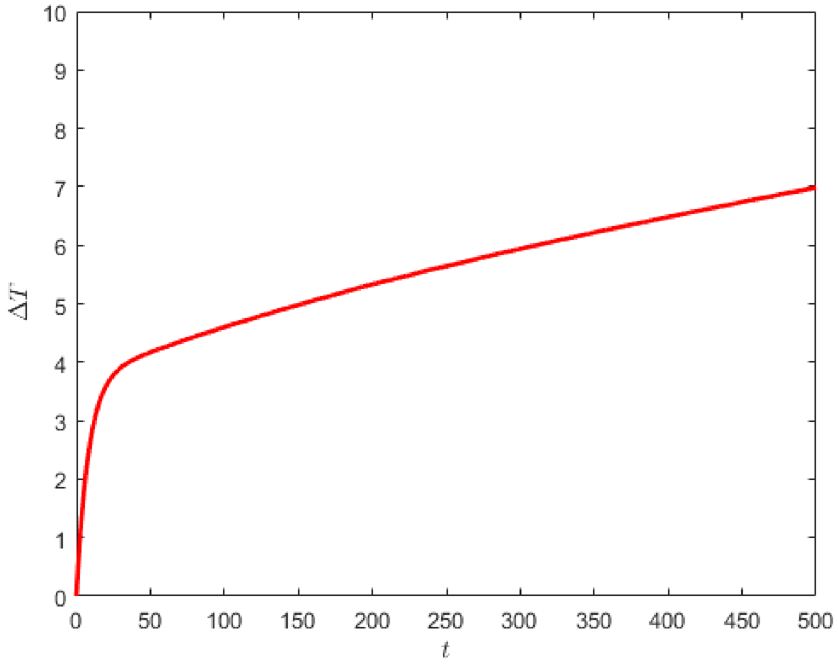
# Pitfalls and problems of linearity assumption

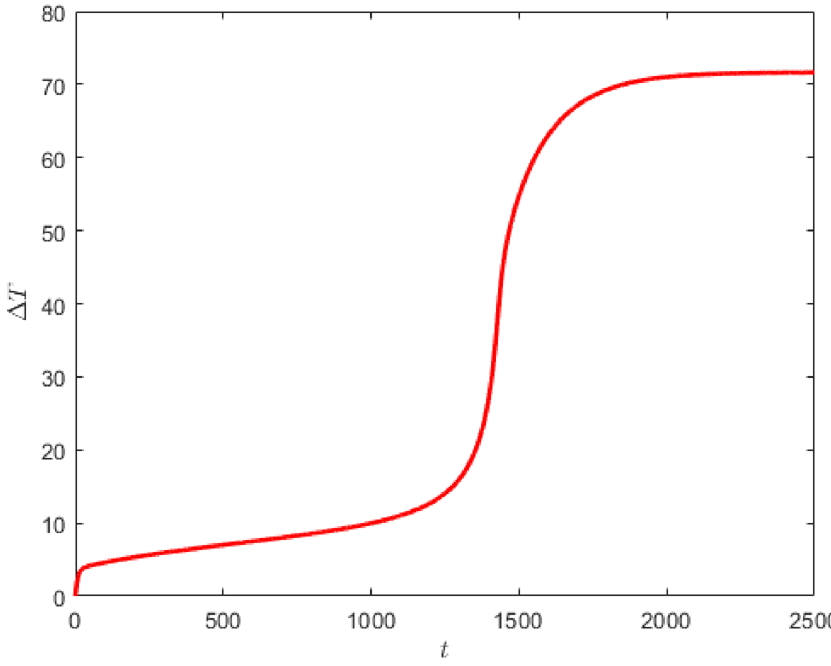



[Bastiaansen, Ashwin, Von der Heydt, 2022]

# Nonlinear Response



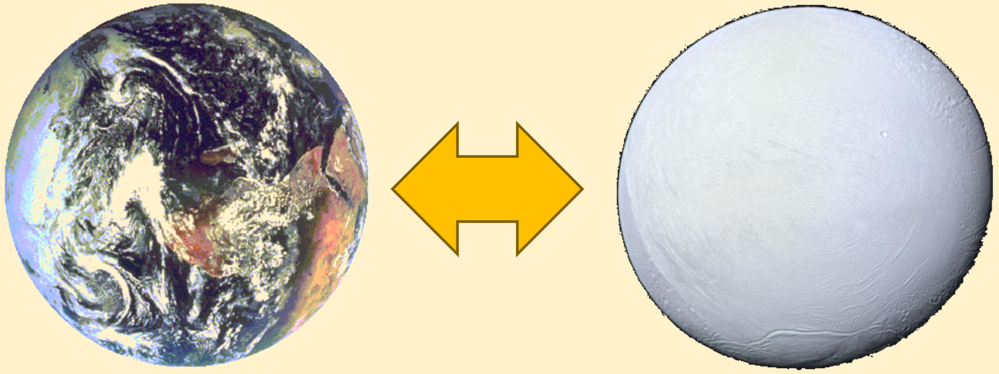




[Bastiaansen, Ashwin, Von der Heydt, 2022]

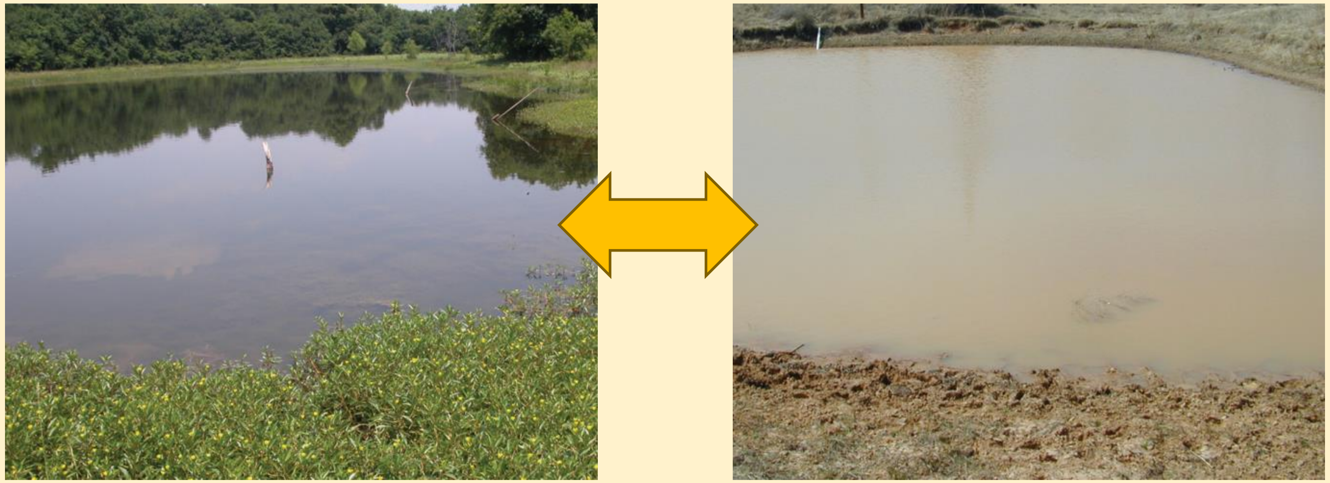
## Tipping Points

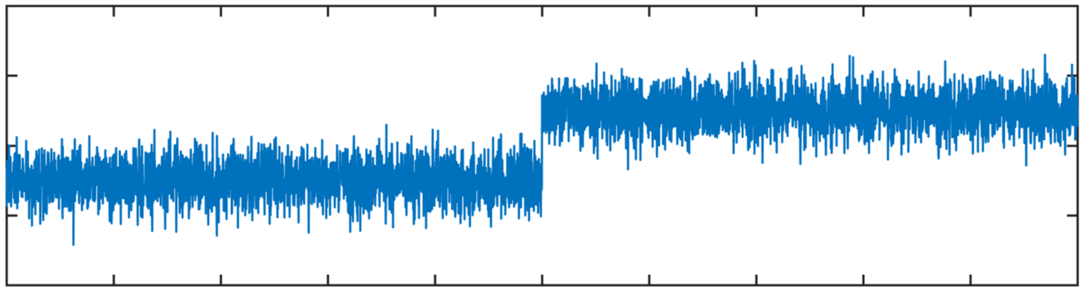
IPCC AR6 (2021) : "a critical threshold beyond which a system reorganizes, often abruptly and/or irreversibly"



Planetary transitions

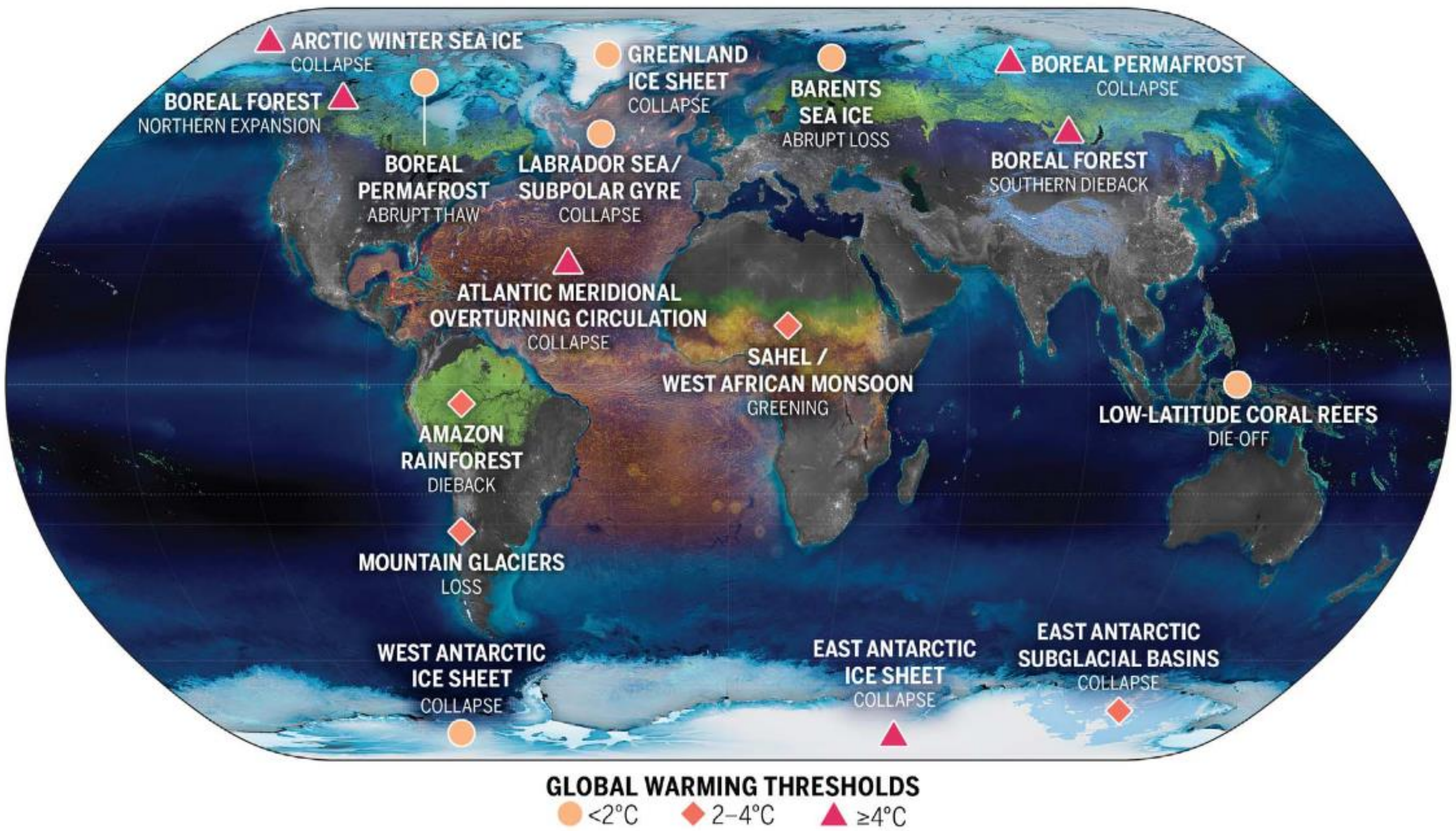
Ecosystem shifts





# Tipping Points

IPCC AR6 (2021) : “a critical threshold beyond which a system reorganizes, often abruptly and/or irreversibly”

