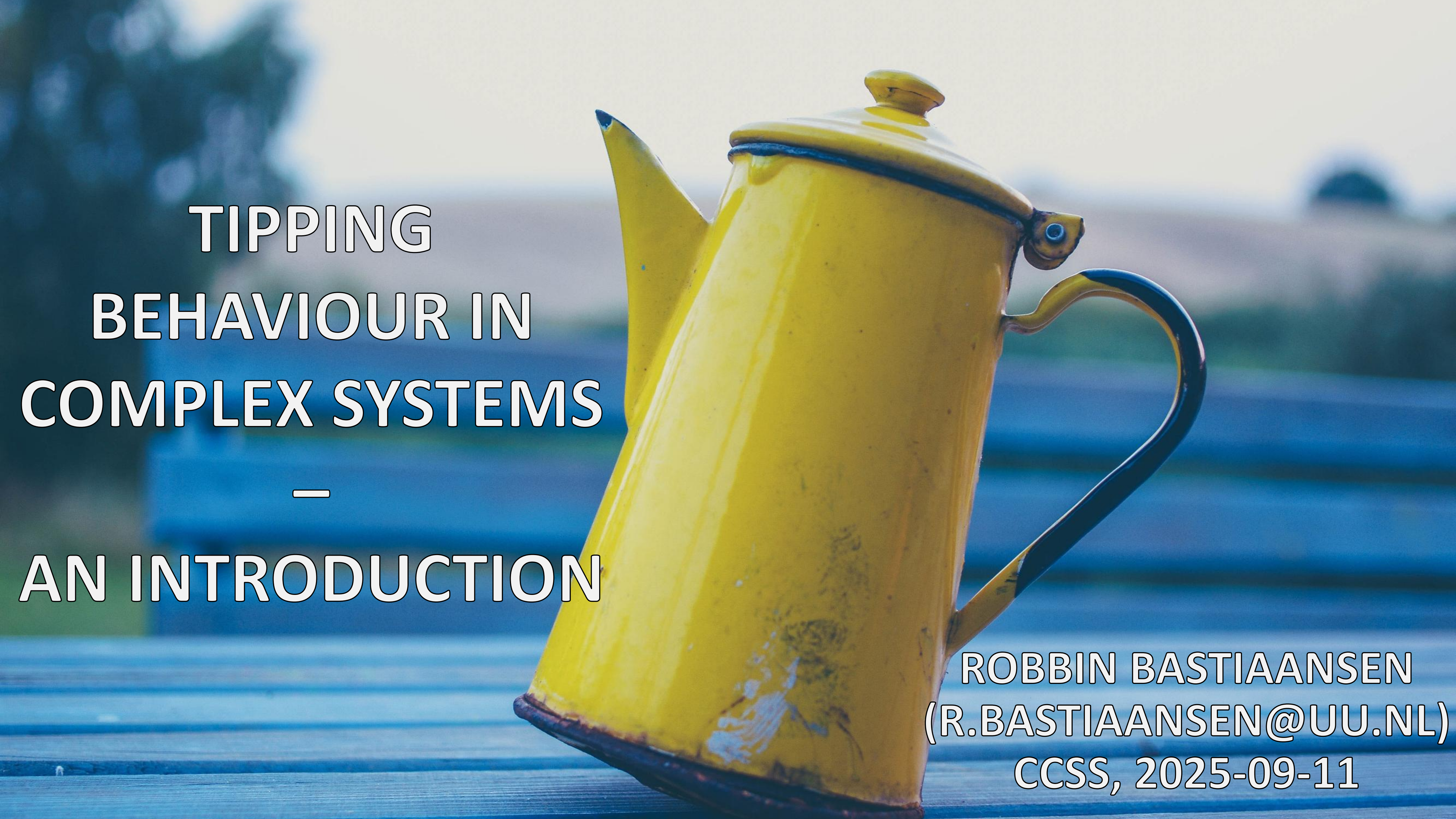
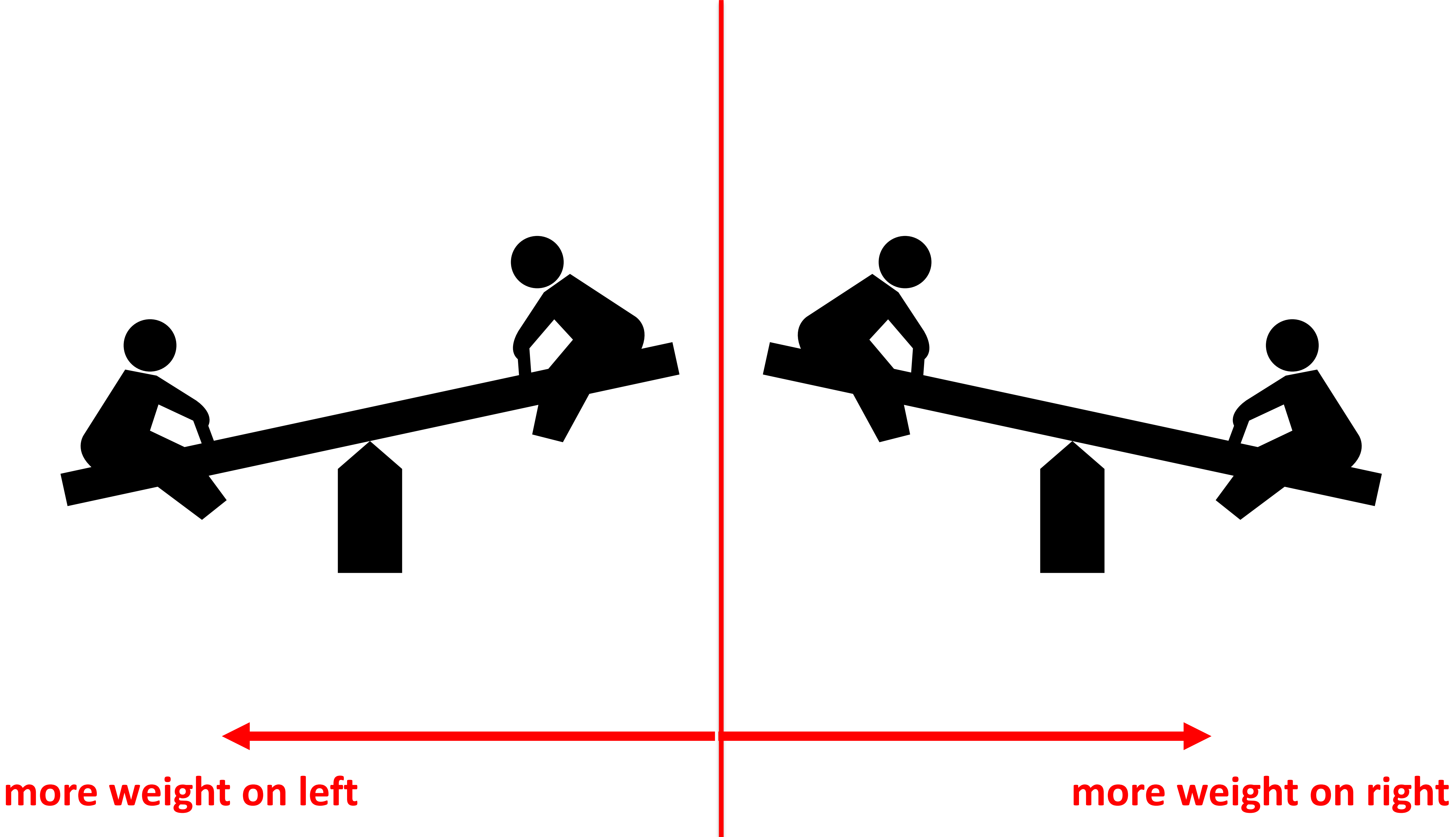


TIPPING BEHAVIOUR IN COMPLEX SYSTEMS – AN INTRODUCTION

ROBBIN BASTIAANSEN
(R.BASTIAANSEN@UU.NL)
CCSS, 2025-09-11

Tipping Points

IPCC AR6 (2021) : "a critical threshold beyond which a system reorganizes, often abruptly and/or irreversibly"


more weight on left
more weight on right

more weight on left

more weight on right

The diffusion of innovation

THE POCKET TELEPHONE: WHEN IT WILL RING!

The latest modern horror in the way of inventions is supposed to be the pocket telephone. We can imagine the moments this instrument will choose for action! —(By W. K. Haselden.)

critical mass achieved: it's everywhere

The diffusion of innovation

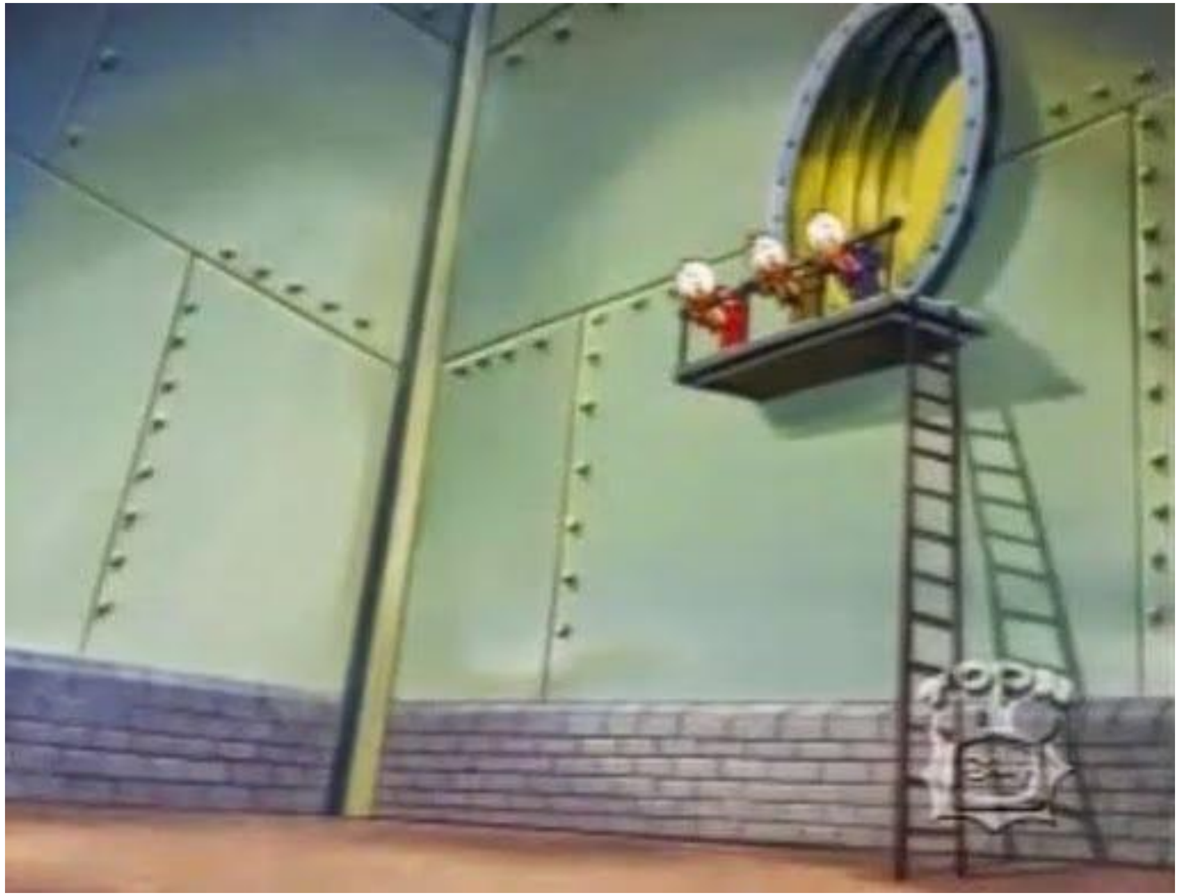
THE POCKET TELEPHONE: WHEN IT WILL RING!