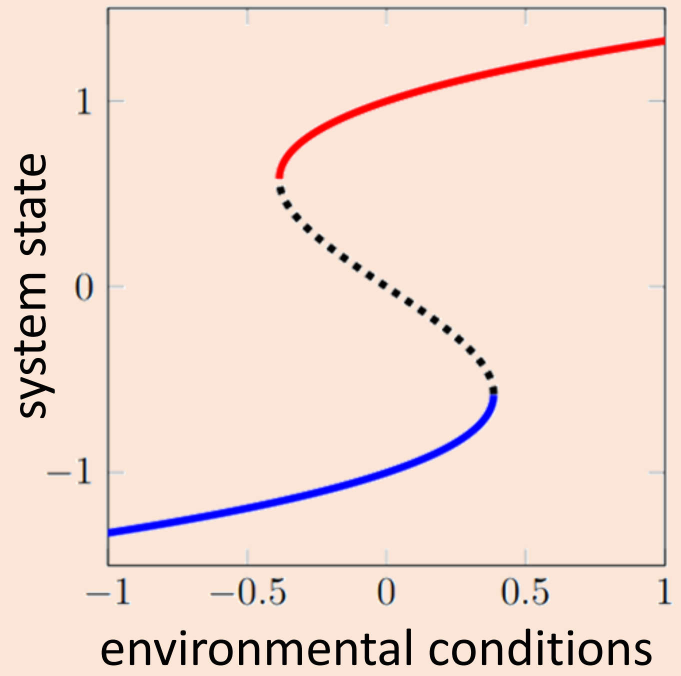


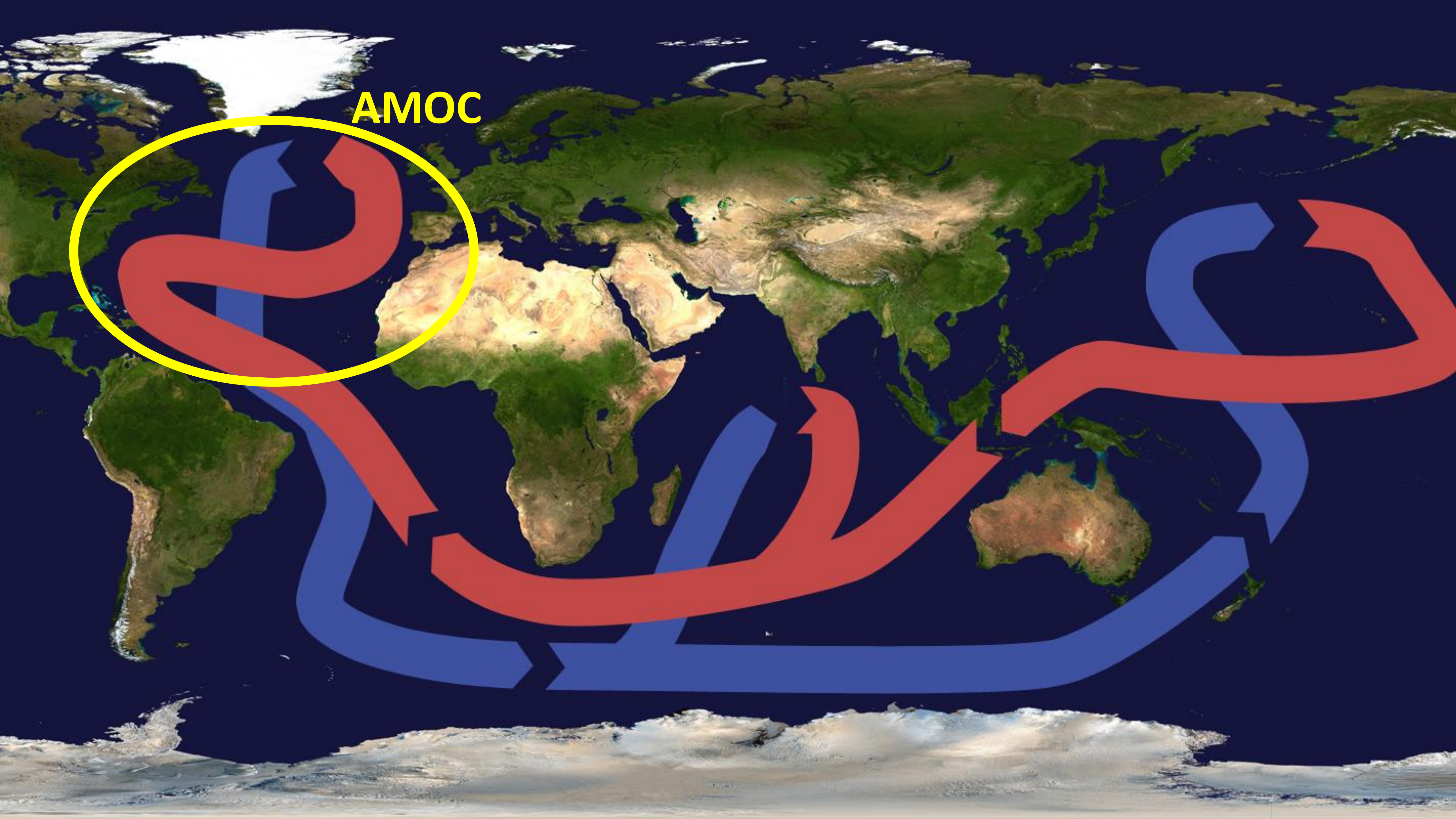

source: McKay et al, 2022

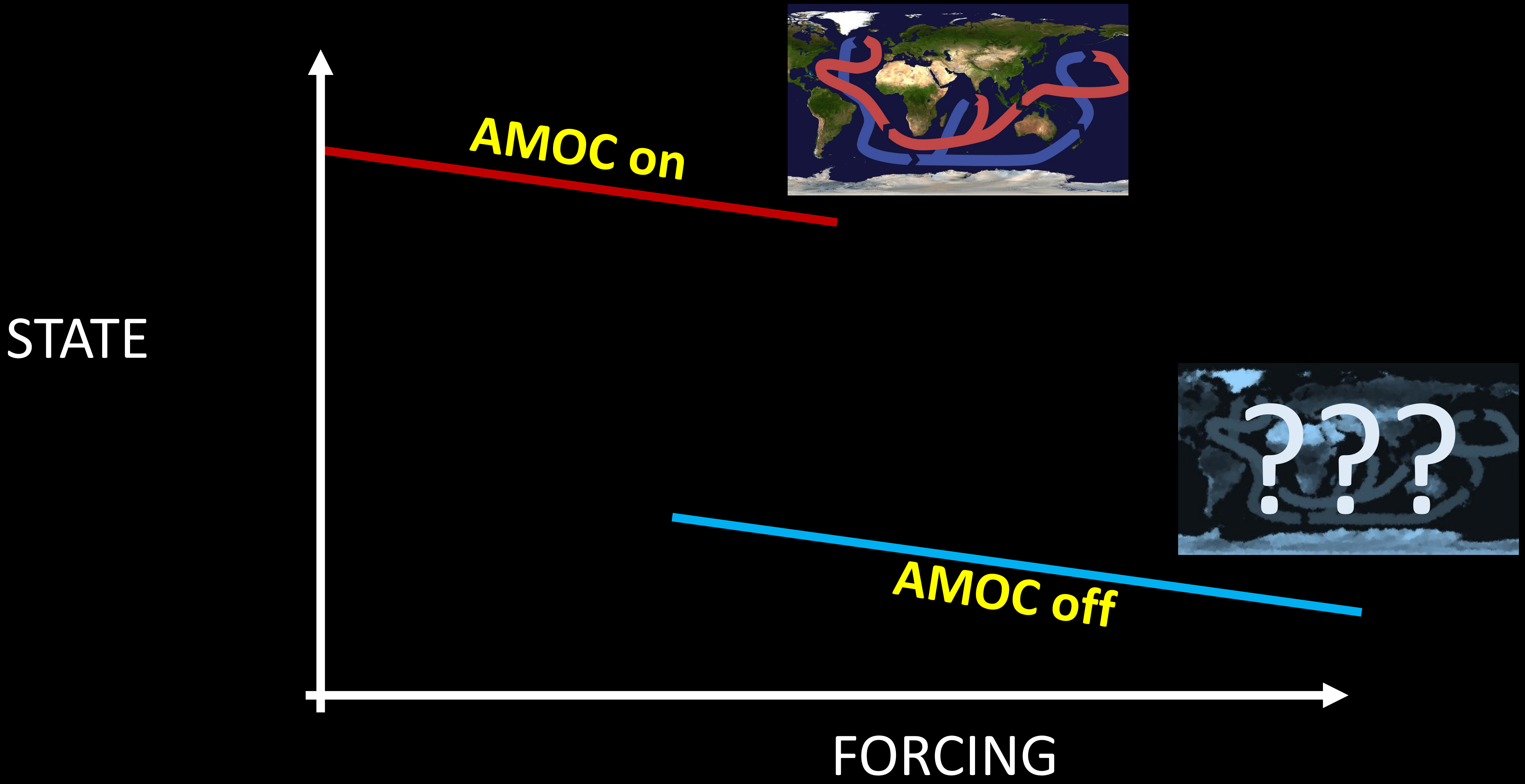
## Mathematics

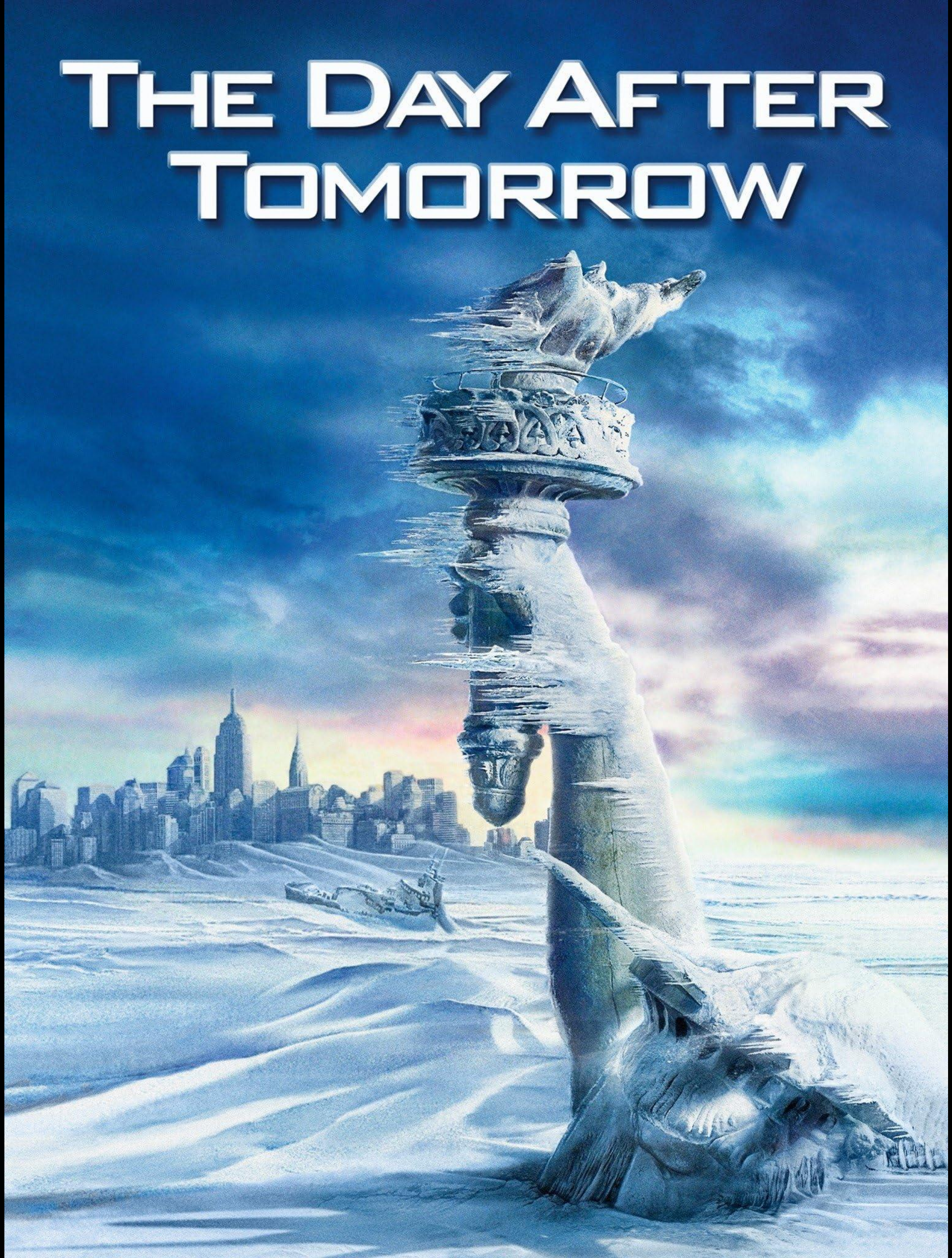
Tipping points ↔ Bifurcations

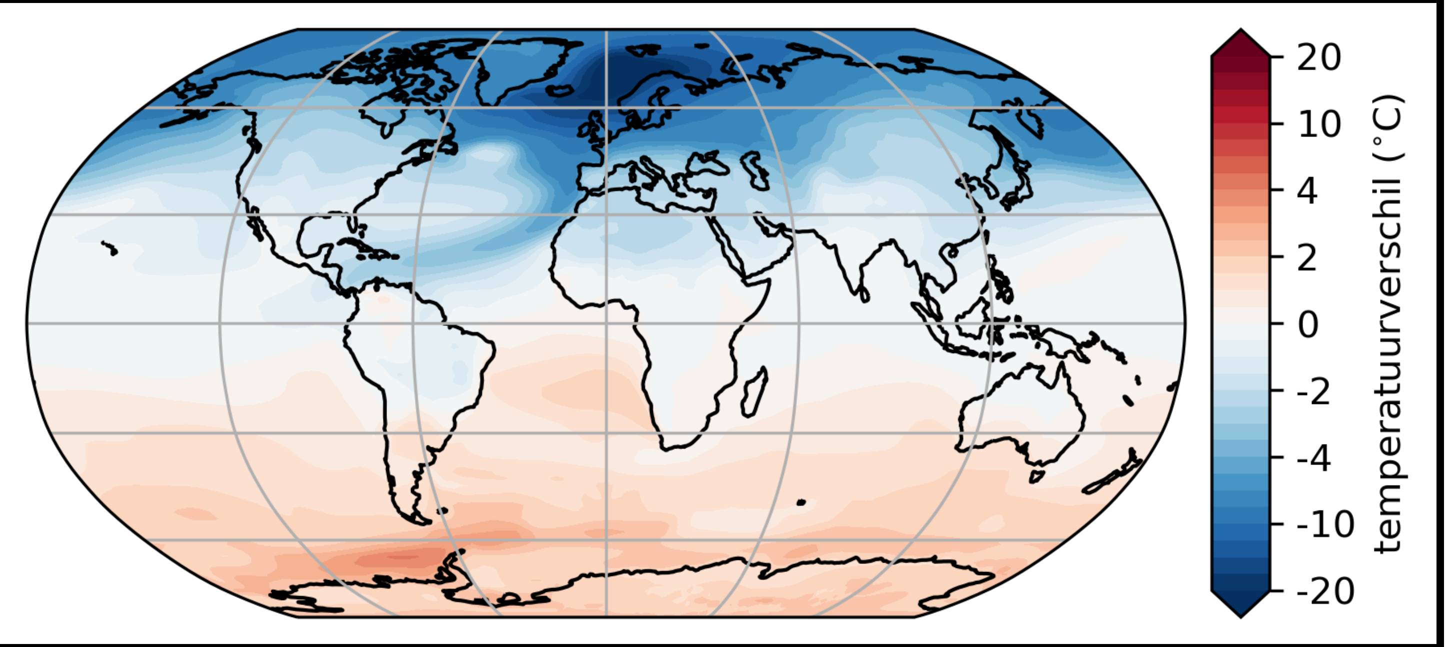
$$\frac{dy}{dt} = f(y, \mu)$$

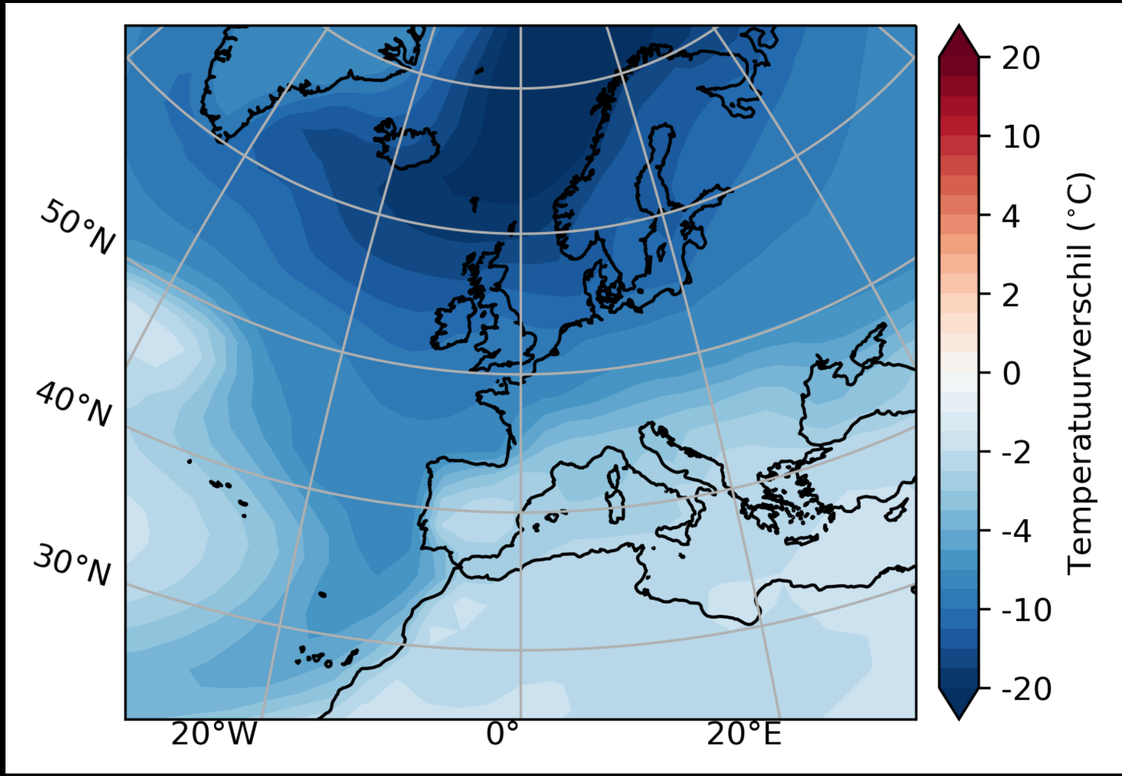



AMOC


AMOC on
AMOC off
???
STATE
FORCING

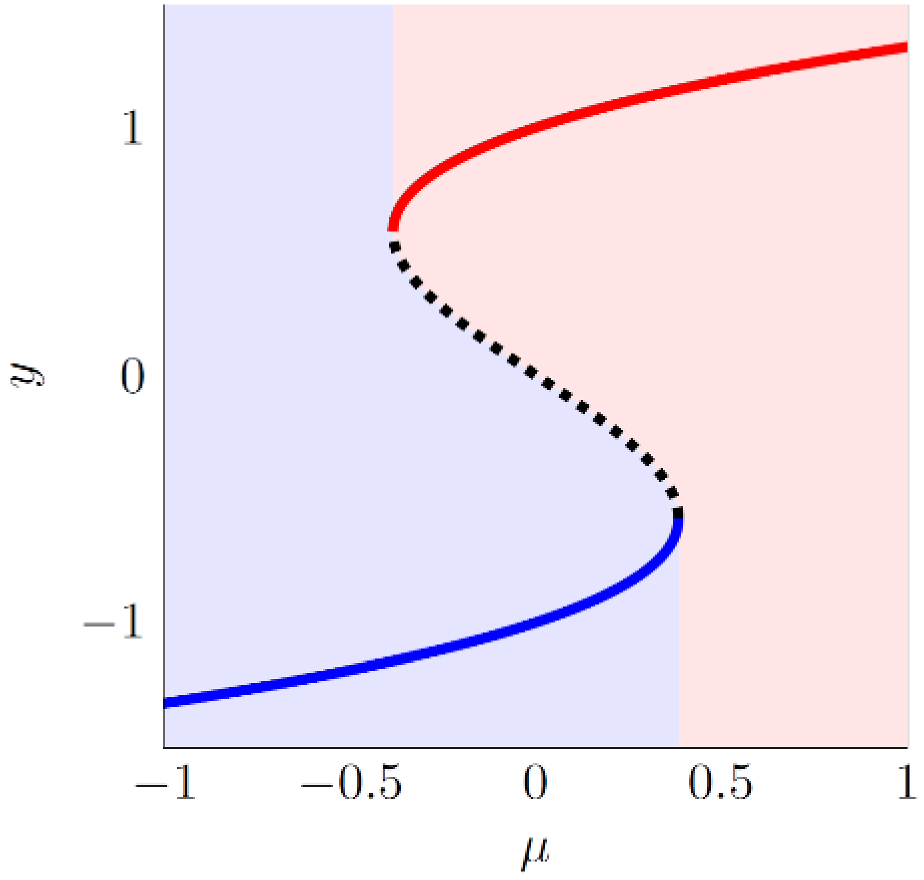

THE DAY AFTER
TOMORROW

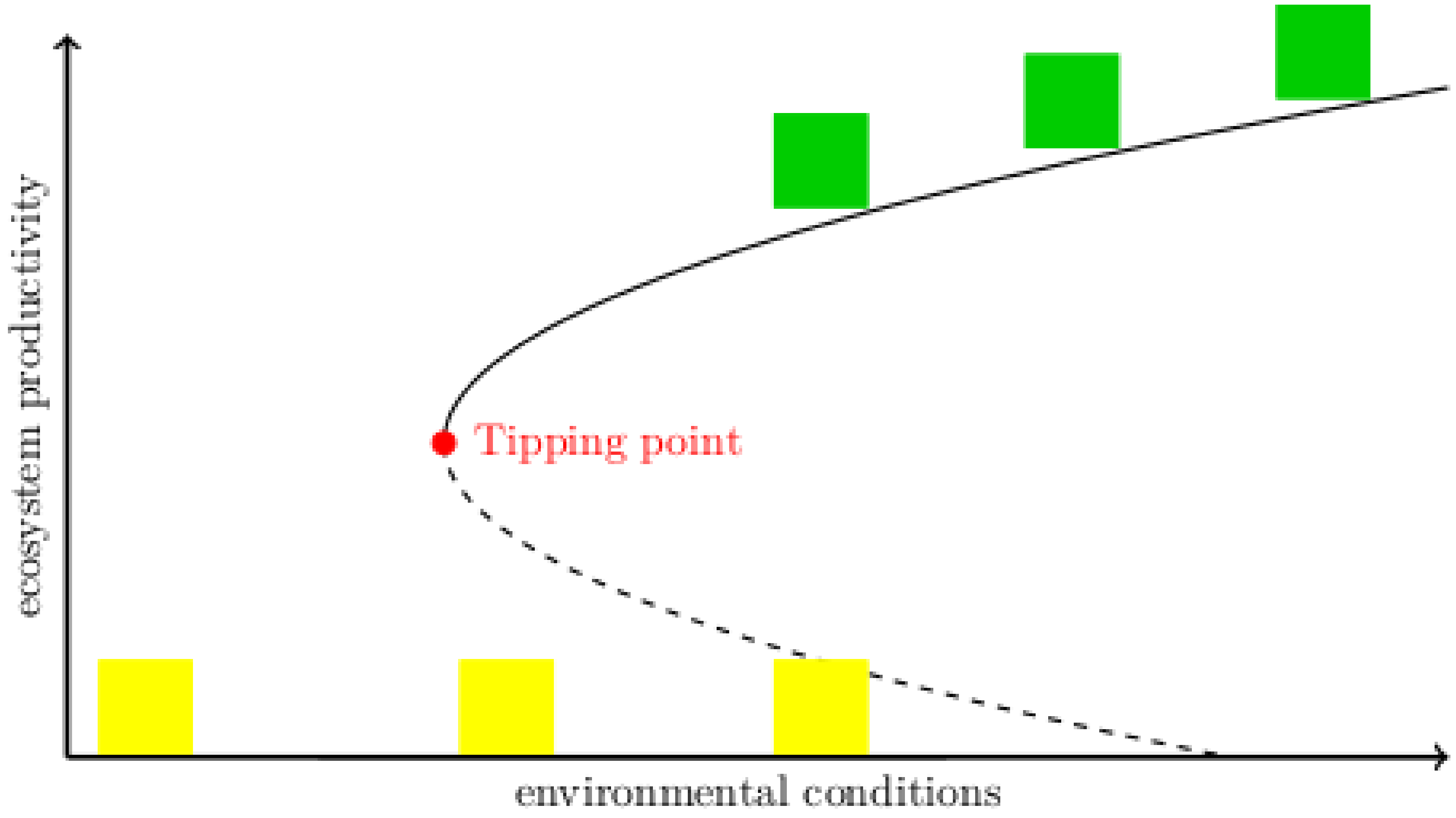