The latest modern horror in the way of inventions is supposed to be the pocket telephone. We can imagine the moments this instrument will choose for action! —(By W. K. Haselden.)

only early adaptors

critical mass achieved: it's everywhere

“How did you go bankrupt? Two ways: Gradually, then suddenly.”
[The Sun Also Rises by Ernest Hemingway]



few bad decisions **few too many**

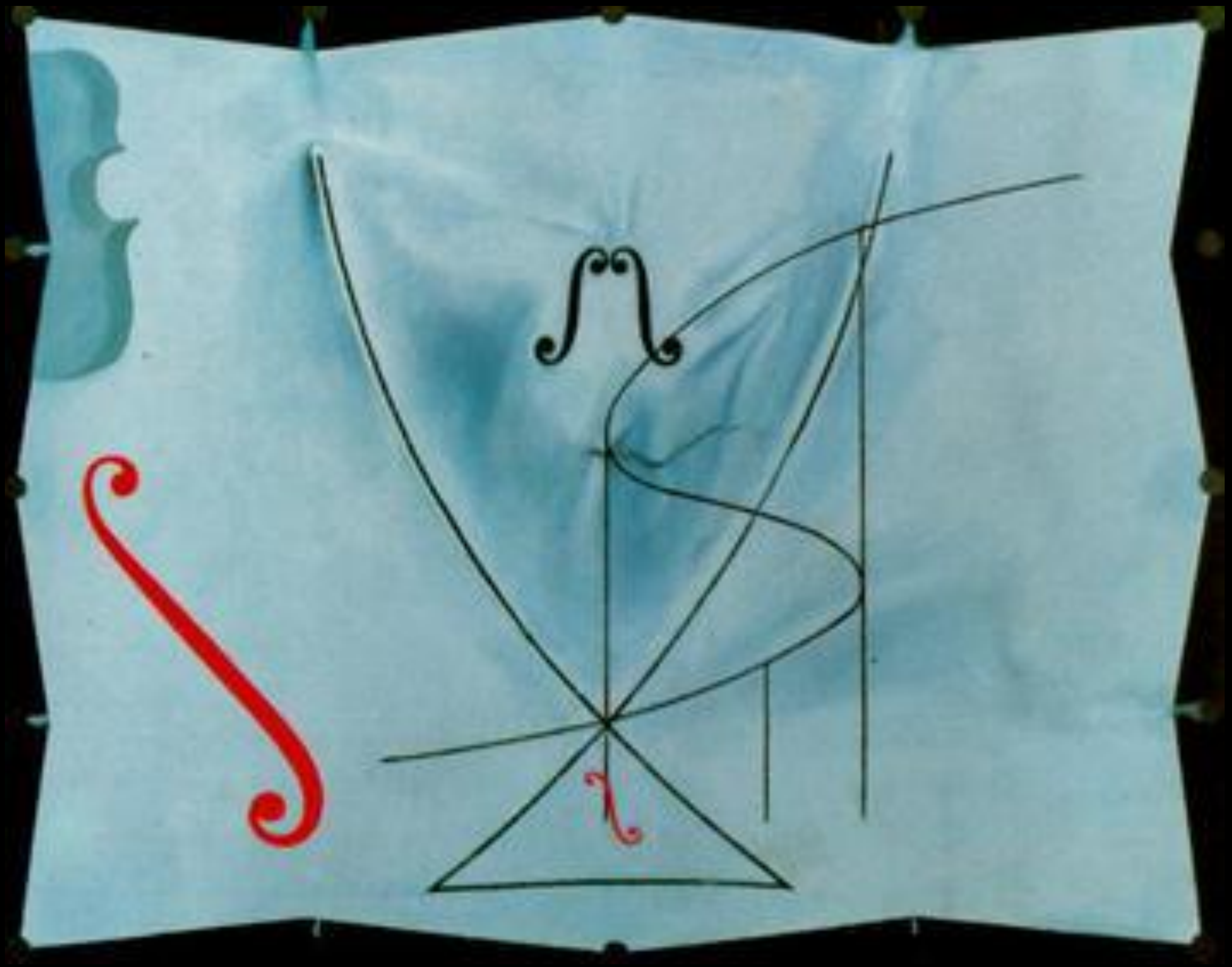
'Intelligent, articulate, thought-provoking'
OBSERVER

THE TIPPING POINT

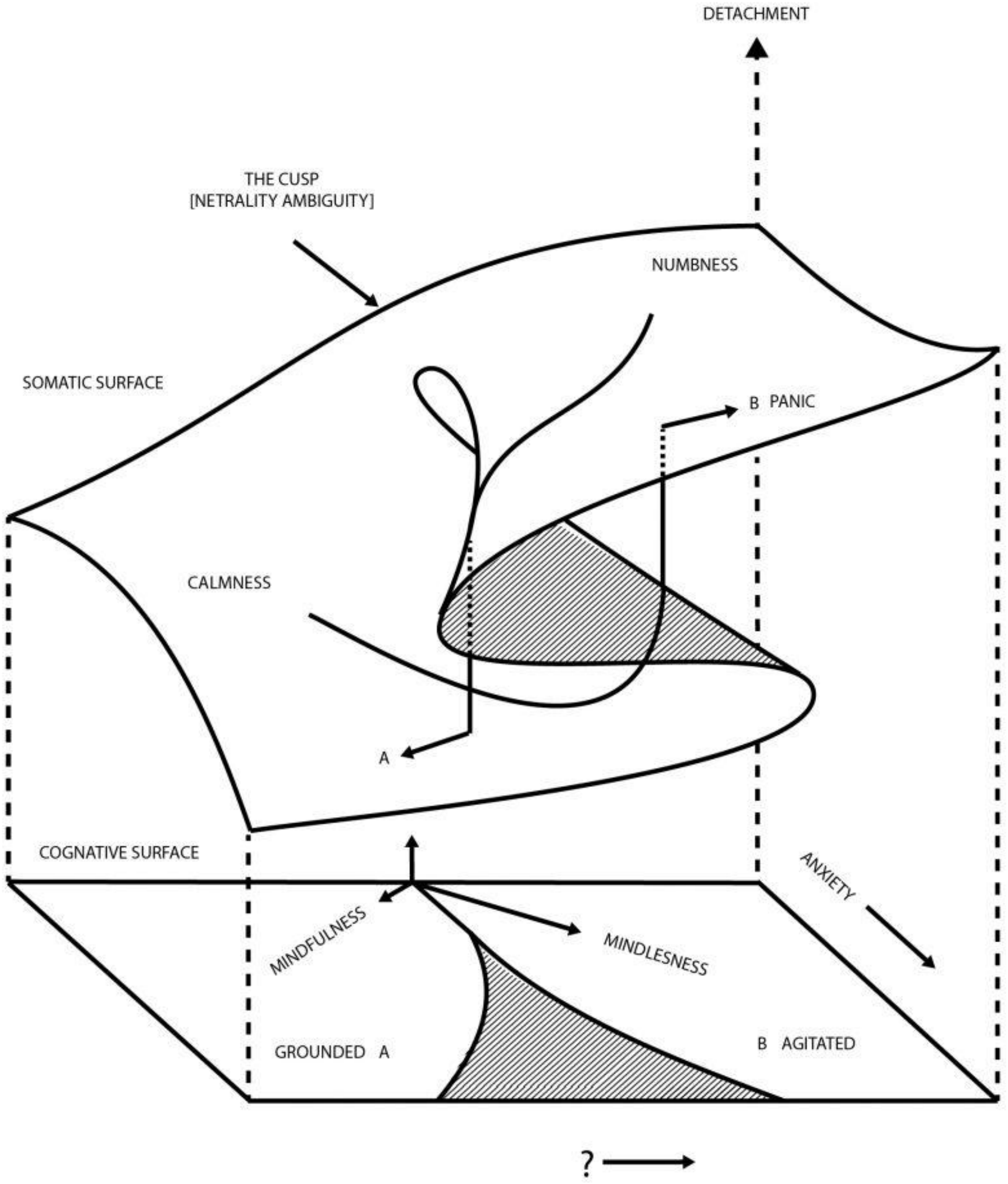
MALCOLM GLADWELL

HOW LITTLE THINGS CAN
MAKE A BIG DIFFERENCE

The International Number One Bestseller

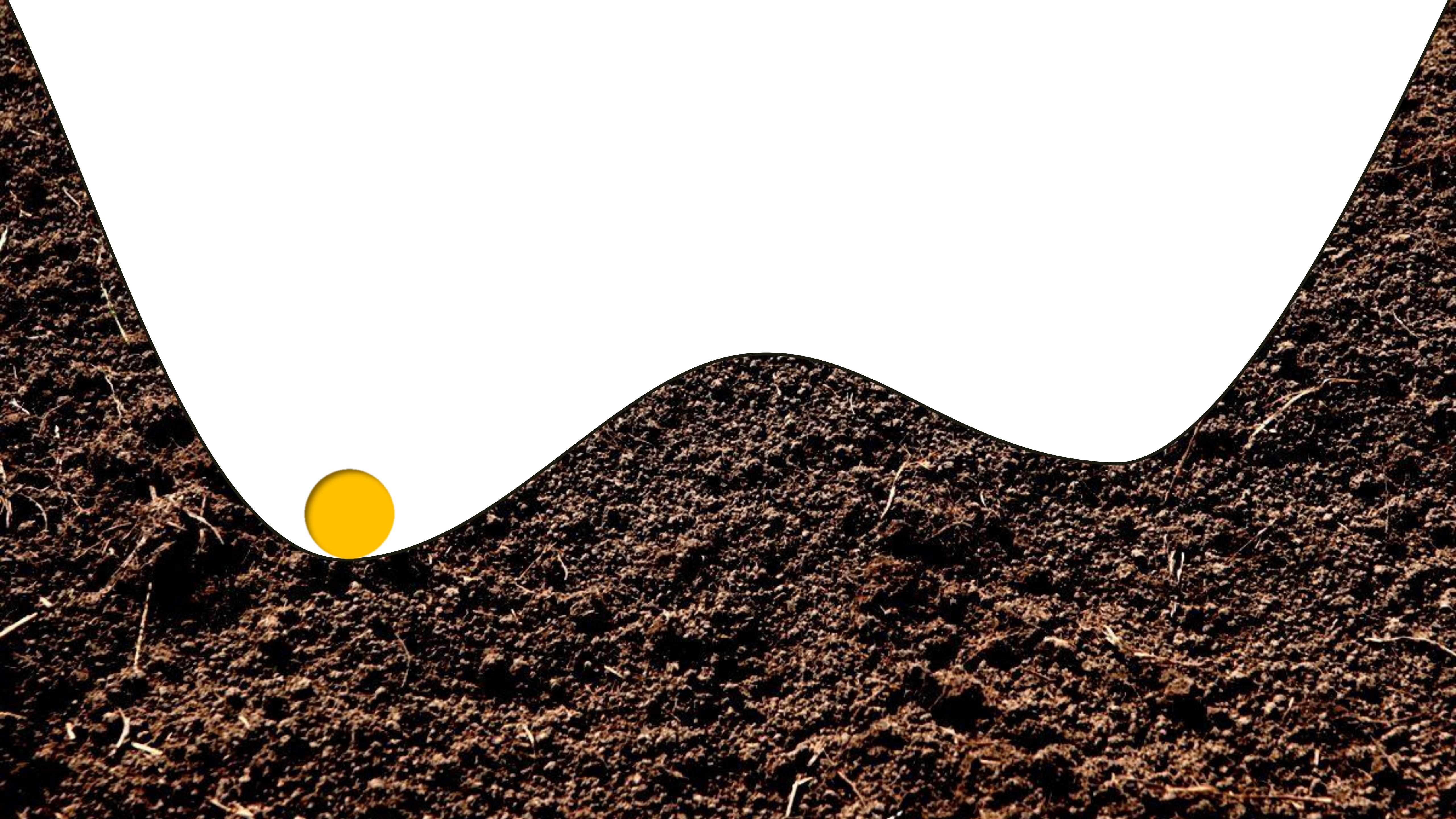


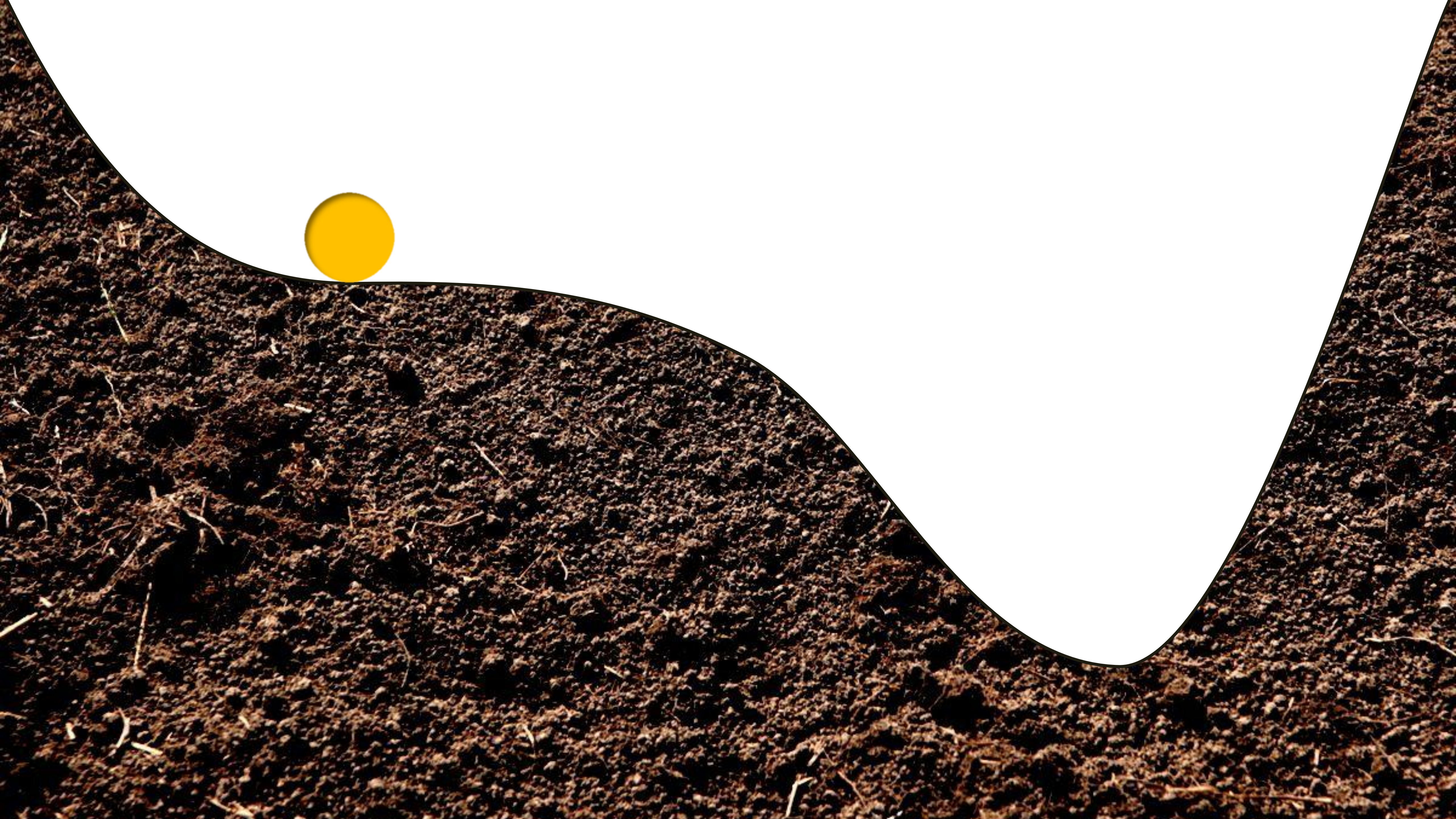
Catastrophe theory

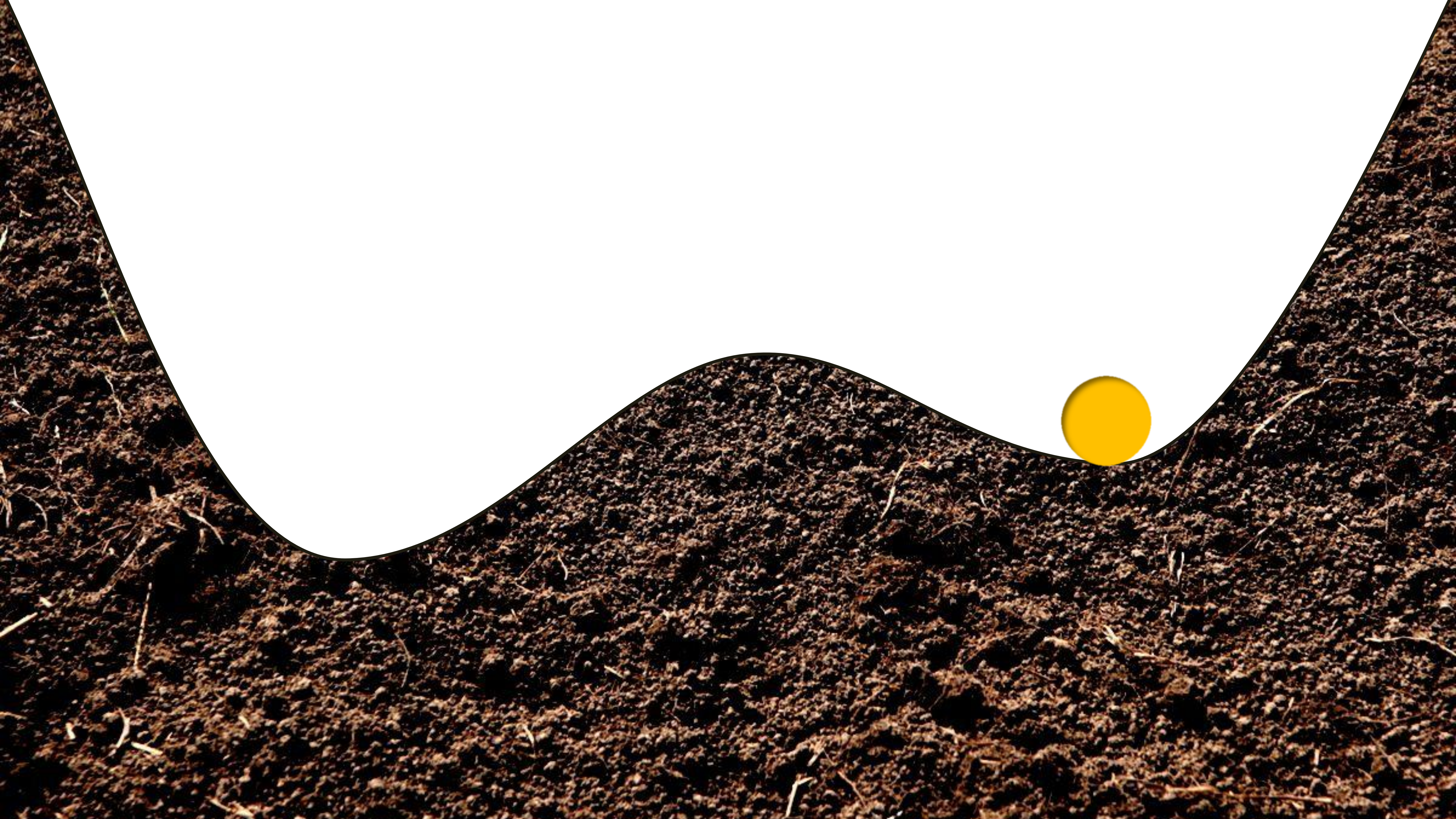


Studies structural changes in the potential function, Energy function or Lyapunov function of a dynamical system

$$\frac{dy}{dt} = -\frac{\delta V(y)}{\delta y}$$

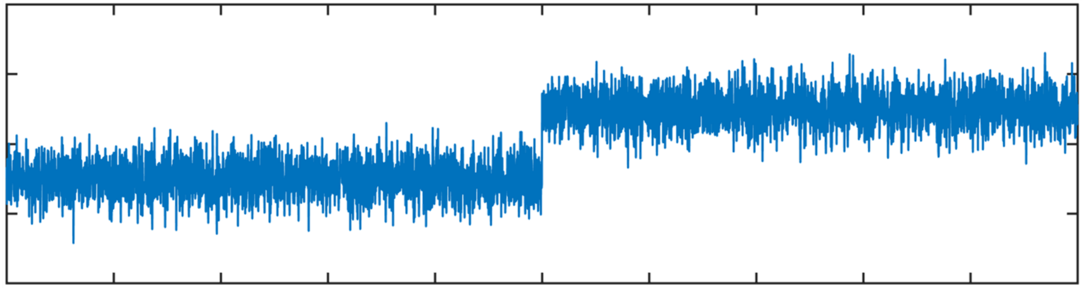


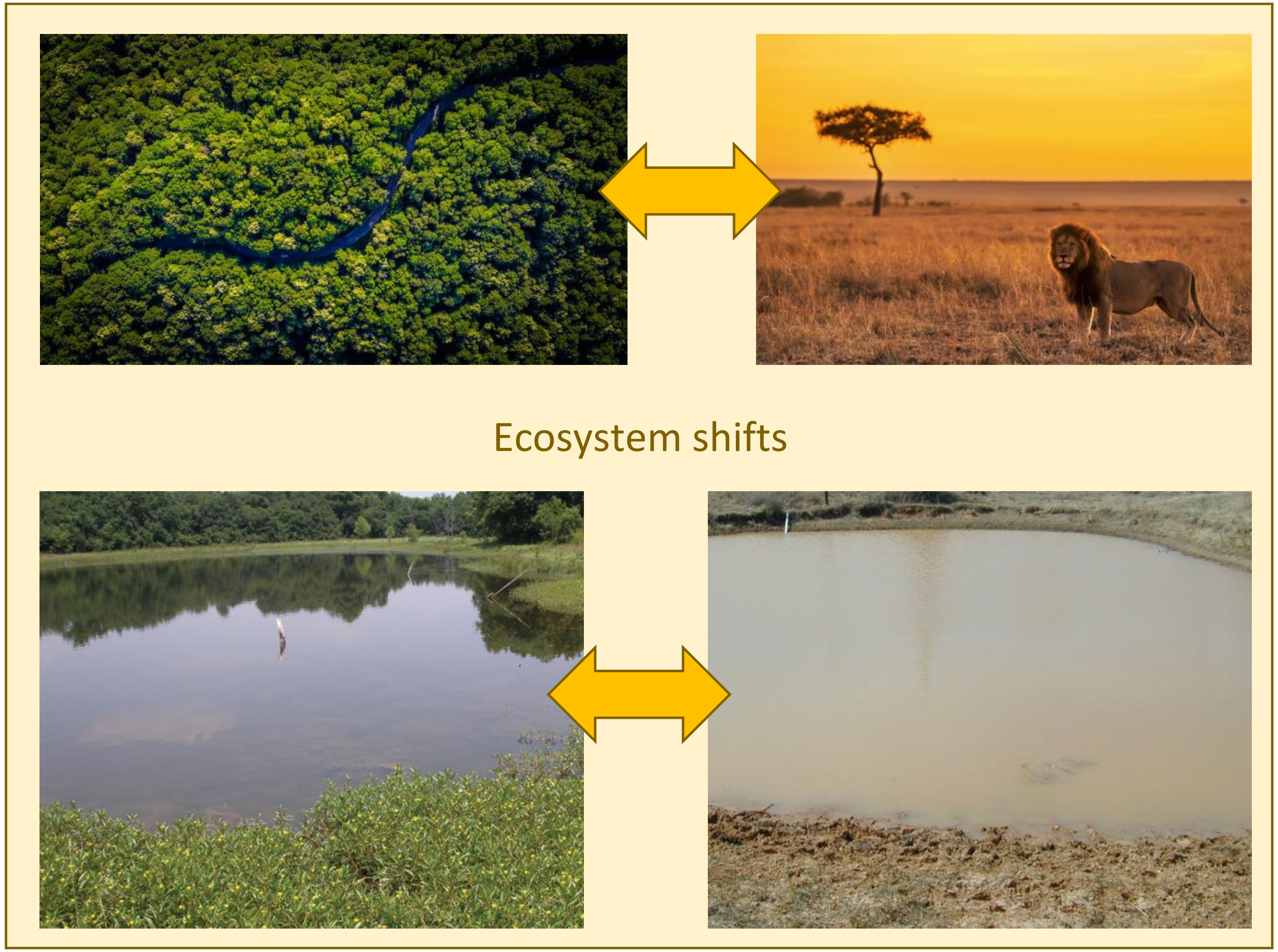




Critical shifts

Classic Literature
[Holling, 1973]
[Noy-Meier, 1975]
[May, 1977]





Critical shifts

Classic Literature

[Holling, 1973]

[Noy-Meier, 1975]