[Van Westen et al, Science Advances. (2024)]

# Classic Theory of Tipping



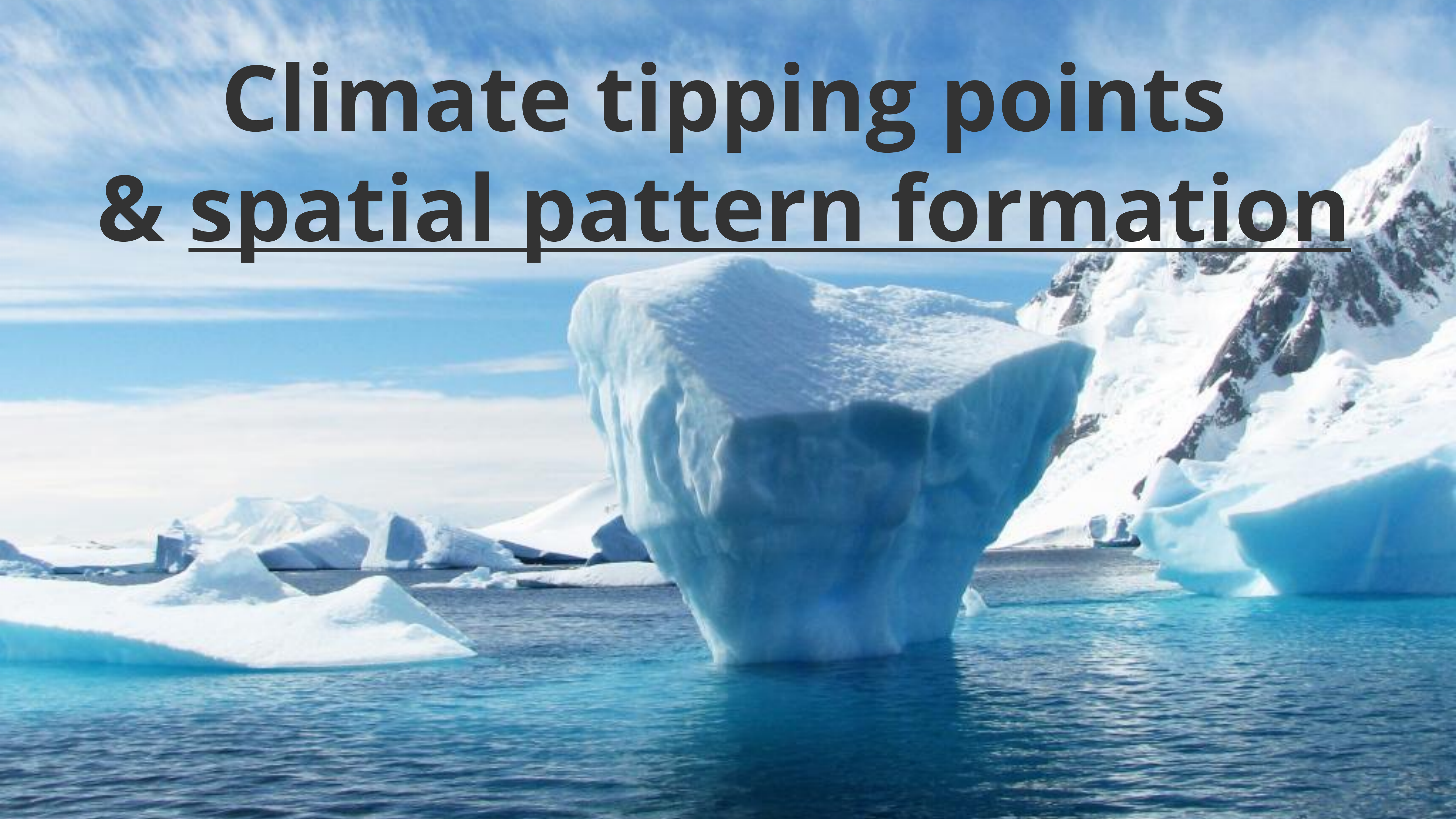

**Canonical example:**

$$\frac{dy}{dt} = y(1 - y^2) + \mu$$

$$\frac{d\vec{y}}{dt} = f(\vec{y}; \mu)$$

$$\begin{cases} \dfrac{du}{dt} = f(u, v; \mu) \\ \dfrac{dv}{dt} = g(u, v; \mu) \end{cases}$$

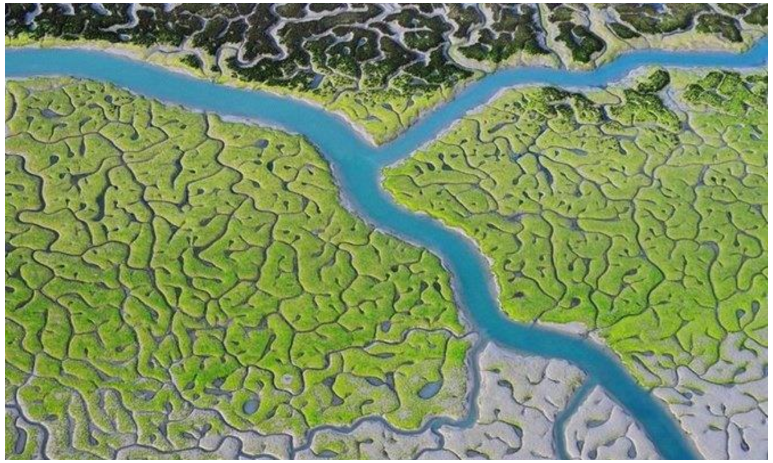
# Climate tipping points & spatial pattern formation



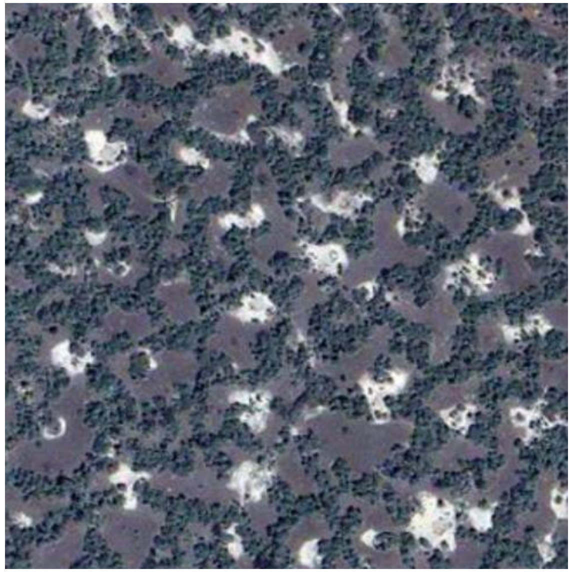
# Examples of spatial patterning – ecosystems

mussel beds

vegetation in coastal systems



marsh formation

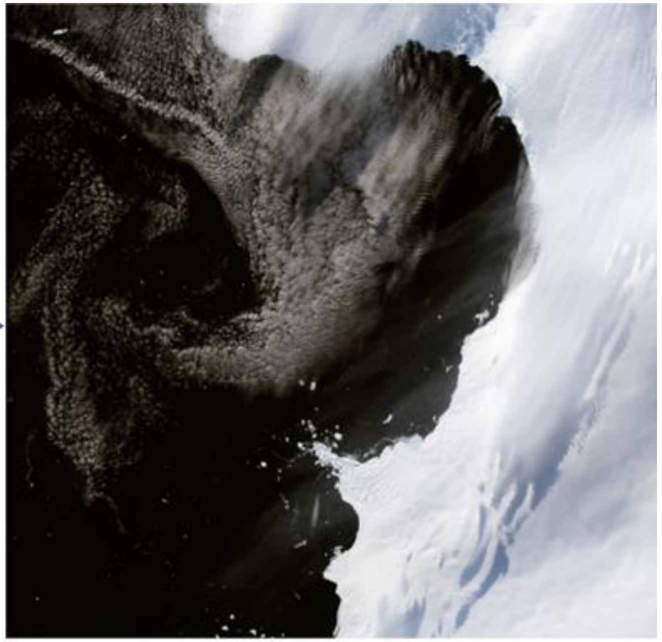


savannas

drylands

tropical forests

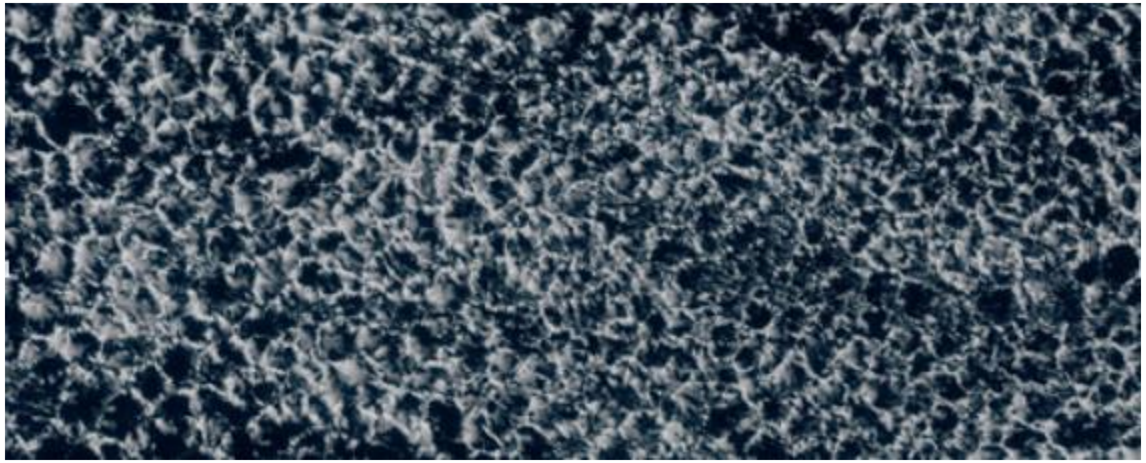
# Examples of spatial patterning – climate



sea-ice & water at Eltanin Bay
[NASA's Earth observatory]

melt ponds

sand dunes



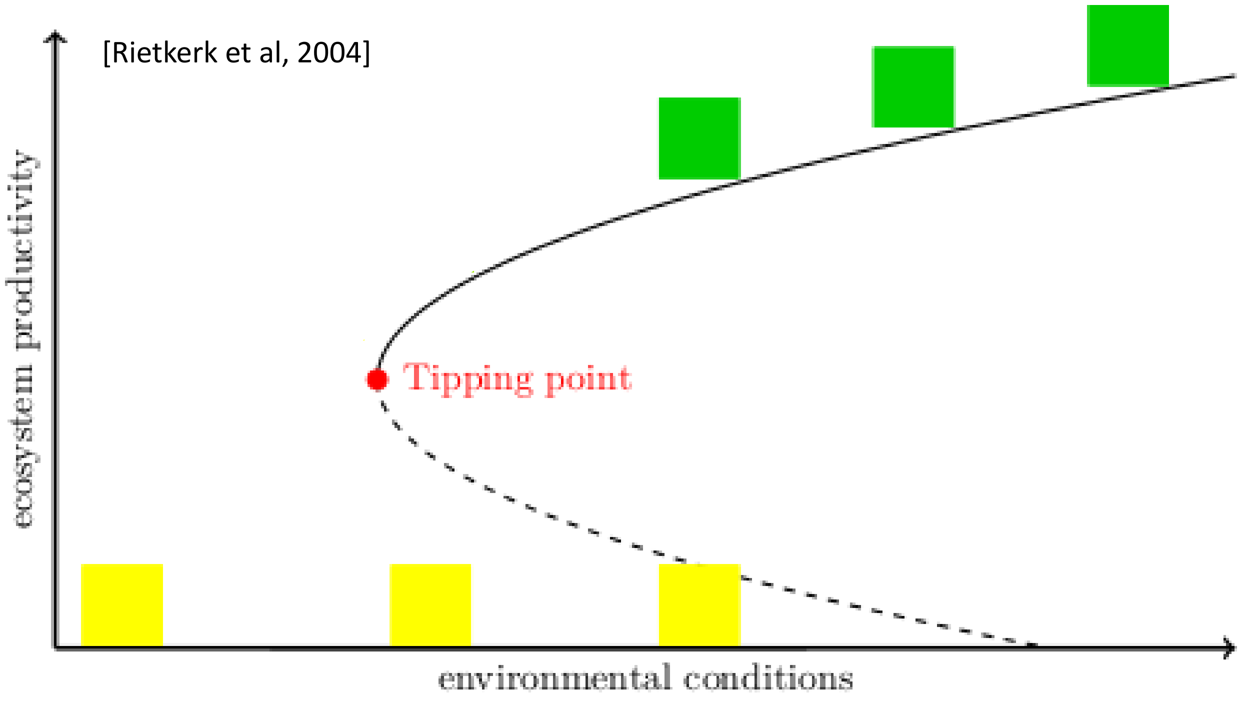
clouds

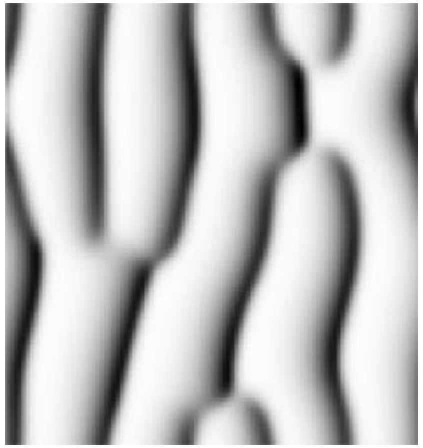
# Patterns in models

Add spatial transport:

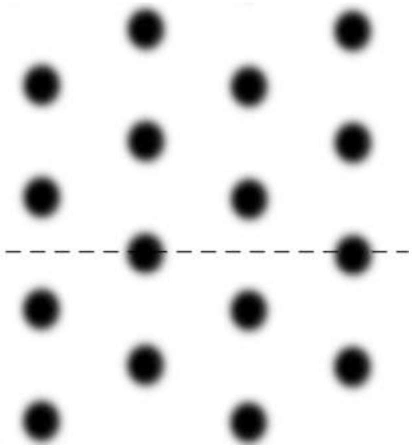
Reaction-Diffusion equations:

$$\begin{cases} \dfrac{du}{dt} = f(u,v) + D_u\,\Delta u \\ \dfrac{dv}{dt} = g(u,v) + D_v\,\Delta v \end{cases}$$

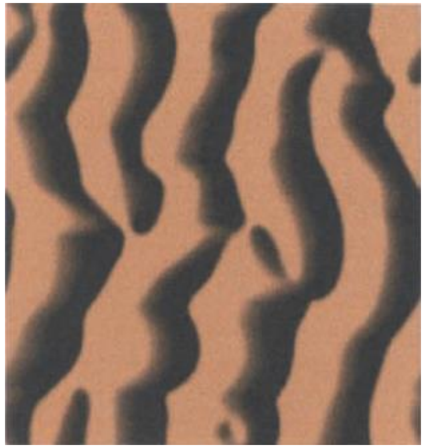


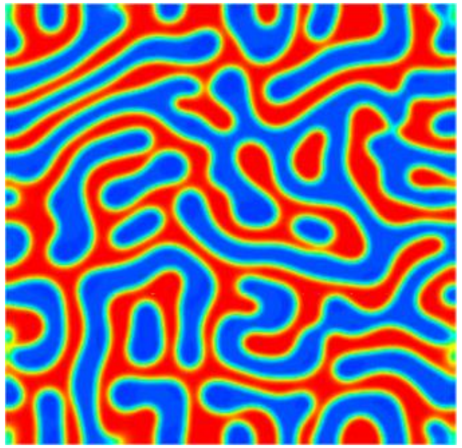
[Klausmeier, 1999]



[Gilad et al, 2004]

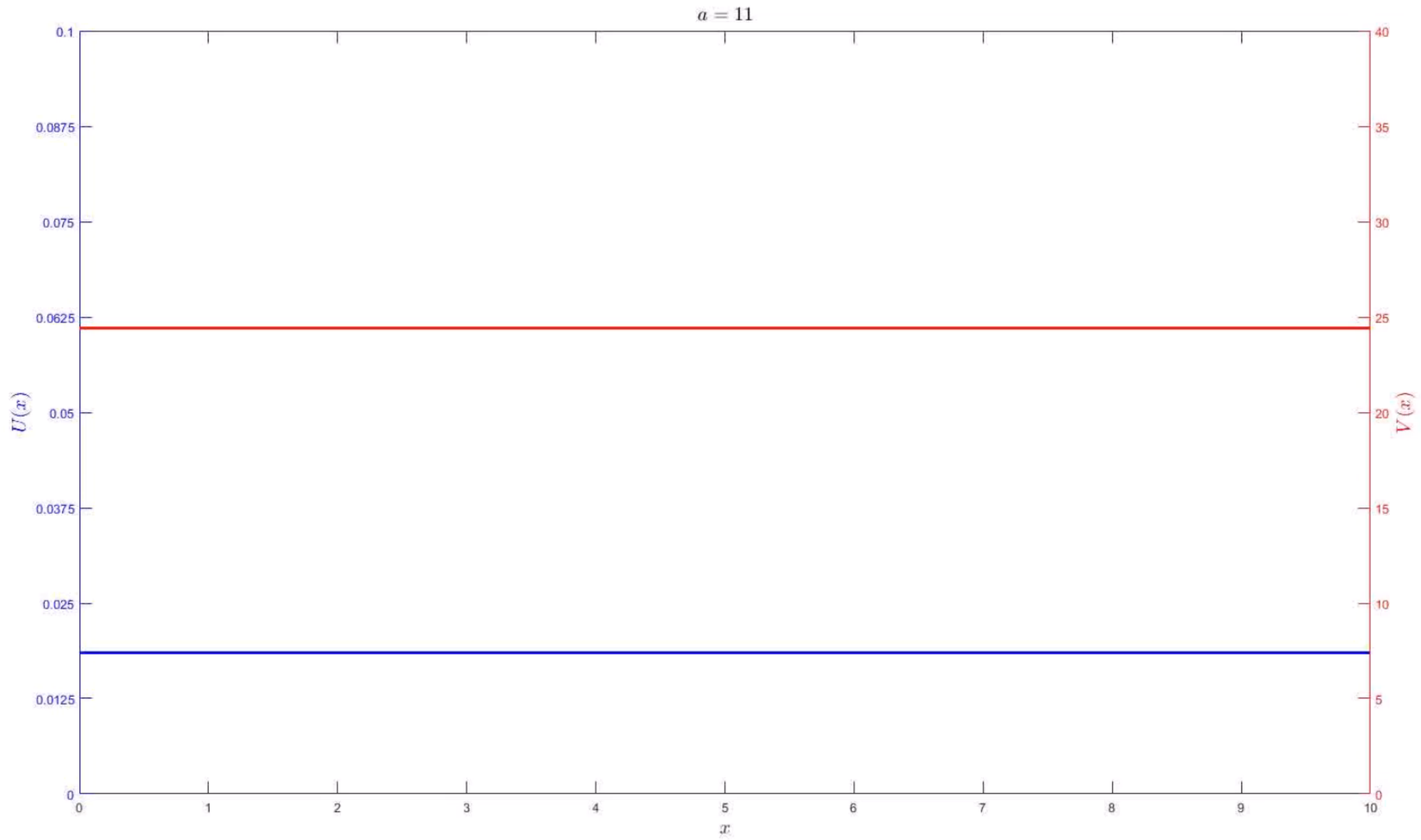


[Rietkerk et al, 2002]



[Liu et al, 2013]

# Spontaneous Pattern Formation



# Turing patterns

$$\begin{cases} \frac{du}{dt} = f(u,v) + D_u \, \Delta u \\ \frac{dv}{dt} = g(u,v) + D_v \, \Delta v \end{cases}$$

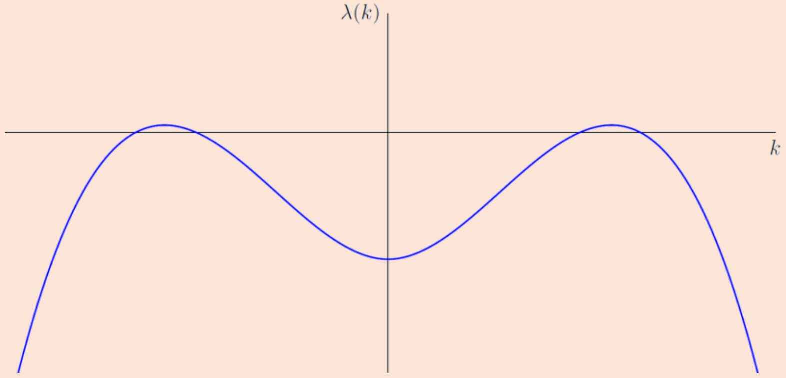
**Turing bifurcation**

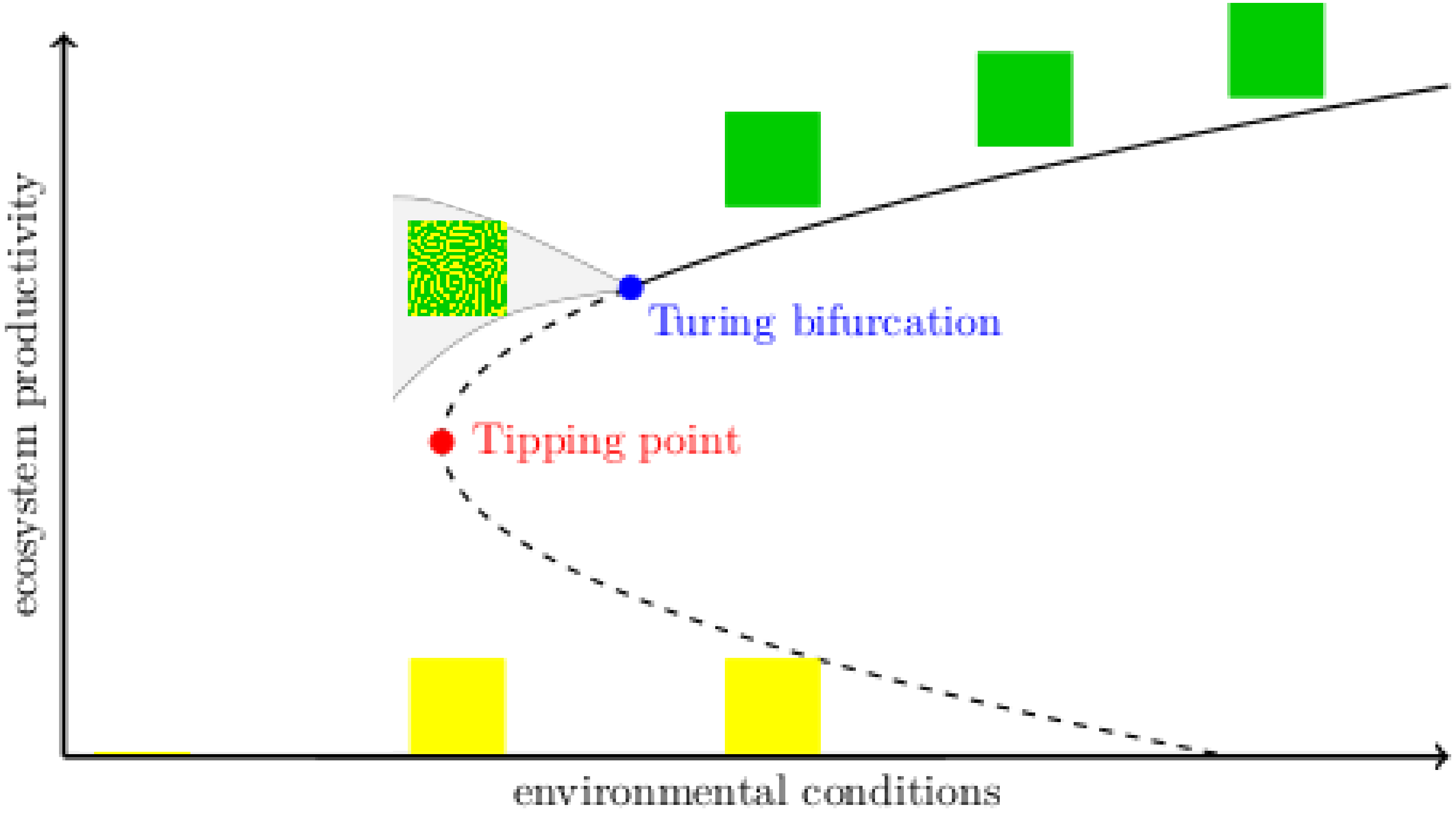
Instability to non-uniform perturbations

$$\binom{u}{v} = \binom{u_*}{v_*} + e^{\lambda t} e^{ikx} \binom{\bar{u}}{\bar{v}}$$