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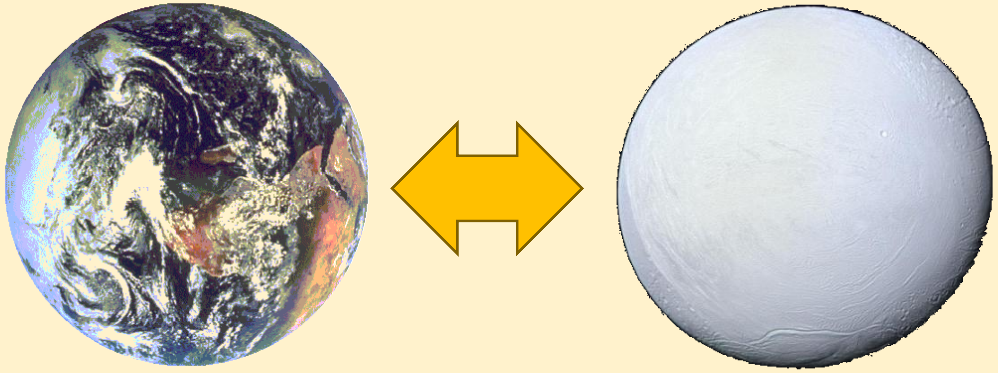
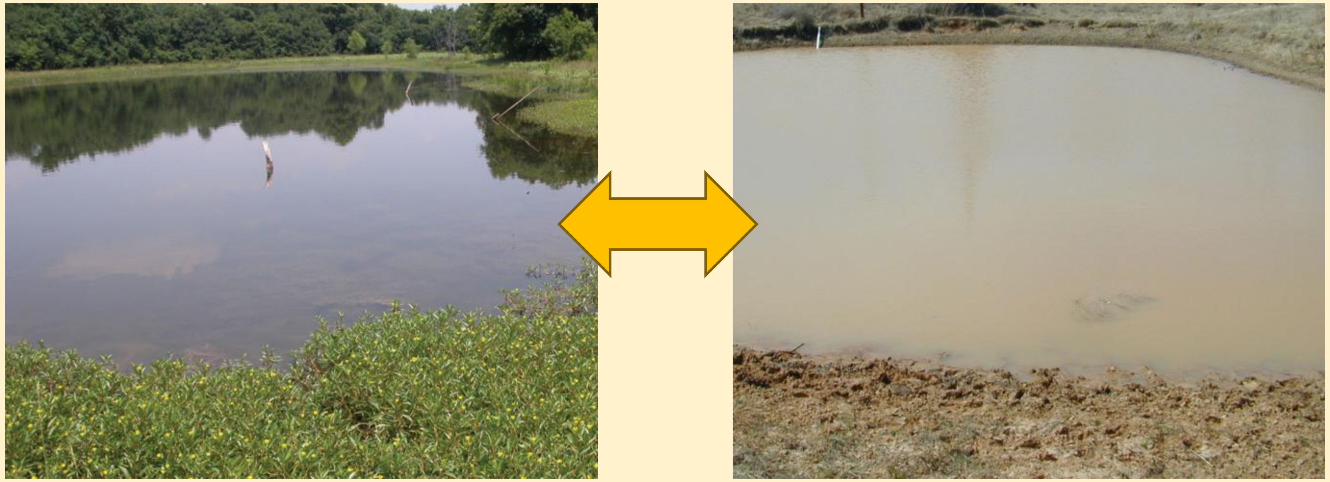


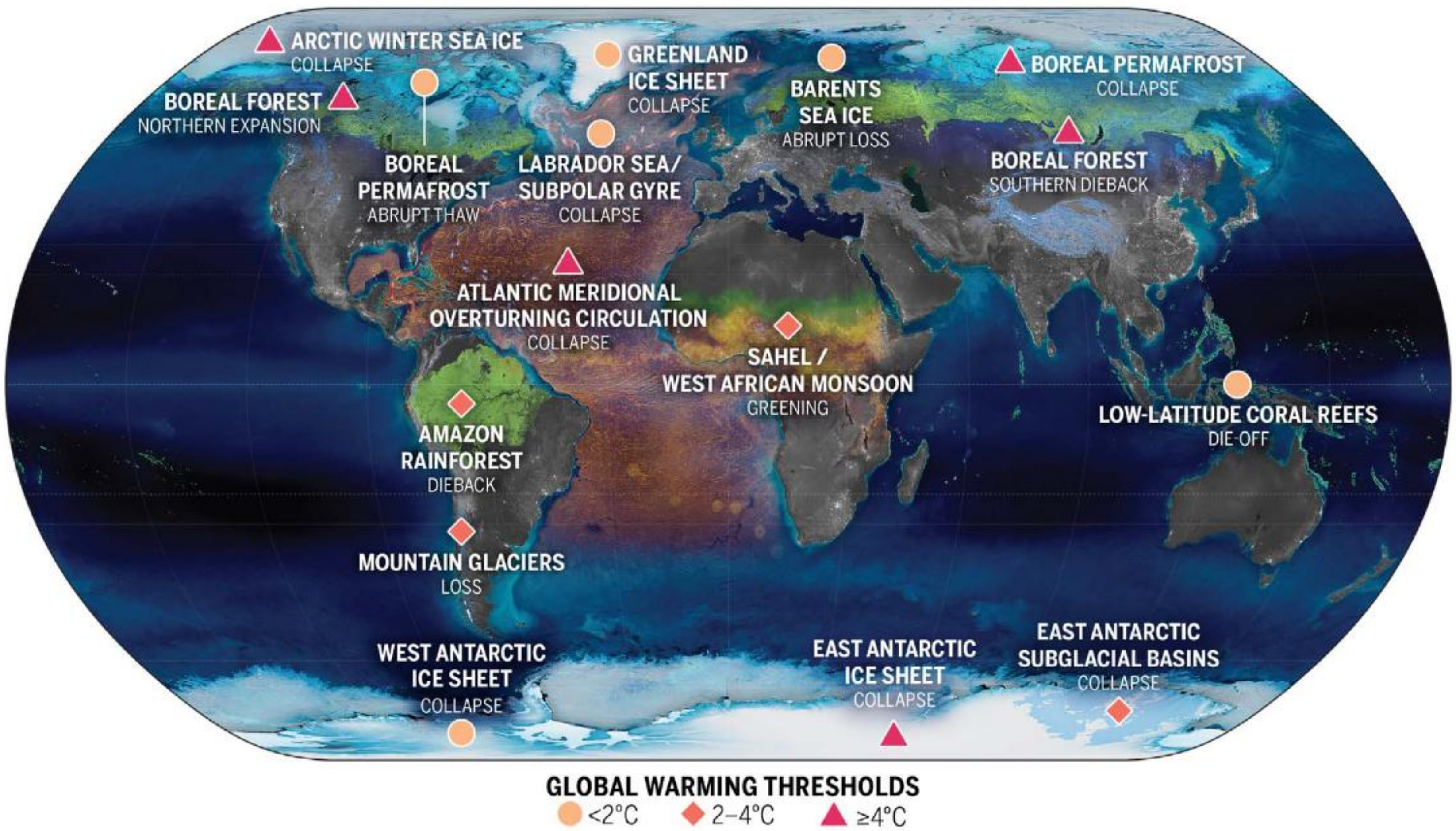
Planetary transitions

Ecosystem shifts



Tipping Points

IPCC AR6 (2021) : "a critical threshold beyond which a system reorganizes, often abruptly and/or irreversibly"

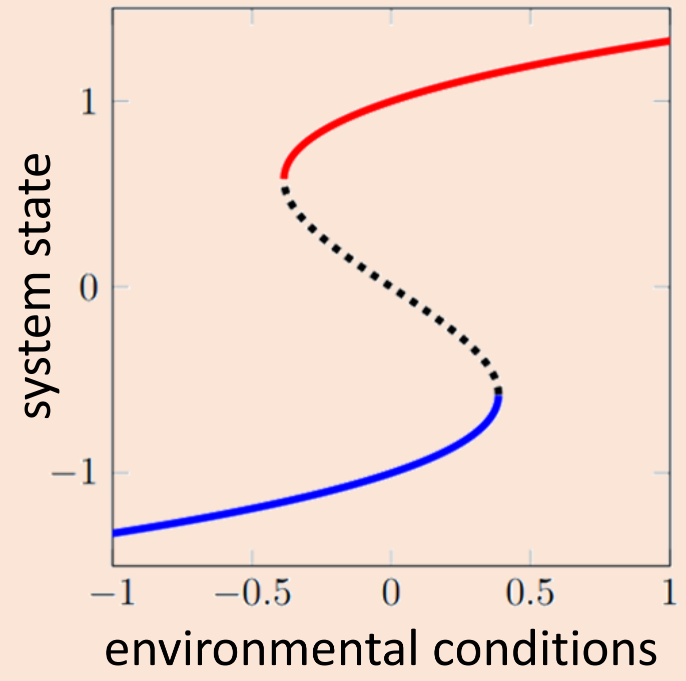


source: McKay et al, 2022

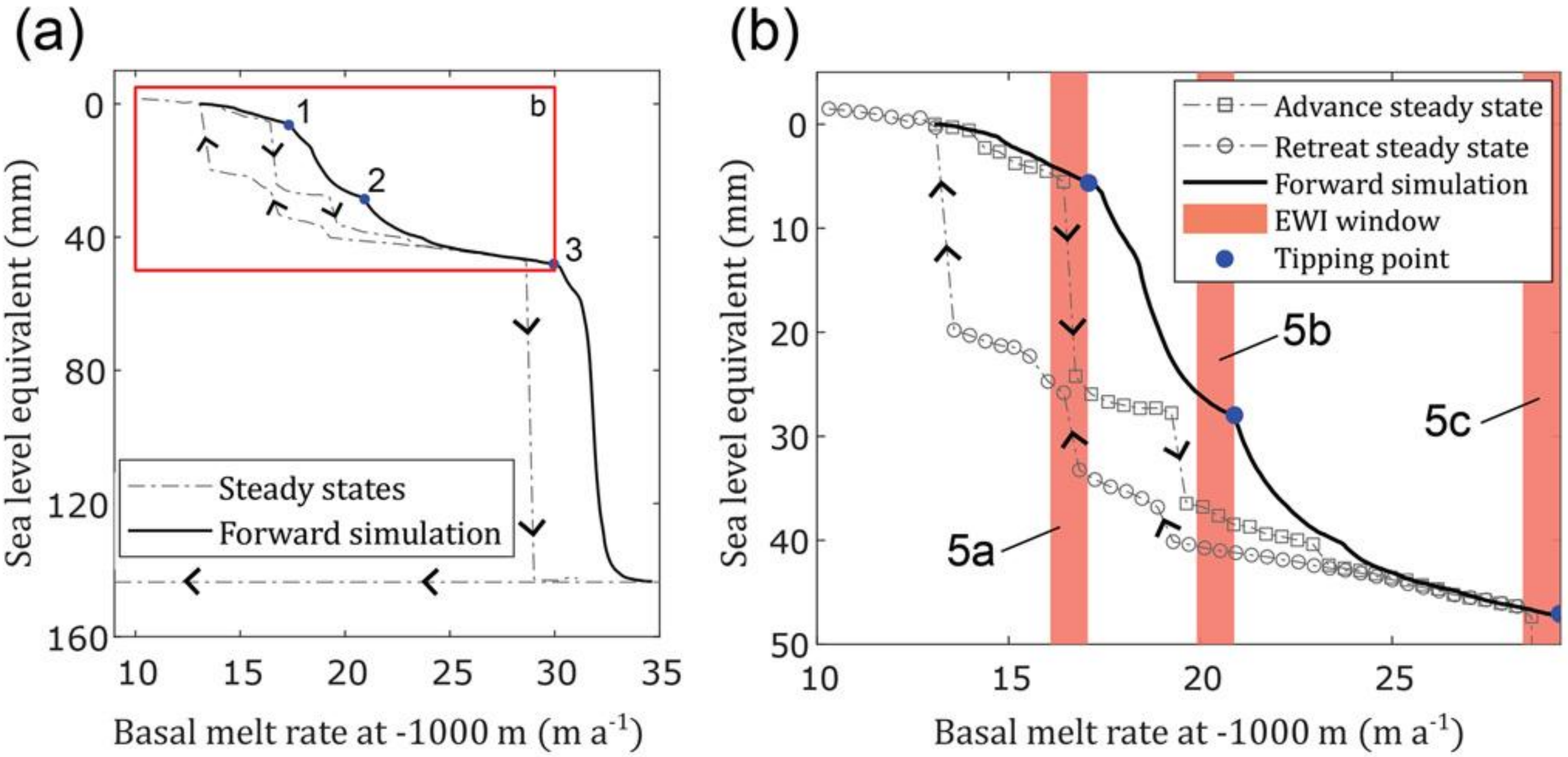
Mathematics

Tipping points ↔ Bifurcations

$$\frac{dy}{dt} = f(y, \mu)$$



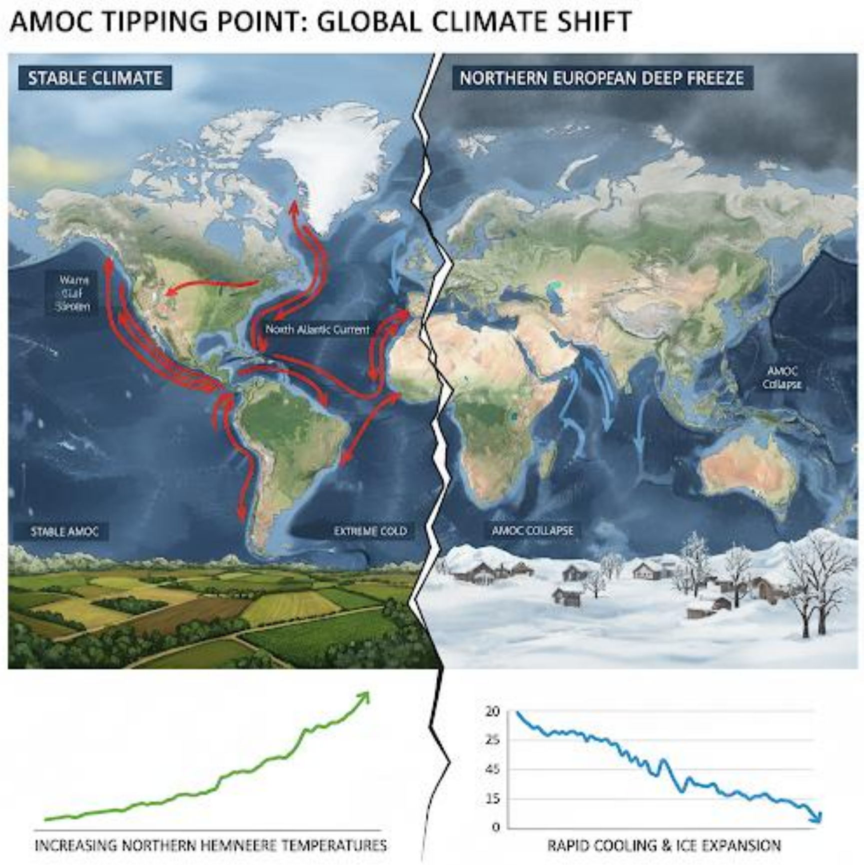
Tipping - Pine Island Glacier, West Antarctica