→ Dispersion relation

$$\lambda(k) = \cdots$$




**Weakly non-linear analysis**

Ginzburg-Landau equation / Amplitude Equation

& Eckhaus/Benjamin-Feir-Newel criterion

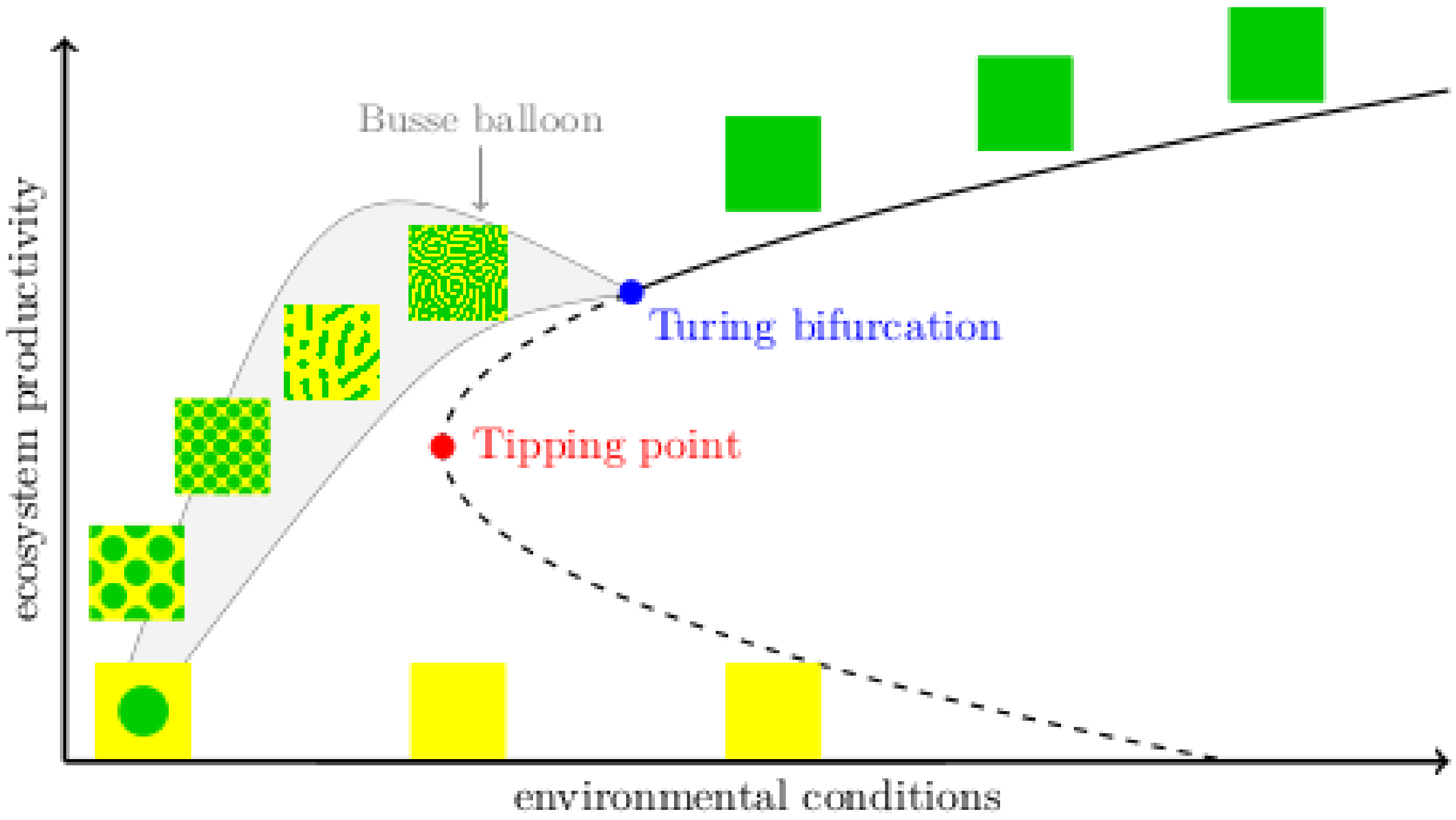
[Eckhaus, 1965; Benjamin & Feir, 1967; Newell, 1974]

# Busse balloon

$$\begin{cases} \dfrac{du}{dt} = f(u,v) + D_u \Delta u \\ \dfrac{dv}{dt} = g(u,v) + D_v \Delta v \end{cases}$$

**Busse balloon**

A model-dependent shape in *(parameter, observable)* space that indicates all stable patterned solutions to the PDE.




**Construction Busse balloon**

Via numerical continuation

few general results on the shape of Busse balloon

**Busse balloon**

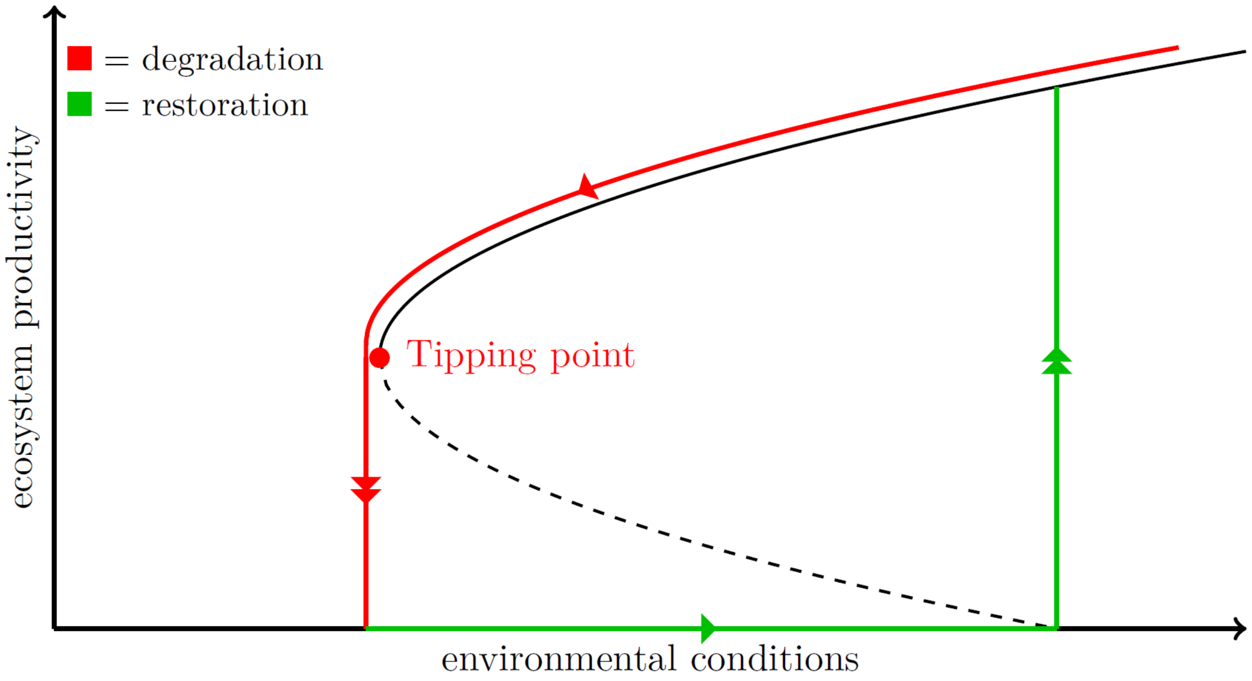
Idea originates from thermal convection

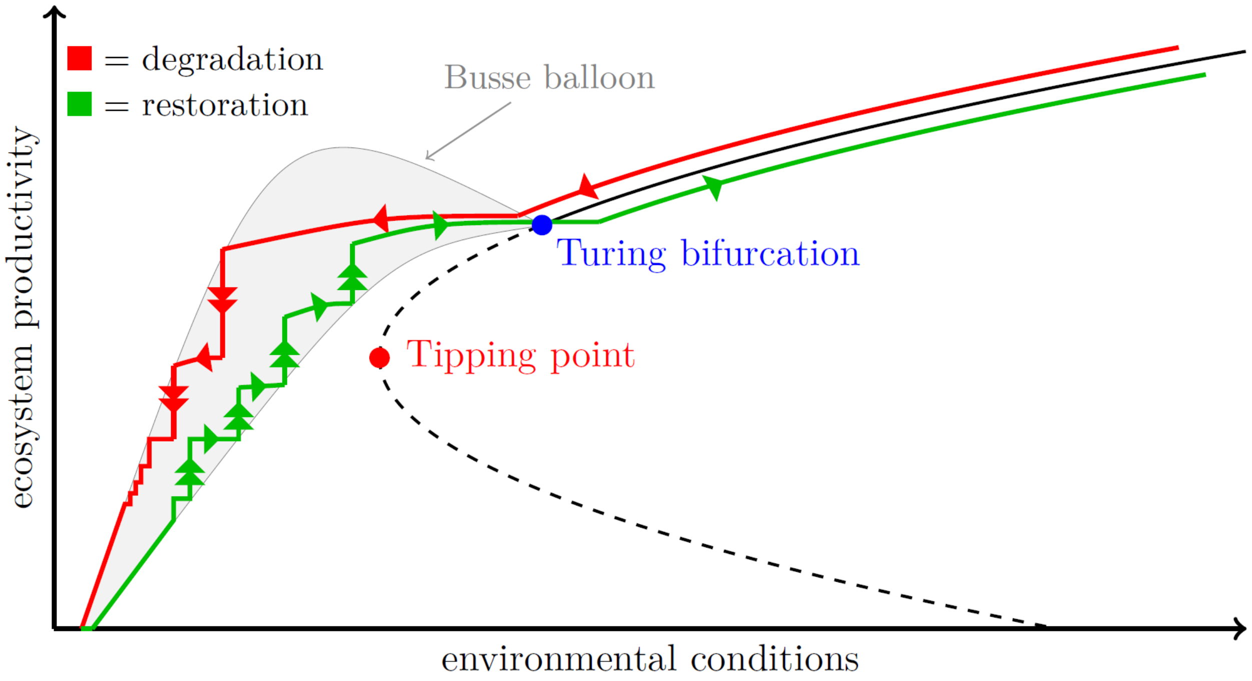
[Busse, 1978]

# Rayleigh Bénard thermal convection

Video source: wikiRigaou (wikimedia commons)

# Tipping of (Turing) patterns

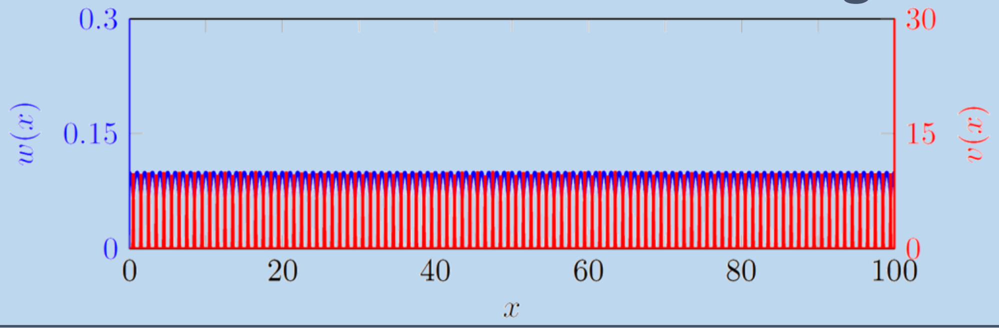


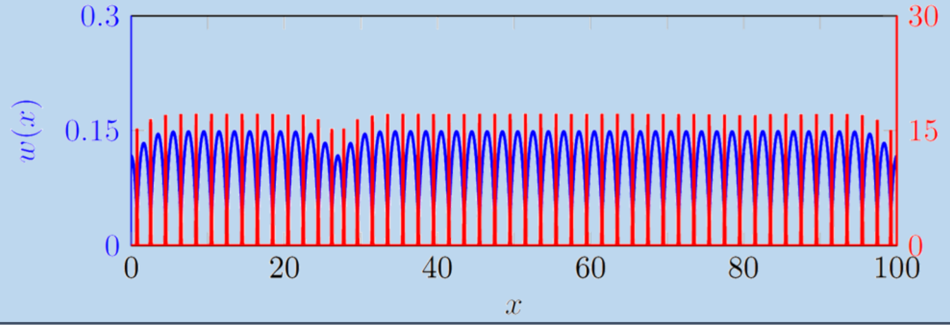




Classic tipping

Tipping of patterns

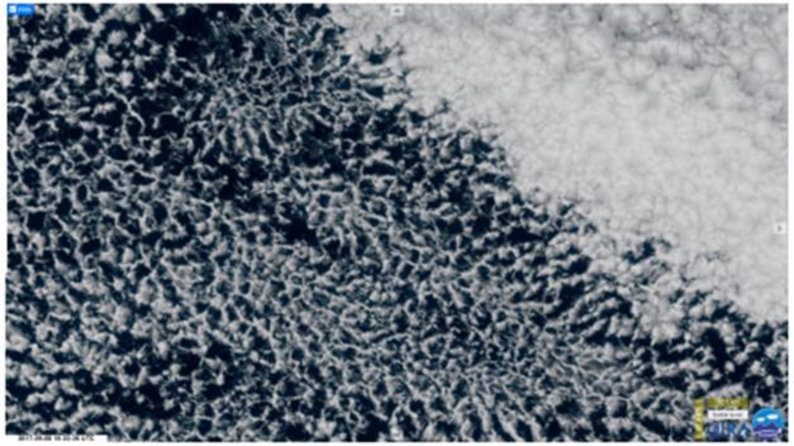
## Degradation of patterns



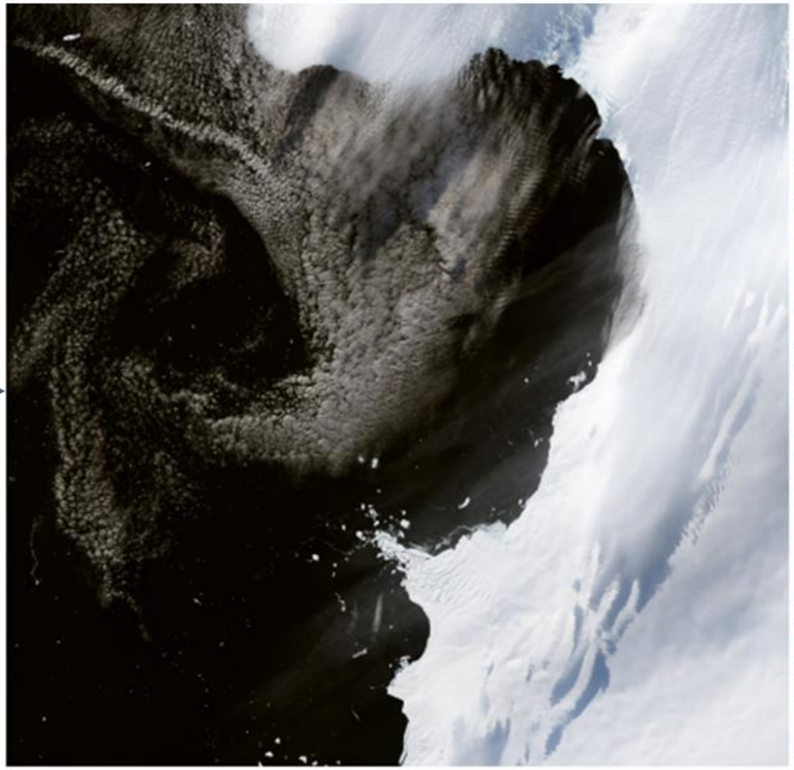
# Examples of spatial patterning – spatial interfaces

tropical forest
& savanna
ecosystems
[Google Earth]

types of
stratocumulus
clouds
[RAMMB/CIRA SLIDER]

sea-ice & water
at Eltanin Bay
[NASA's Earth observatory]