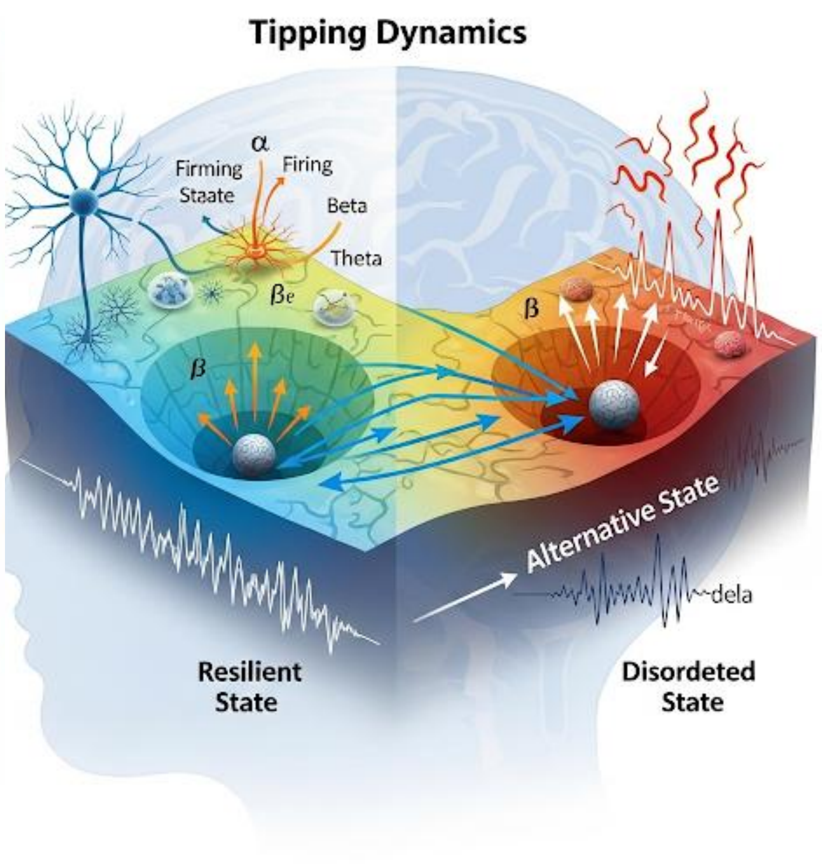
[Rosier et al, 2021]

Tipping in other systems

Power grid



Ocean circulation



Brain dynamics

Images by Gemini for visualisation. They do not have scientific meaning. I guess the tipping point for AI singularity is still far away…

Classic Theory of Tipping

$$\frac{d\vec{y}}{dt} = f(\vec{y};\mu)$$

Canonical example:

$$\frac{dy}{dt} = y(1 - y^2) + \mu$$

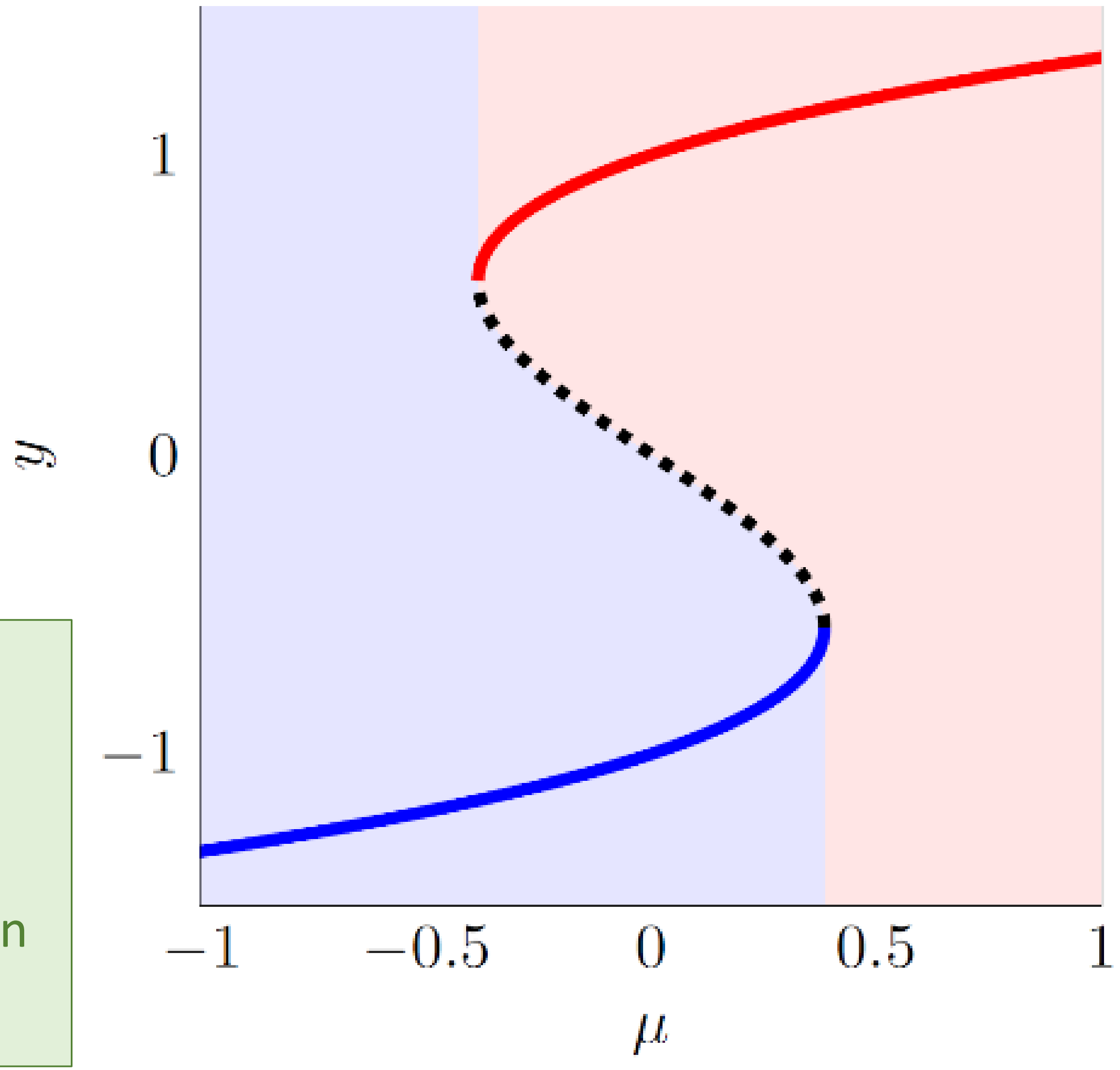
Tipping

[Ashwin et al, 2012]

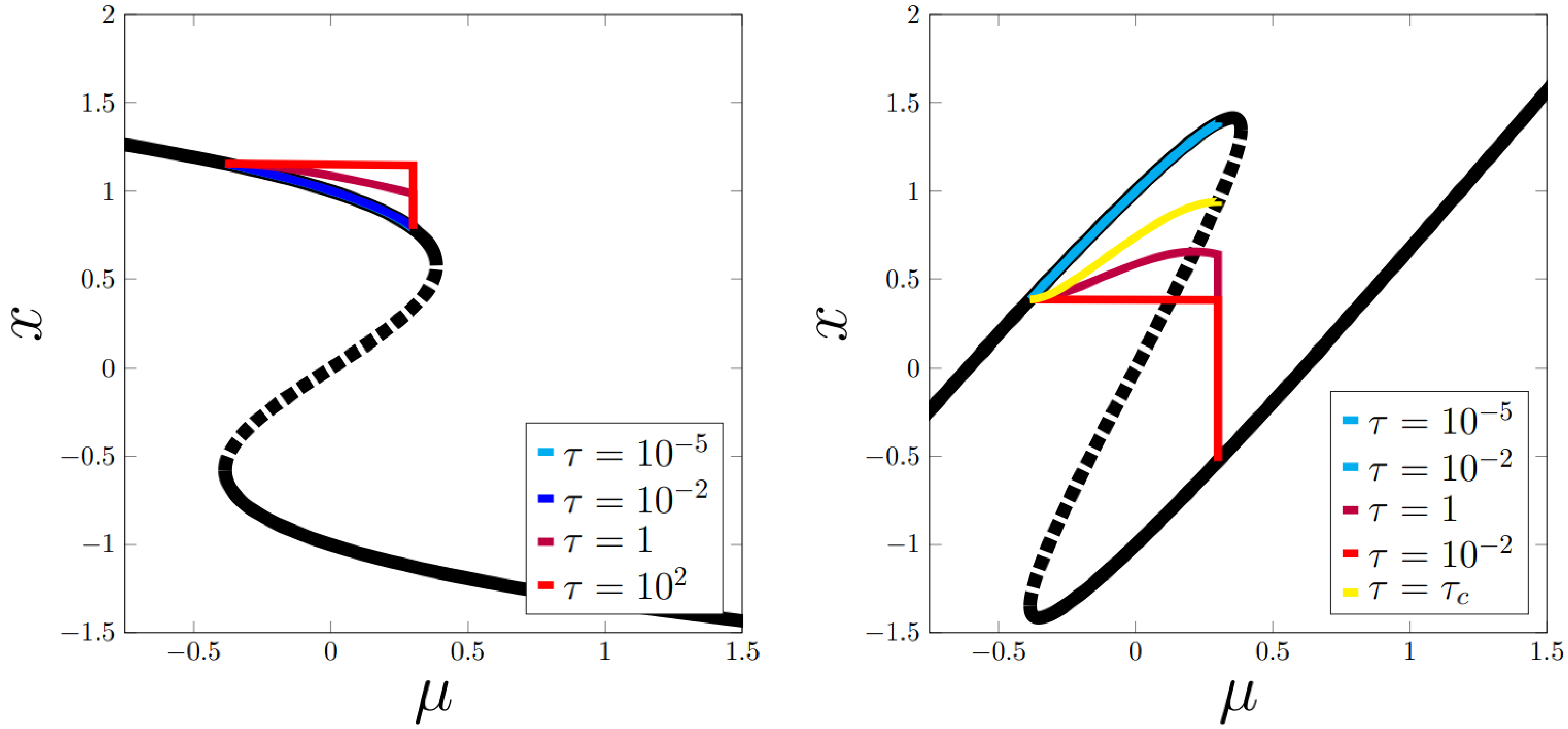
Bifurcation-Tipping : Basin disappears

Noise-Tipping : Forced outside Basin

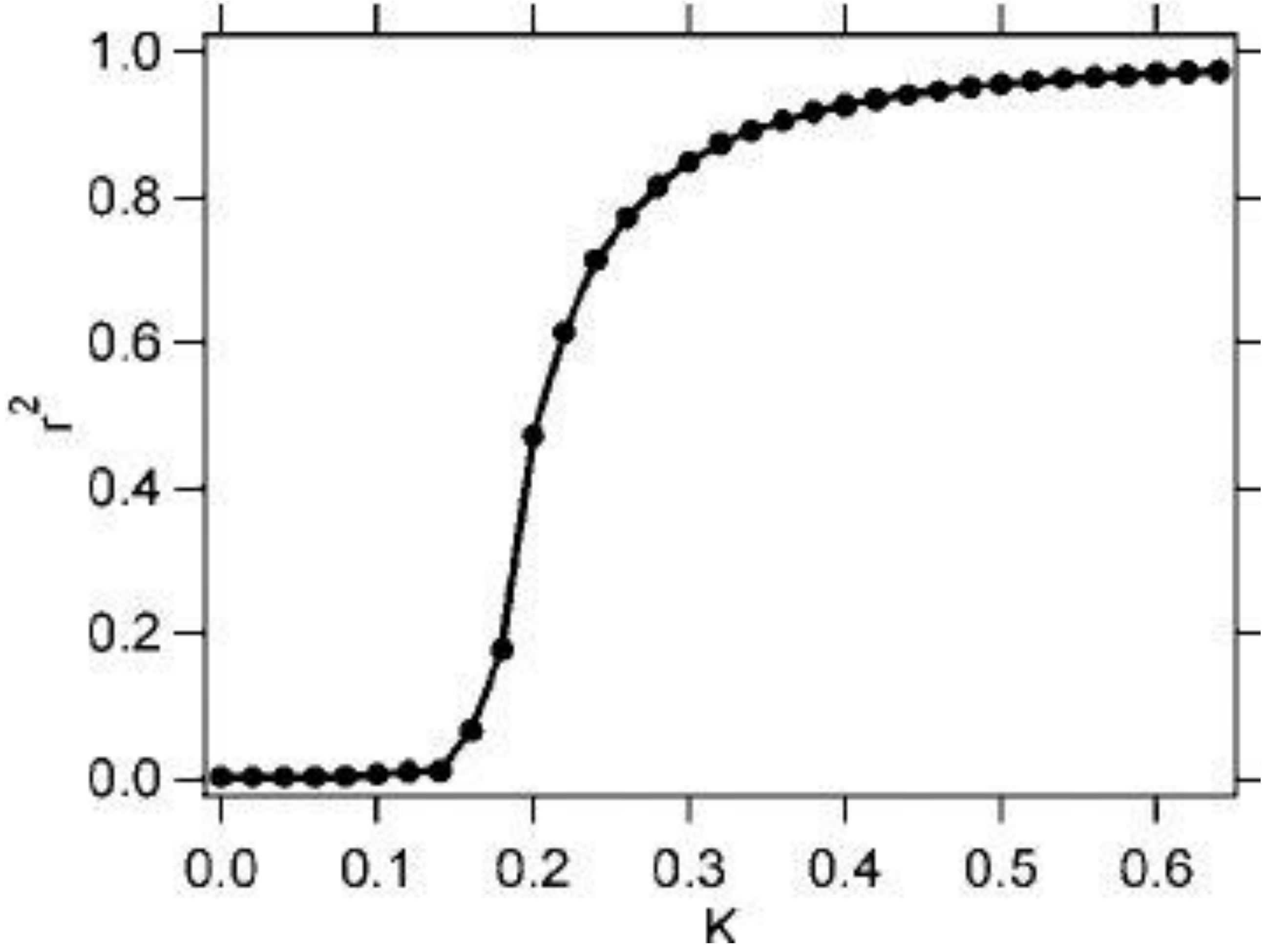
Rate-Tipping : *(more complicated)*



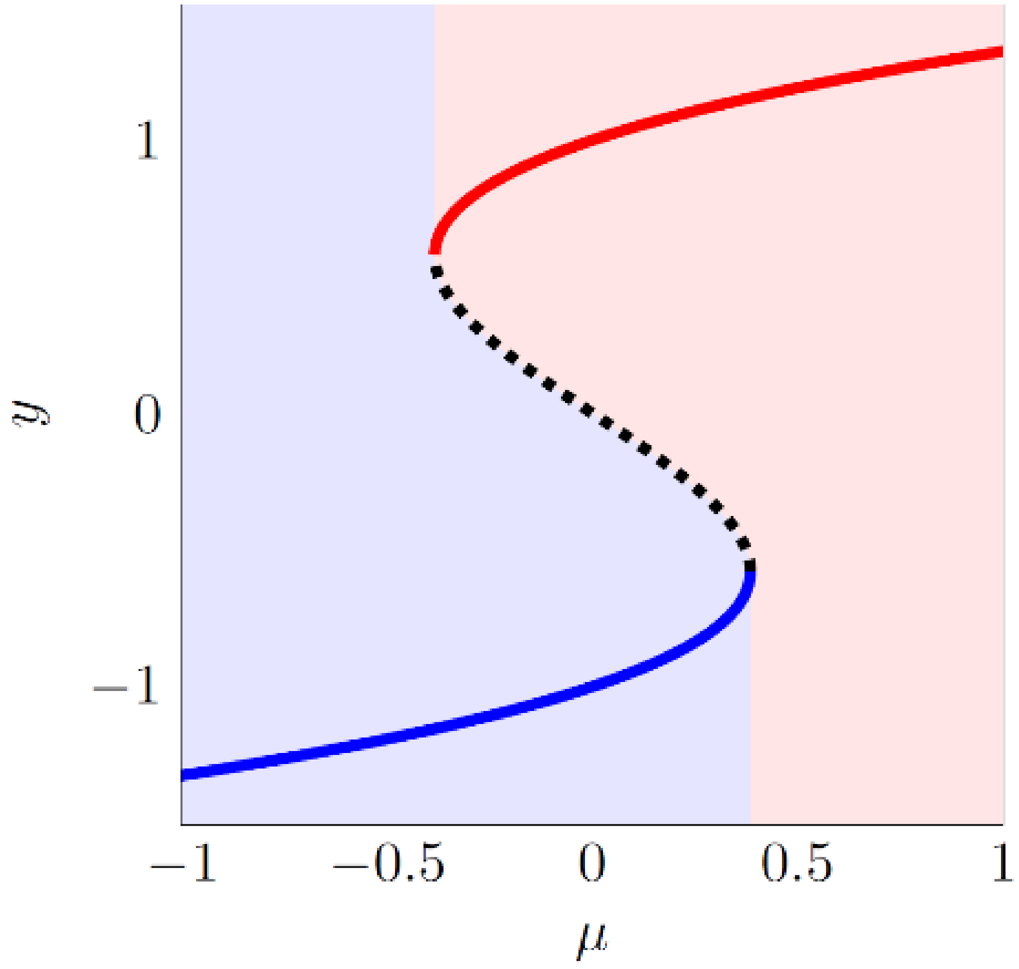
Rate-Tipping



Phase transitions



Kuramoto model [English, 2007]



$$\frac{dy}{dt} = y(1 - y^2) + \mu$$

Timescales

Systems with multiple time scales

$$\frac{dx}{dt} = f(x;\mu)$$ ← Internal Dynamics

$$\frac{d\mu}{dt} = \tau g(\tau t)$$ ← Parameter Drift

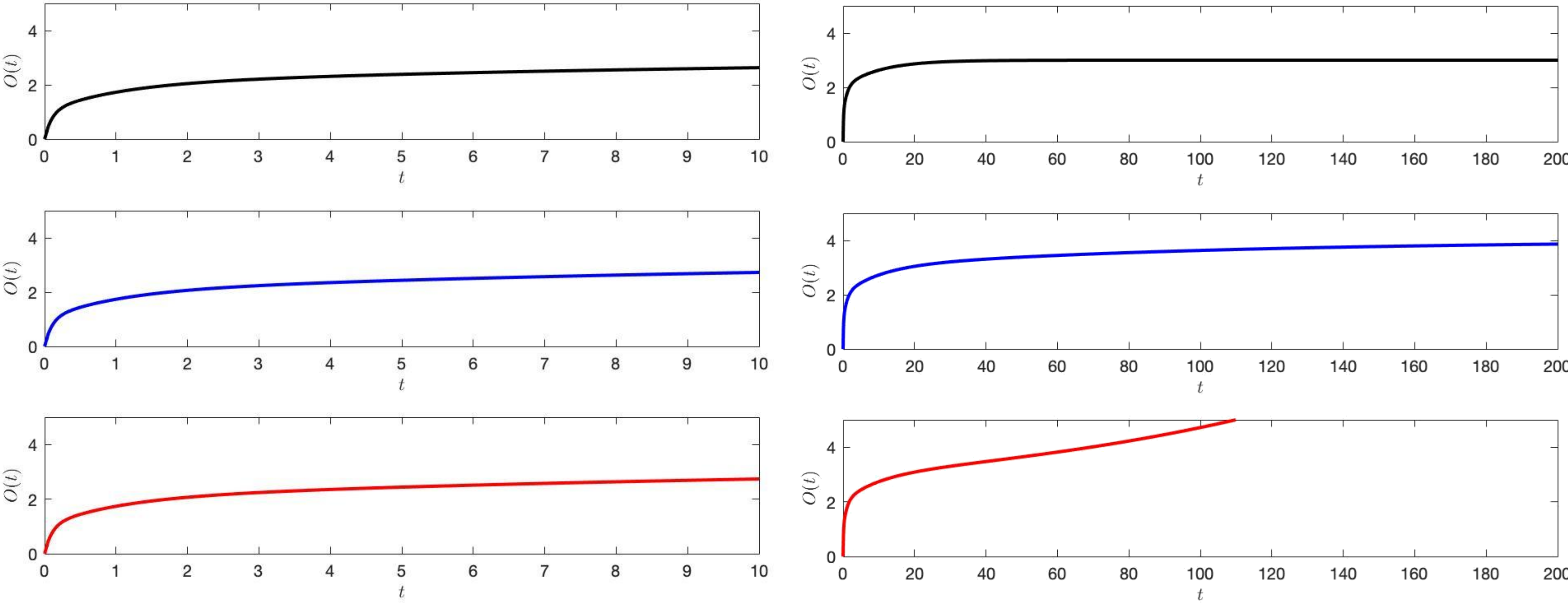
Time Scale Separation (both τ)

$\tau \ll 1$: forcing slow compared to system dynamics → B-tipping

$\tau \gg 1$: forcing fast compared to system dynamics → S-tipping

$\tau = \mathcal{O}(1)$: forcing comparable to system dynamics →R-tipping

Importance of timescales



[Bastiaansen, Ashwin, Von der Heydt, 2023]