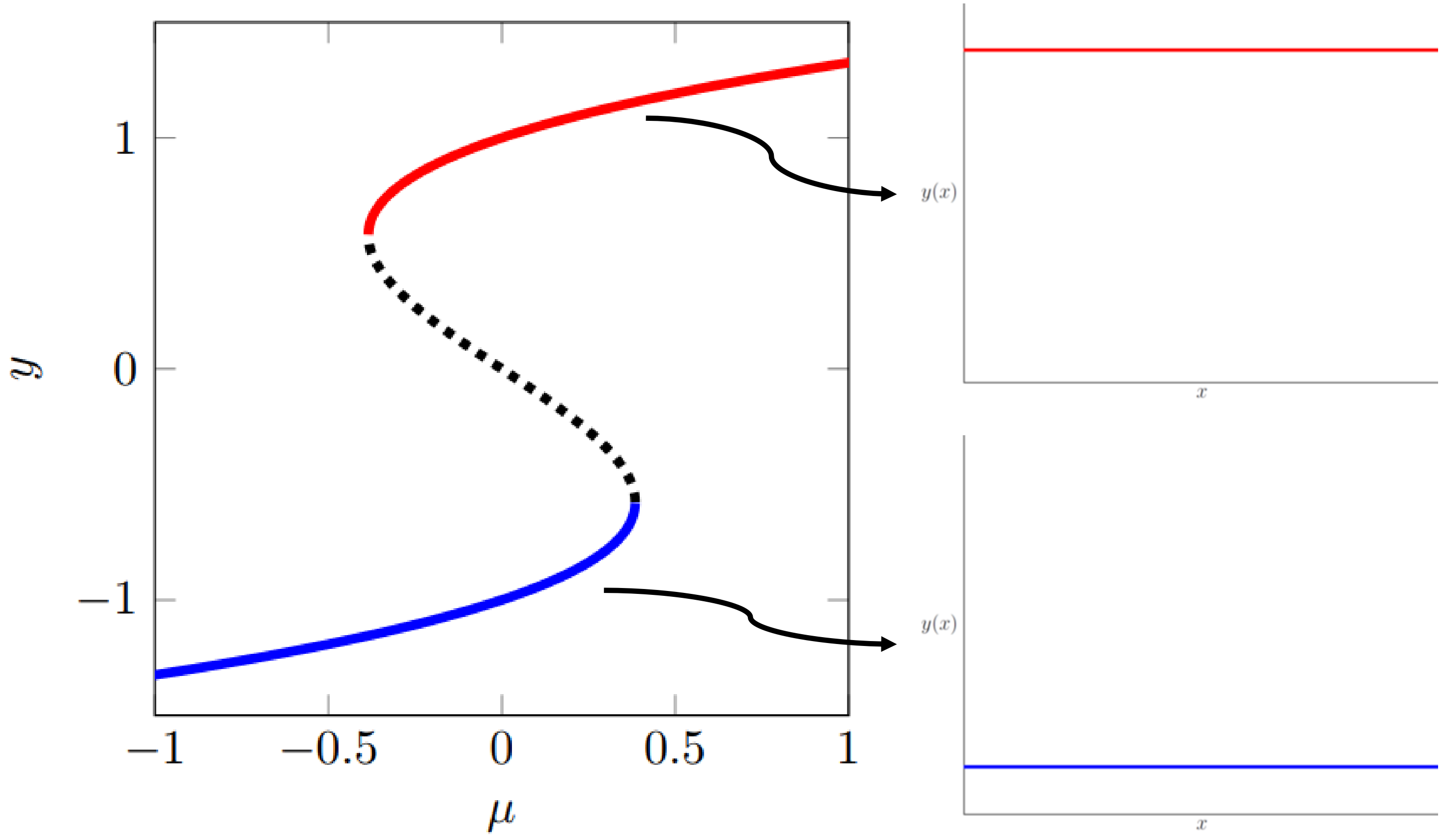


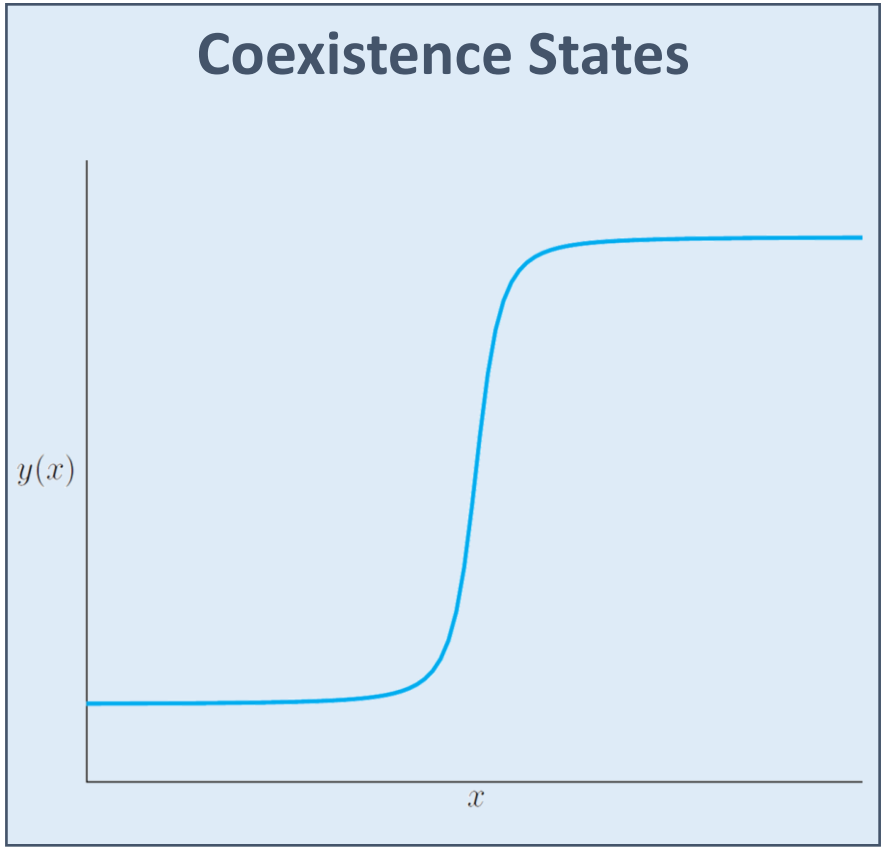
algae bloom
in Lake St. Clair
[NASA's Earth observatory]

# Coexistence states

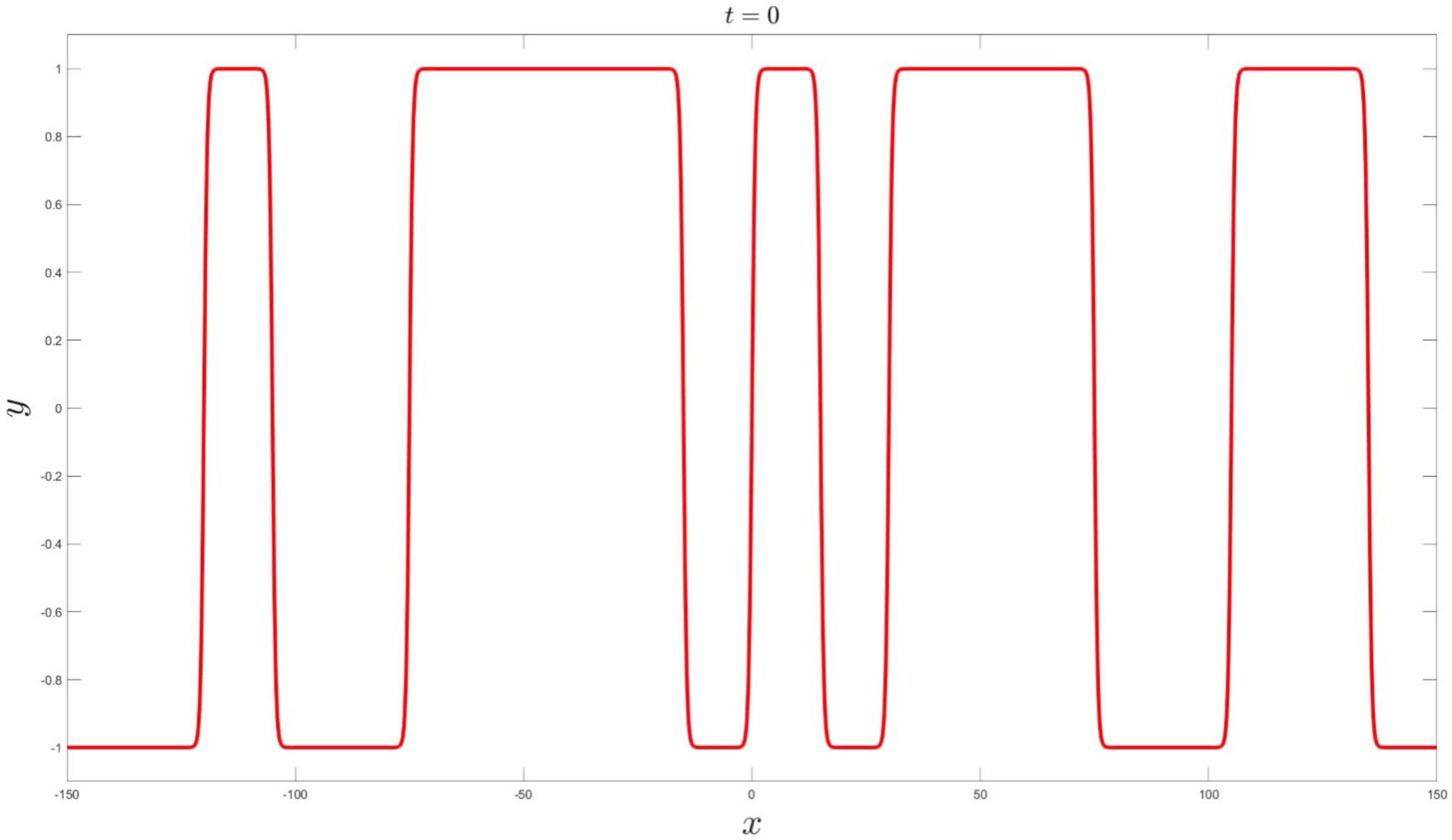
Bistable (Allen-Cahn/Nagumo) equation:

$$\frac{\partial y}{\partial t} = y(1 - y^2) + \mu + D\,\frac{\partial^2 y}{\partial x^2}$$



# Dynamics of $\frac{\partial y}{\partial t} = D\ \frac{\partial^2 y}{\partial x^2} + y\left(1 - y^2\right) + \mu$

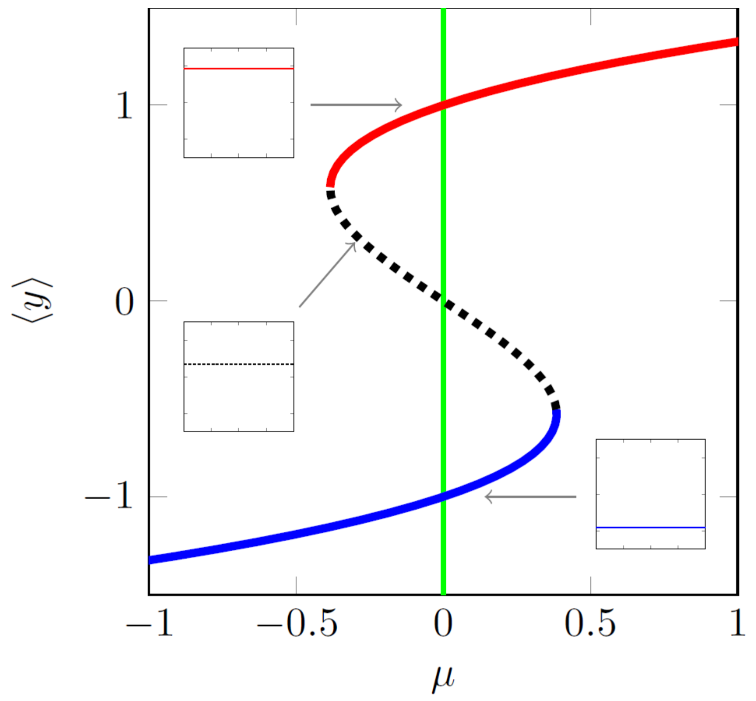


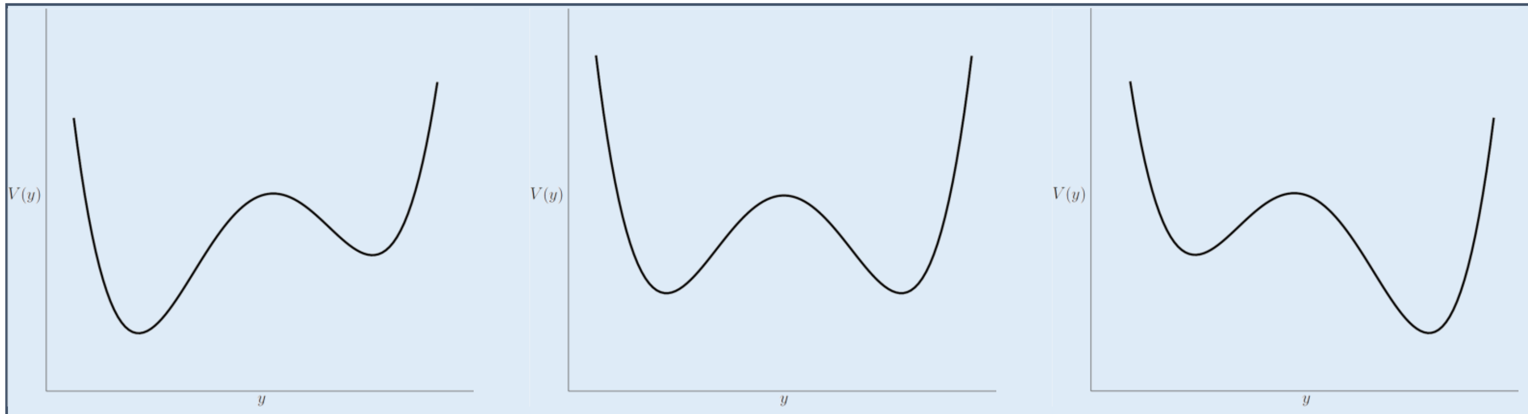
# Front Dynamics

$$\frac{\partial y}{\partial t} = D\frac{\partial^2 y}{\partial x^2} + f(y;\mu)$$