EXAMPLE 1: Multiscale Global Energy Balance Model

$$C\frac{dT}{dt} = Q_0(1 - \alpha) - \epsilon(T)\sigma T^4 + \mu$$

Short-Wave

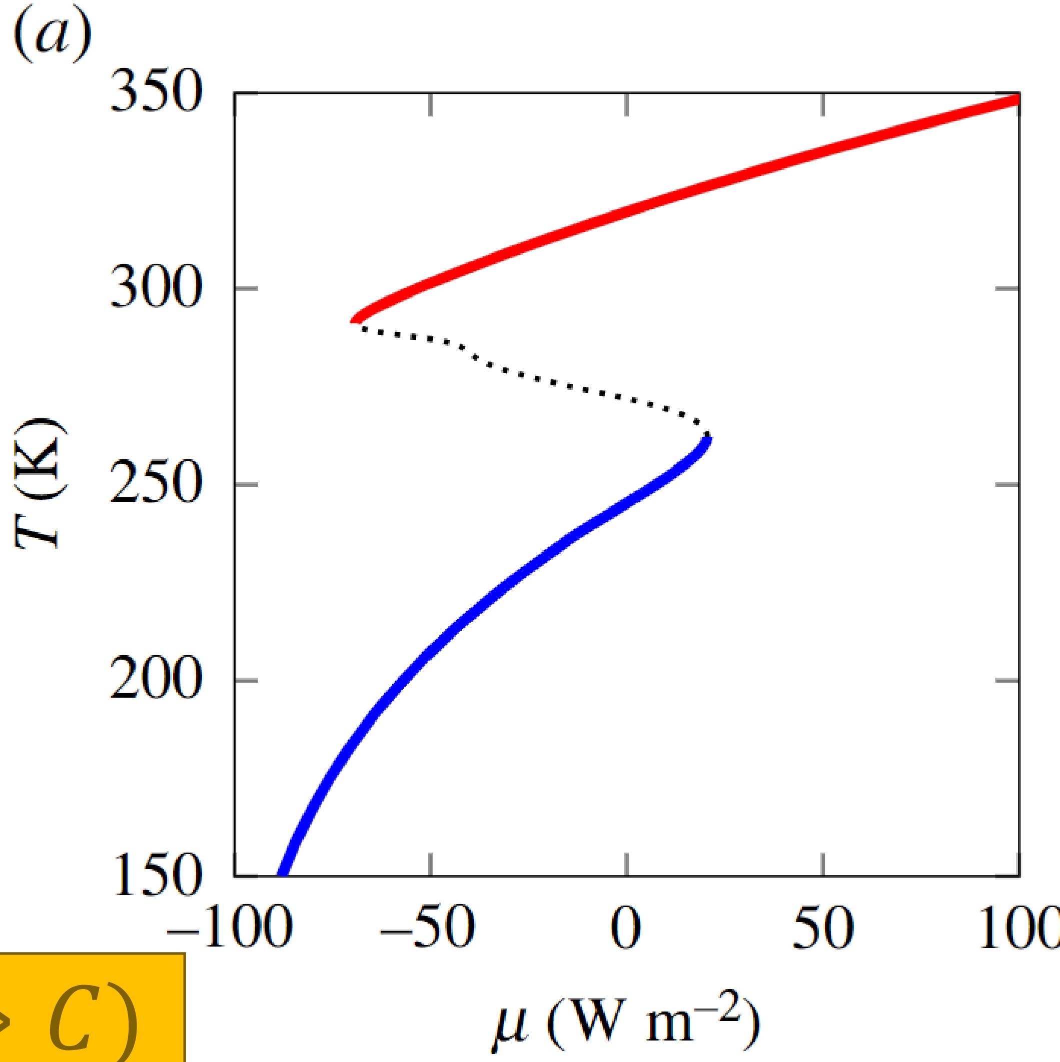
Long-Wave

CO2

$$\tau_\alpha \frac{d\alpha}{dt} = \alpha_0(T) - \alpha$$

Dynamic albedo

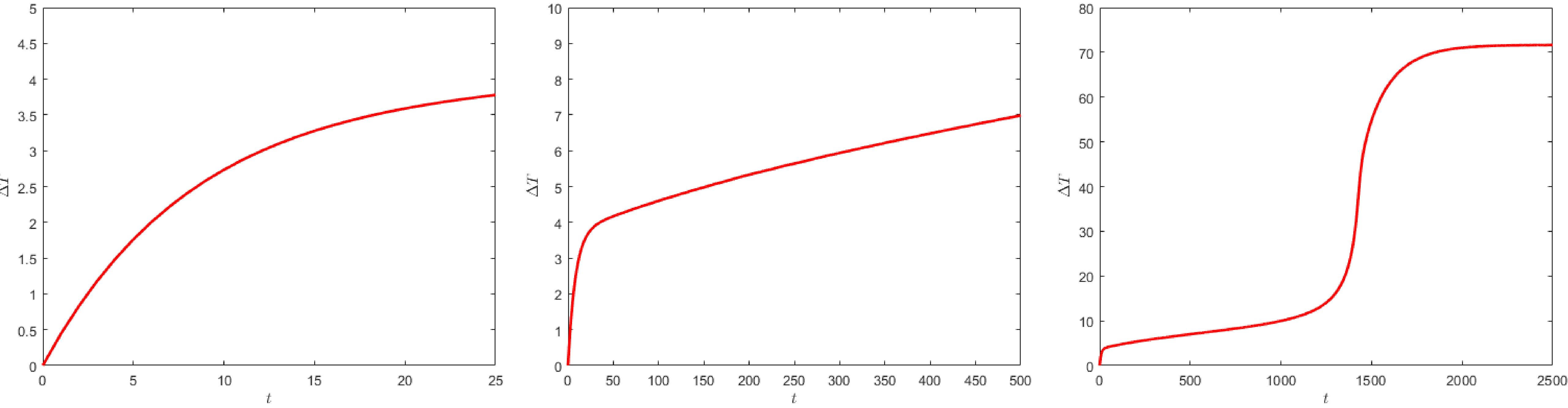
Internal Time Scale Separation ($\tau_\alpha \gg C$)



Abrupt 4xCO2 forcing experiment

- Initialize for μ_0 (initial CO2-levels)
- Change to μ_1 (4xCO2 levels)

→ Look at dynamics

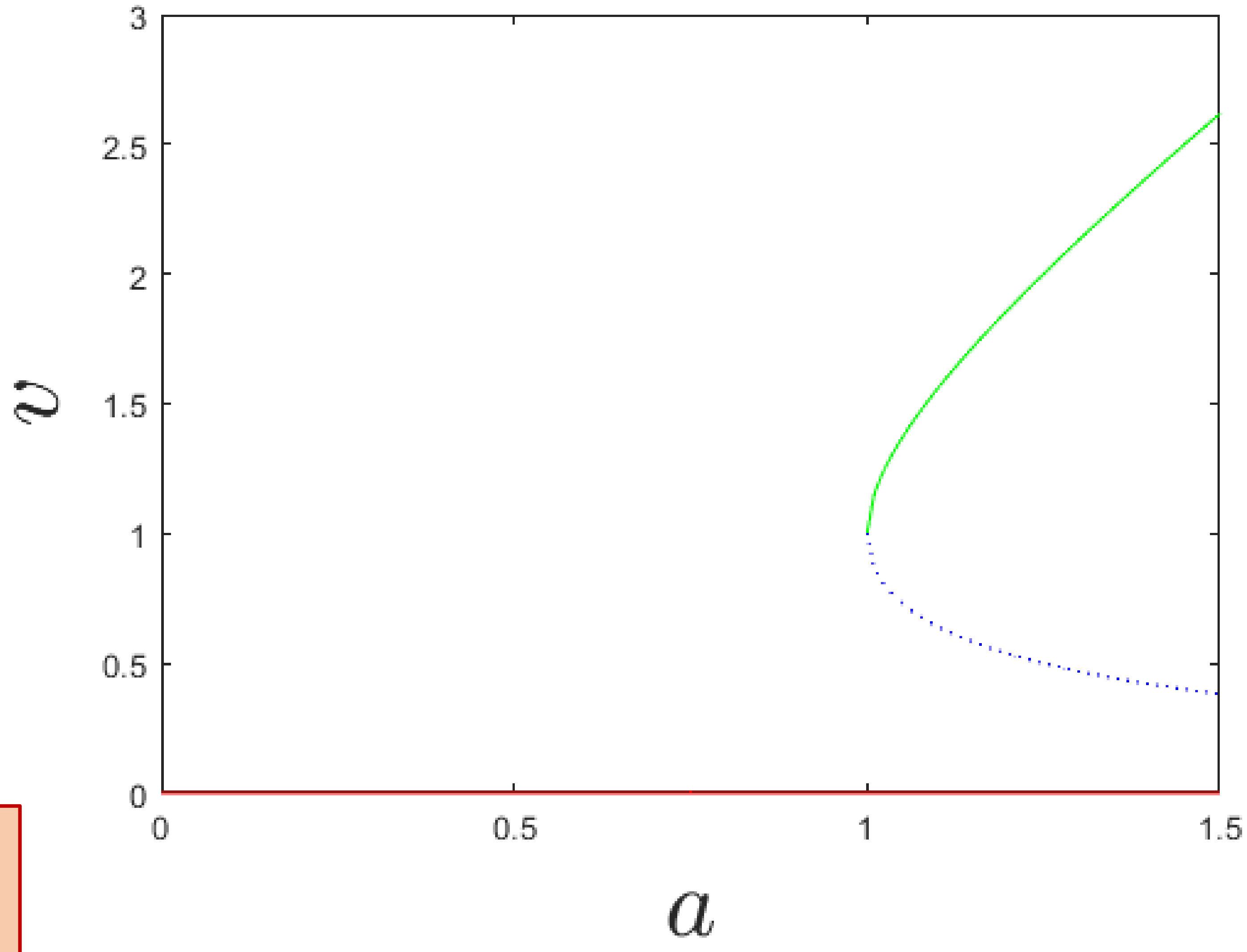


[Bastiaansen, Ashwin, Von der Heydt, 2023]