Potential function $V(y;\mu)$:

$$\frac{\partial V}{\partial y}(y;\mu) = -f(y;\mu)$$



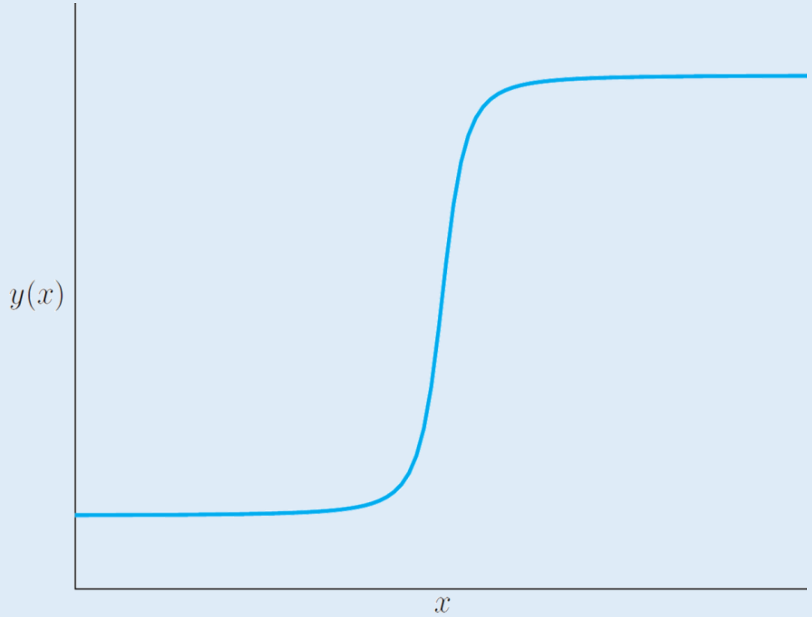




moves right

stationary

moves left

**Maxwell Point $\mu_{maxwell}$**



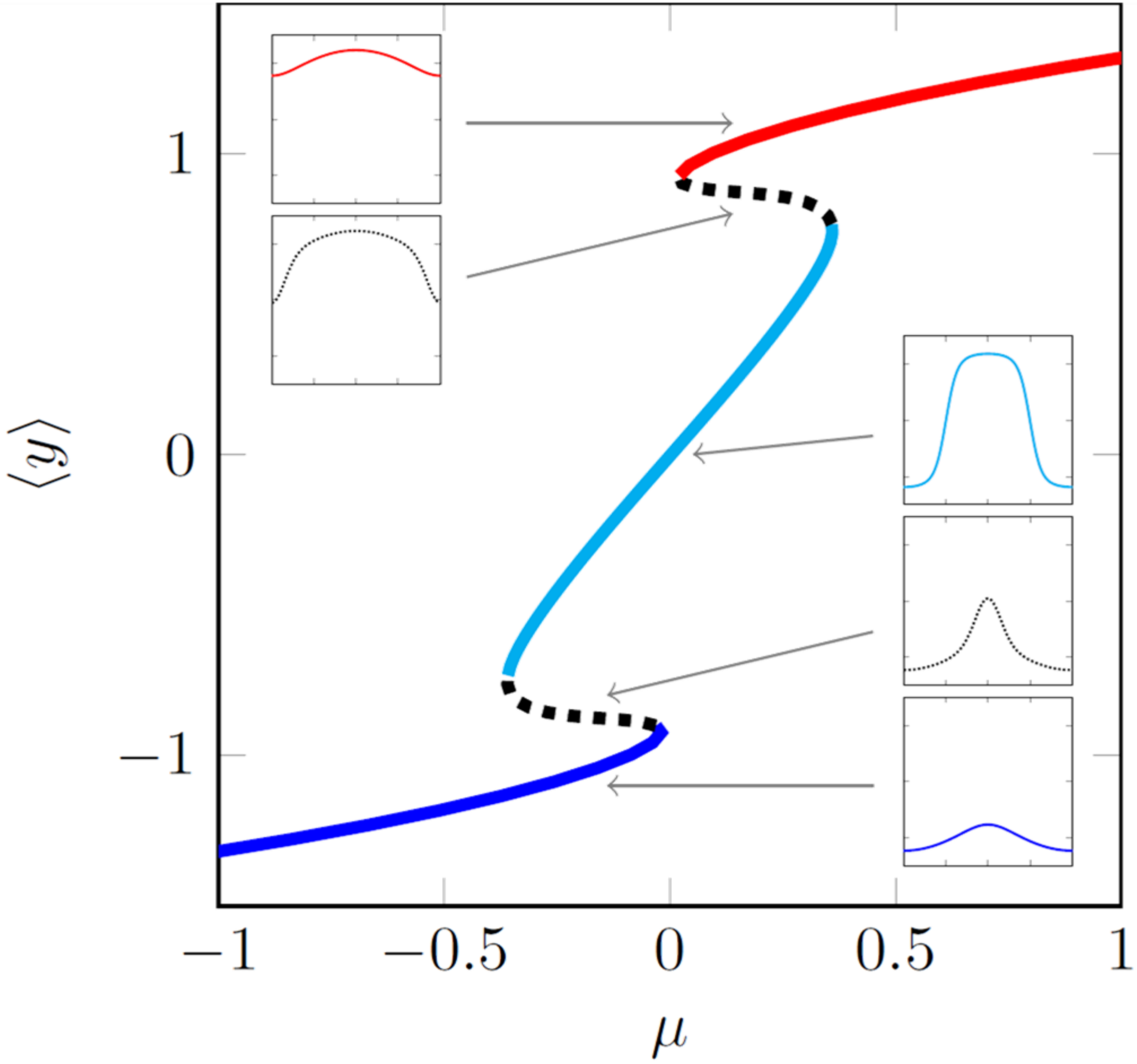
# Adding Spatial Heterogeneity

$$\frac{\partial y}{\partial t} = D\frac{\partial^2 y}{\partial x^2} + f(y, x; \mu)$$

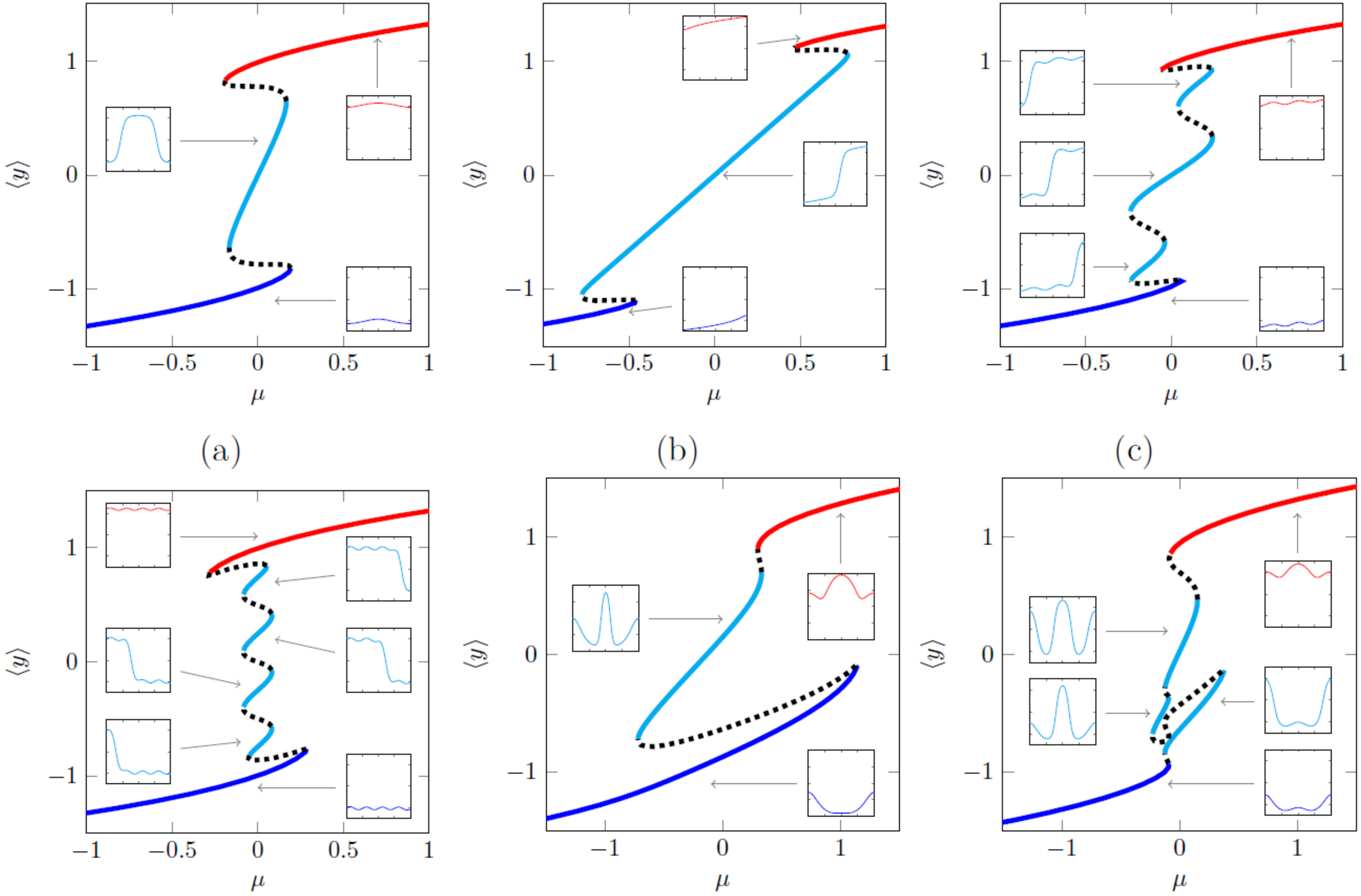
Now, the **local** difference in potentials determines the front movement

New behaviour:

- Multi-fronts can be stationary
- Maxwell point is smeared out

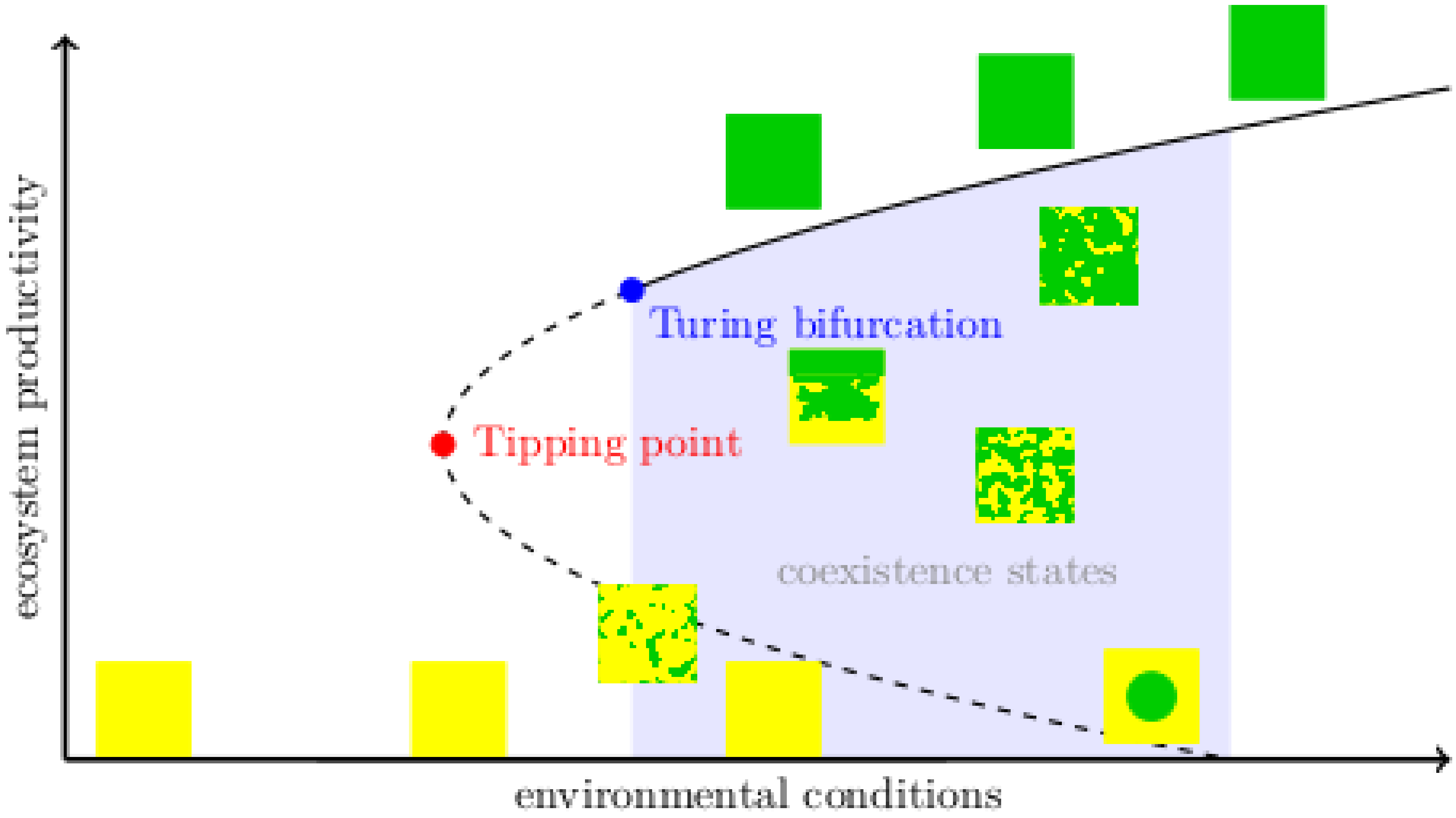


# Other Spatial Heterogeneities



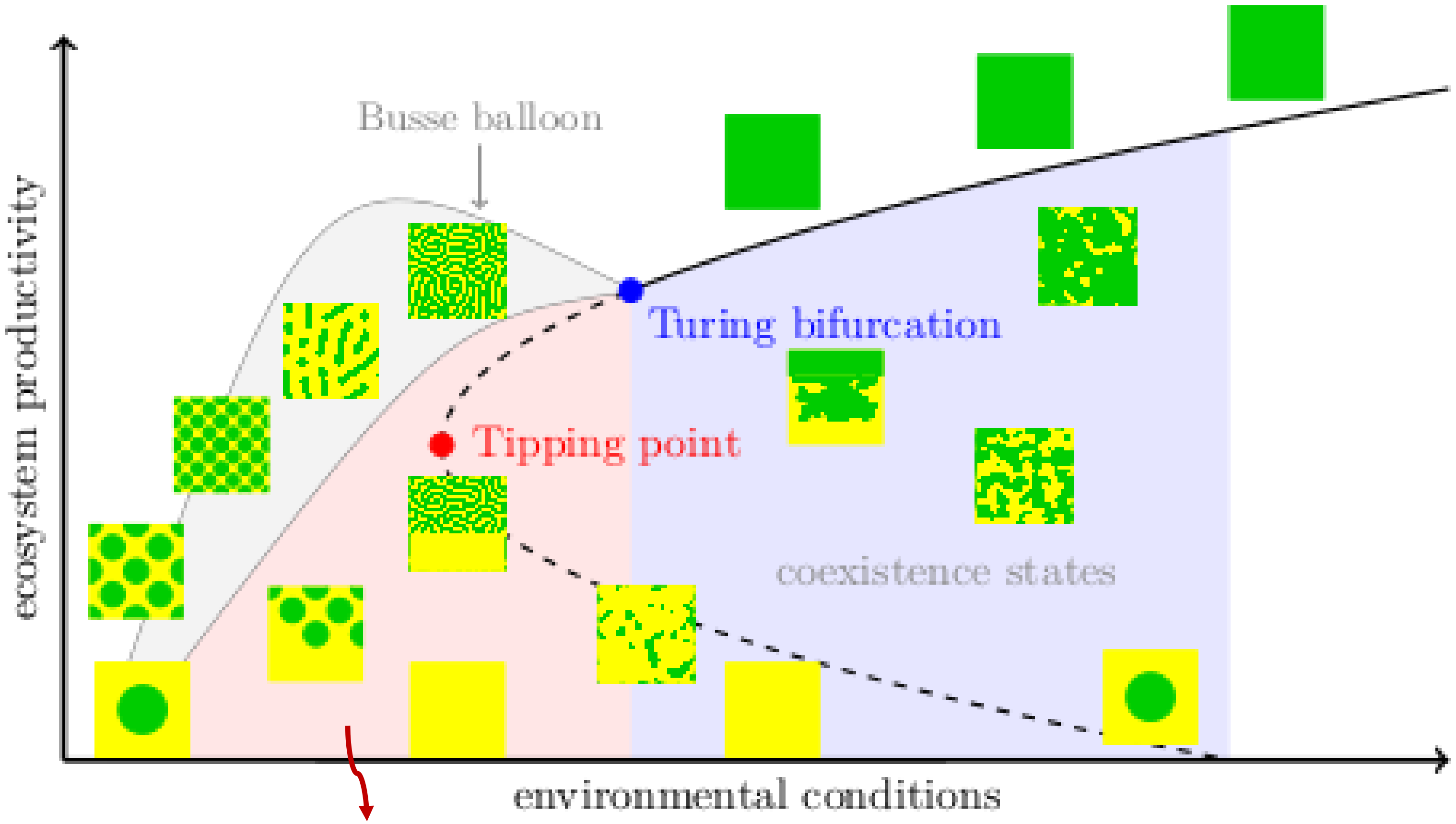

[Bastiaansen, Dijkstra, von der Heydt, 2022]

# Coexistence states in bifurcation diagram

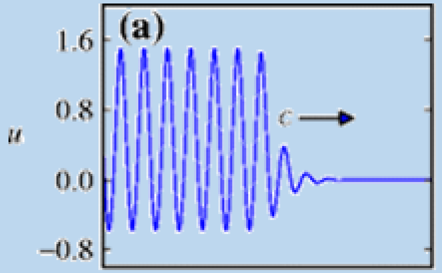


# Tipping in Spatially Extended Systems

# “Bifurcation Diagram” for spatially extended systems



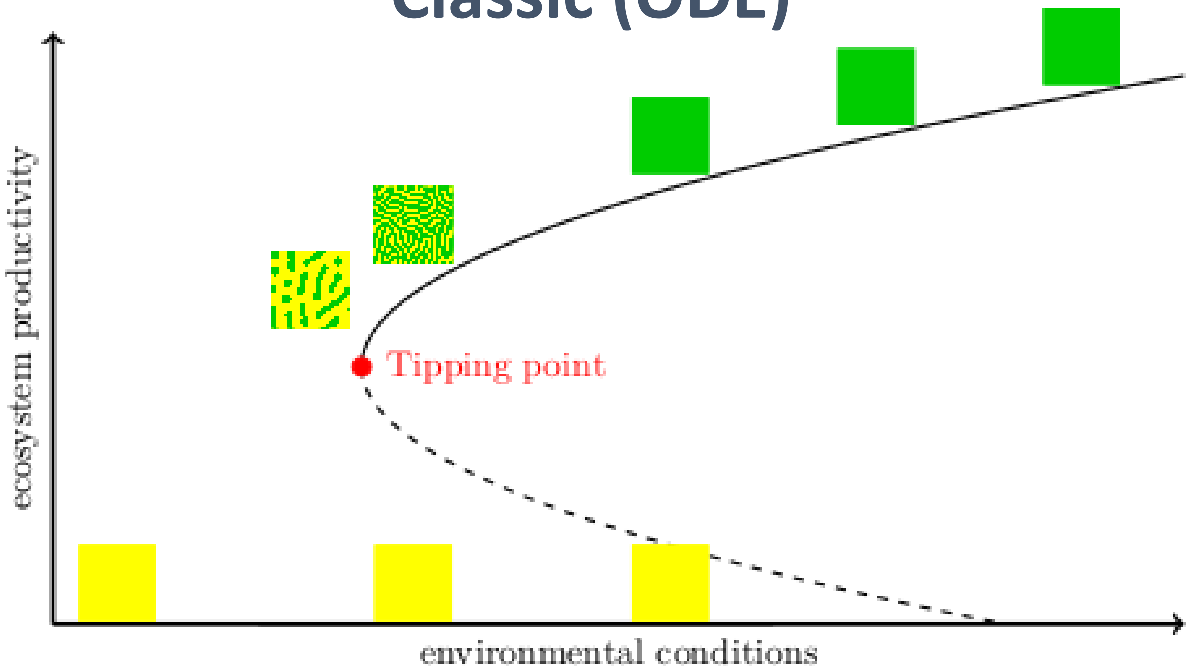

Coexistence states between patterned and uniform states also exist




[Bel et al, 2012]

# What if the system tips?

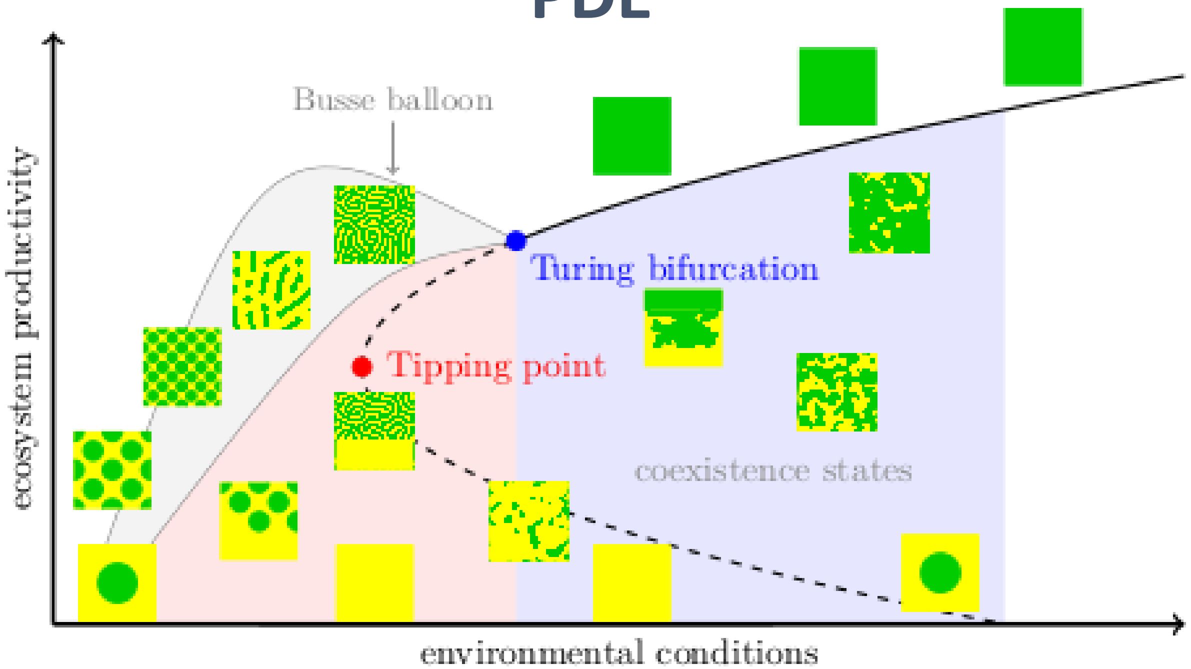
## Classic (ODE)




Crossing a Tipping Point:
→ Always full reorganization

**Early Warning Signals**
signal for <u>WHEN</u>

## PDE




Crossing a bifurcation:
Now also possible:
→ Spatial reorganization (Turing patterns)
→ Fragmented tipping (coexistence states)

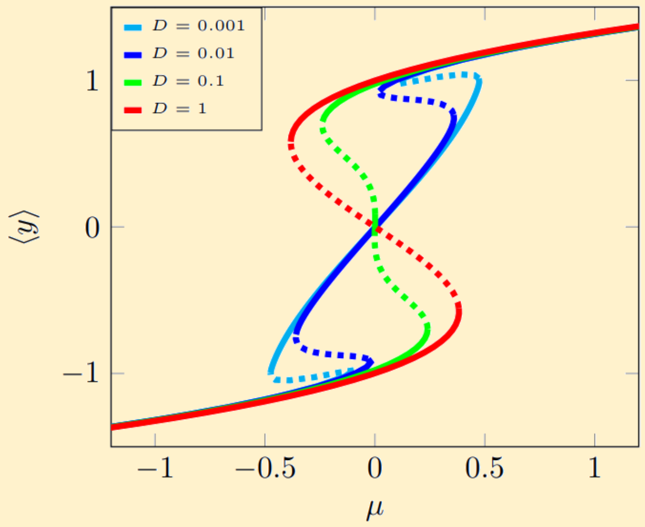
**Early Warning Signals**
need to signal <u>WHEN</u> & <u>WHAT</u>

# Do systems always behave like this? (a.k.a. the small print)

No.

Well-mixed systems

Spatially confined systems




→ Such systems (again) behave like ODEs ←

But even in other systems terms & conditions apply:
System-specific knowledge is required!

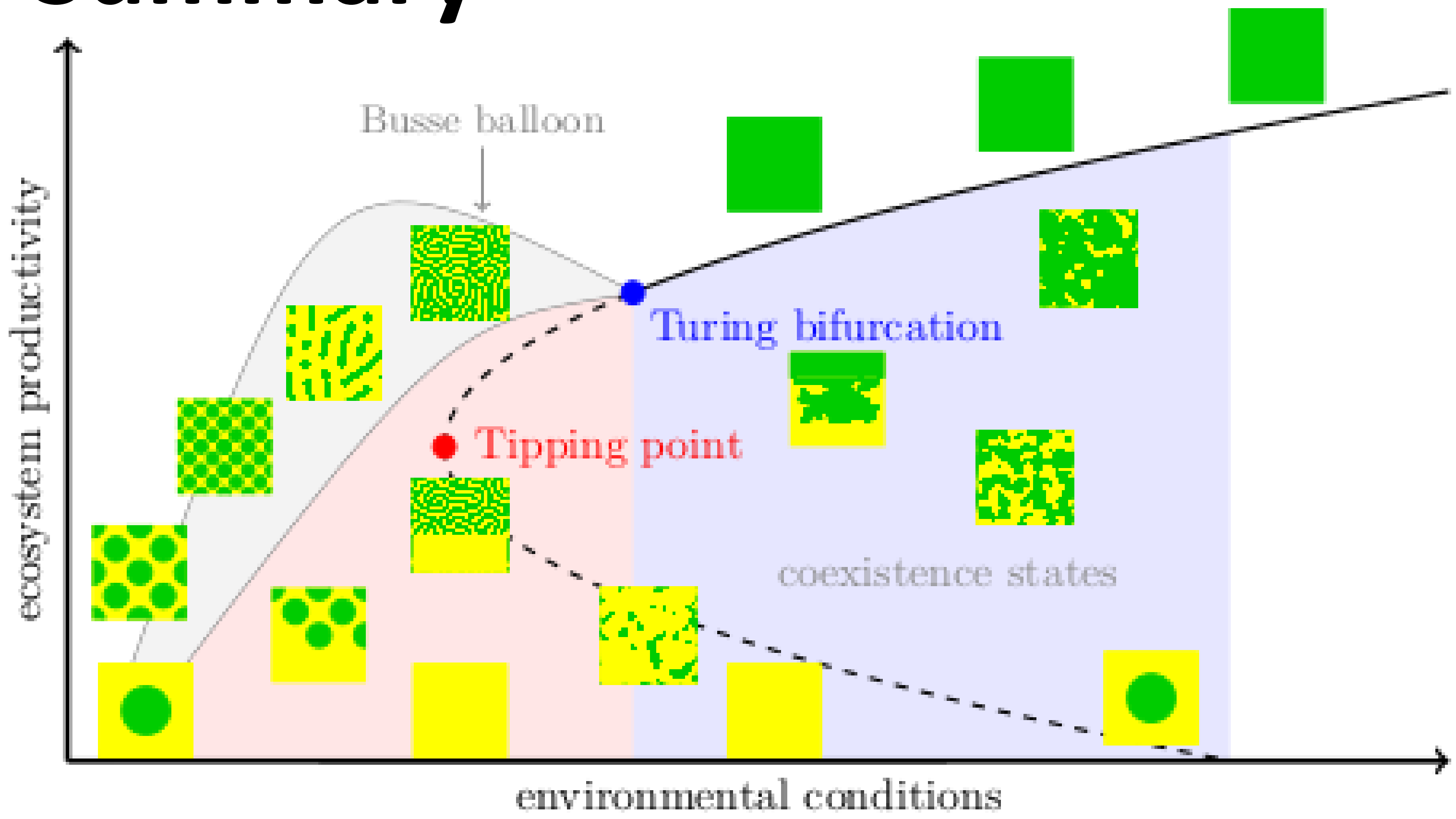
# Summary

## Climate tipping points
- ⚠ Sudden rapid changes
- ⚠ Drastic effects

## Spatial patterns:
- Turing patterns
- Coexistence states

## Tipping can be more subtle:
- Spatial reorganization
- Fragmented tipping

## Dynamics of patterns is:
- Slow pattern adaptation
- Fast pattern degradation





THANKS TO:

Swarnendu Banerjee, Mara Baudena, Paul Sanders, Alexandre Bouvet, Martina Chirilus-Bruckner, Vincent Deblauwe, Aurora Faure Ragani, Arjen Doelman, Henk Dijkstra, Maarten Eppinga, Sjoerd Terpstra, Anna von der Heydt, Olfa Jaïbi, Johan van de Koppel, Swinda Falkena, Stéphane Mermoz, Max Rietkerk, Eric Siero, Tasso Kaper, Koen Siteur