EXAMPLE 2: Time scale of feedback

$$\frac{du}{dt} = a - u - uv^2$$

$$\frac{dv}{dt} = uv^2 - mv$$



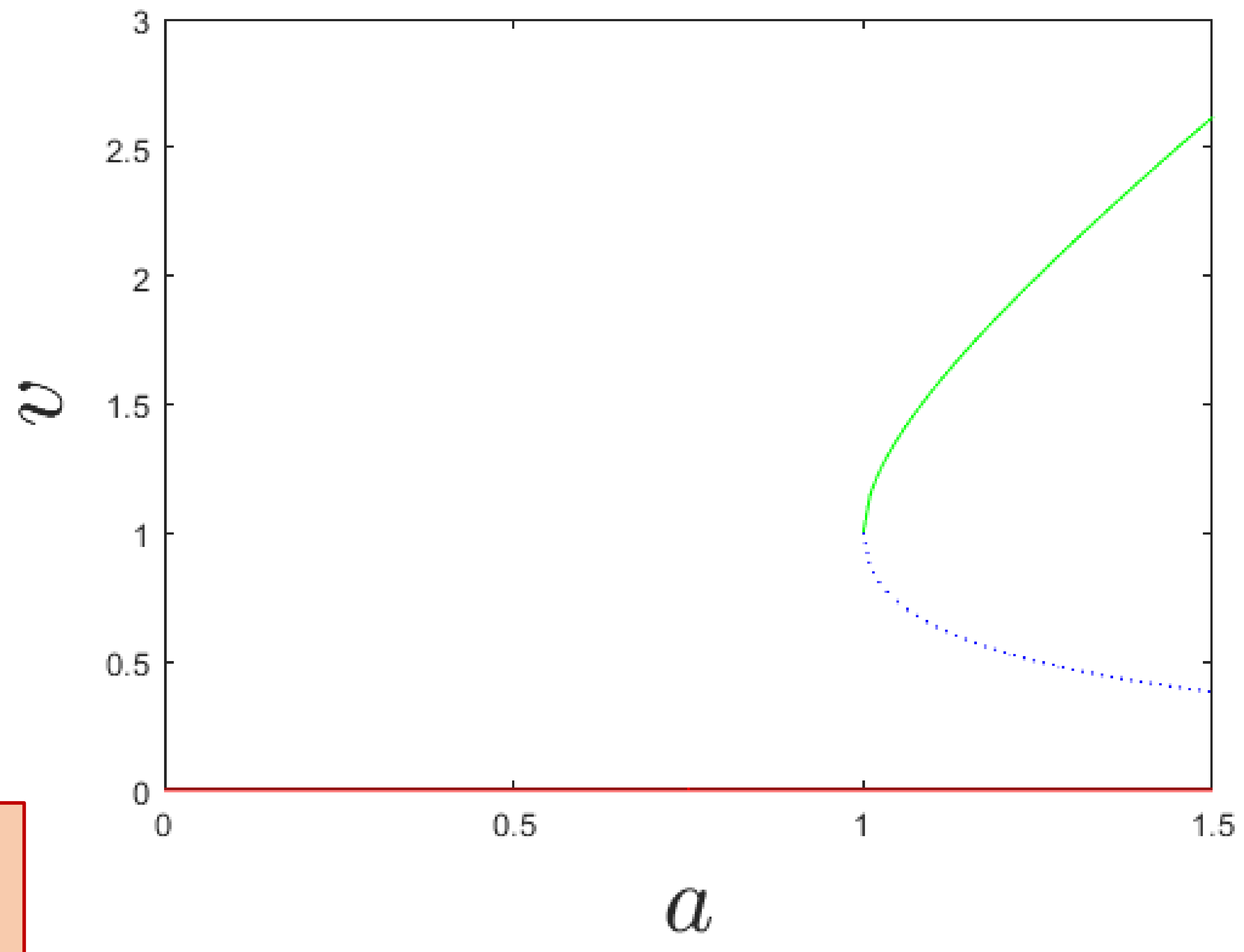
Parameters:
$m = 0.5$

EXAMPLE 2: Time scale of feedback

$$\frac{du}{dt} = a - u - uvs$$

$$\frac{dv}{dt} = uvs - mv$$

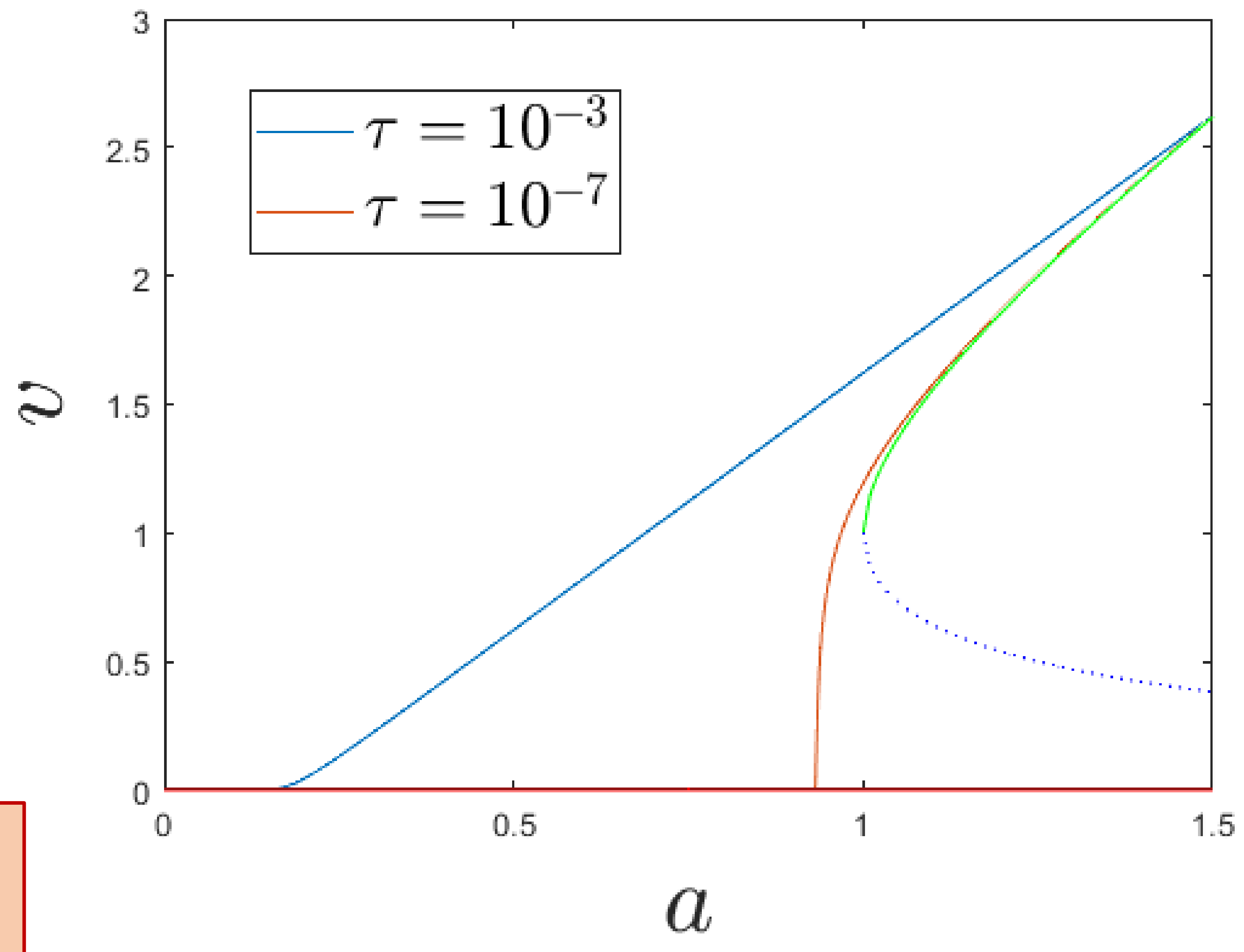
$$\frac{ds}{dt} = \tau_{INT}(v - s)$$



Parameters:
$m = 0.5$
$\tau_{INT} = 10^{-5}$

EXAMPLE 2: Time scale of feedback

$$\frac{du}{dt} = a - u - uvs$$
$$\frac{dv}{dt} = uvs - mv$$
$$\frac{ds}{dt} = \tau_{INT}(v - s)$$
$$\frac{da}{dt} = -\tau$$



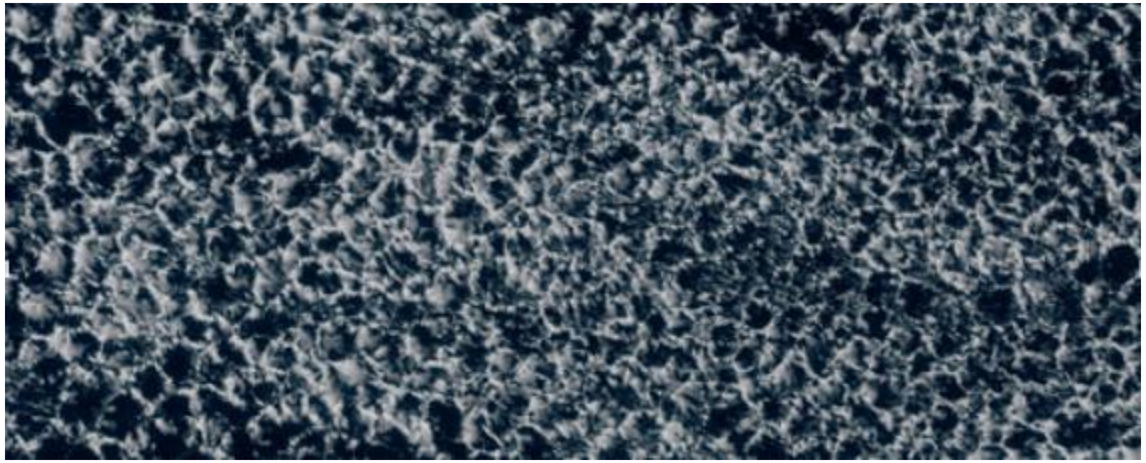
Parameters:
$m = 0.5$
$\tau_{INT} = 10^{-5}$

Spatial effects

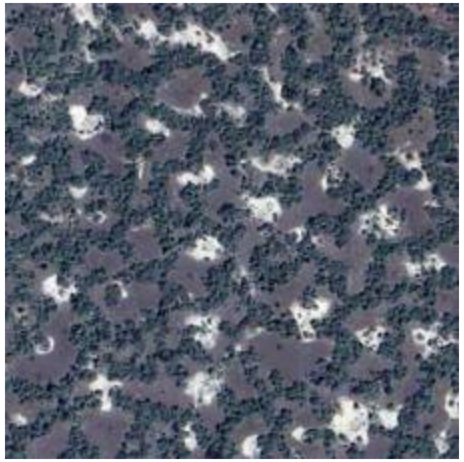
Spatially extended systems

Examples of spatial patterning – regular patterns

mussel beds

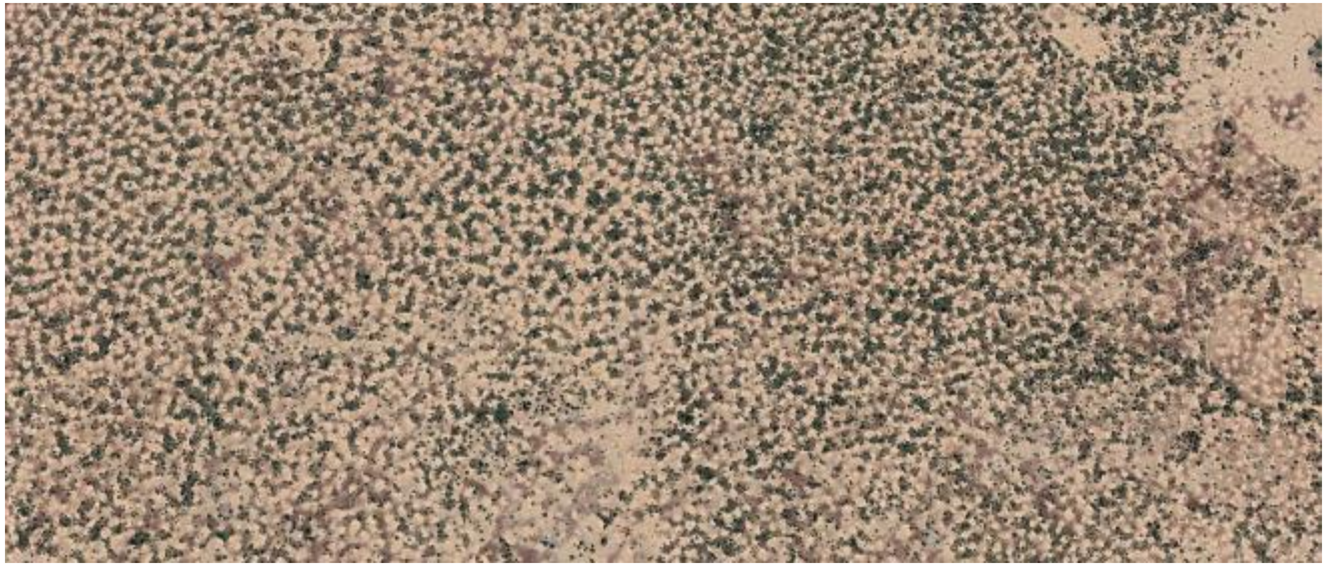


clouds



savannas

melt ponds

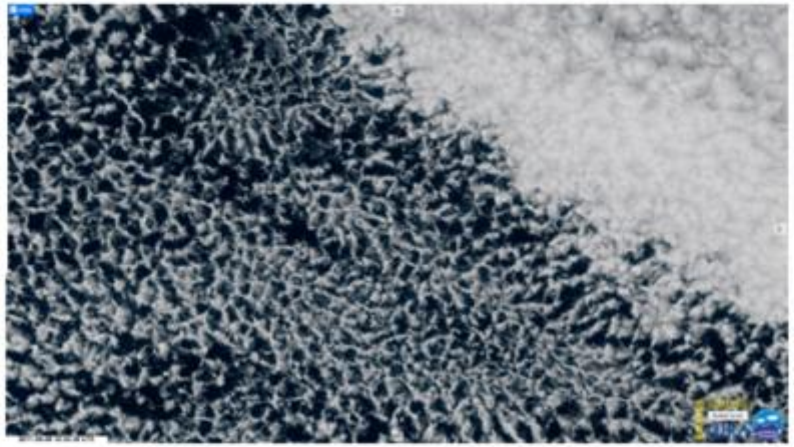


drylands

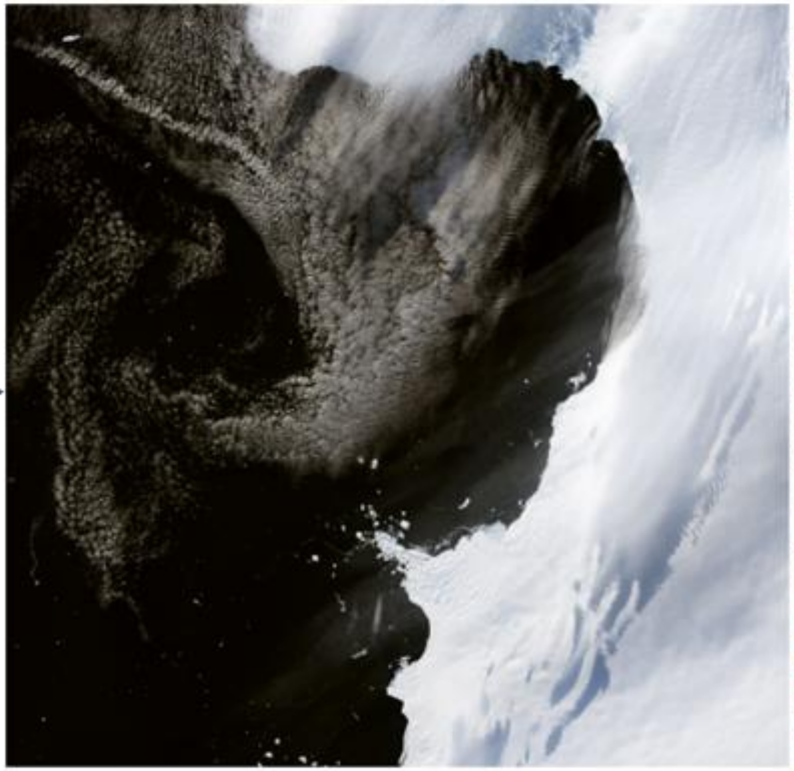
Examples of spatial patterning – spatial interfaces

tropical forest & savanna ecosystems
[Google Earth]

types of stratocumulus clouds
[RAMMB/CIRA SLIDER]

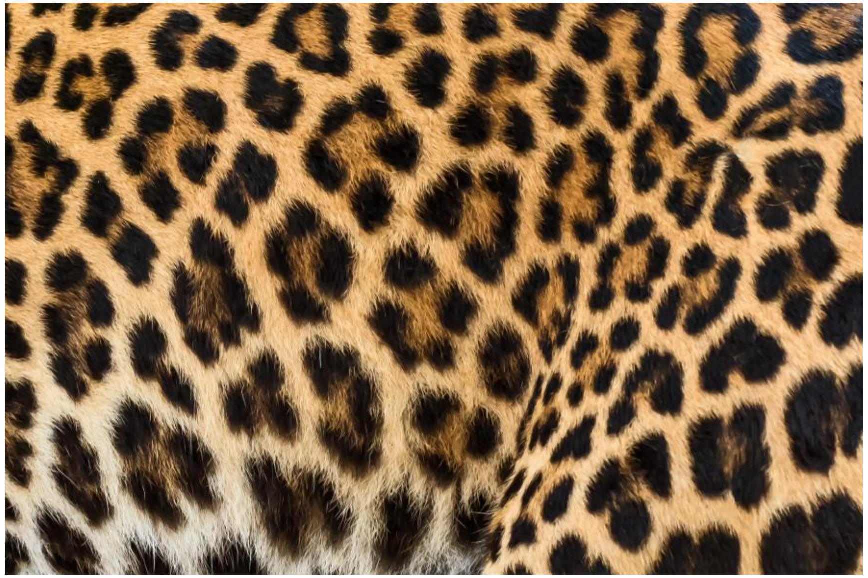


sea-ice & water at Eltanin Bay
[NASA's Earth observatory]

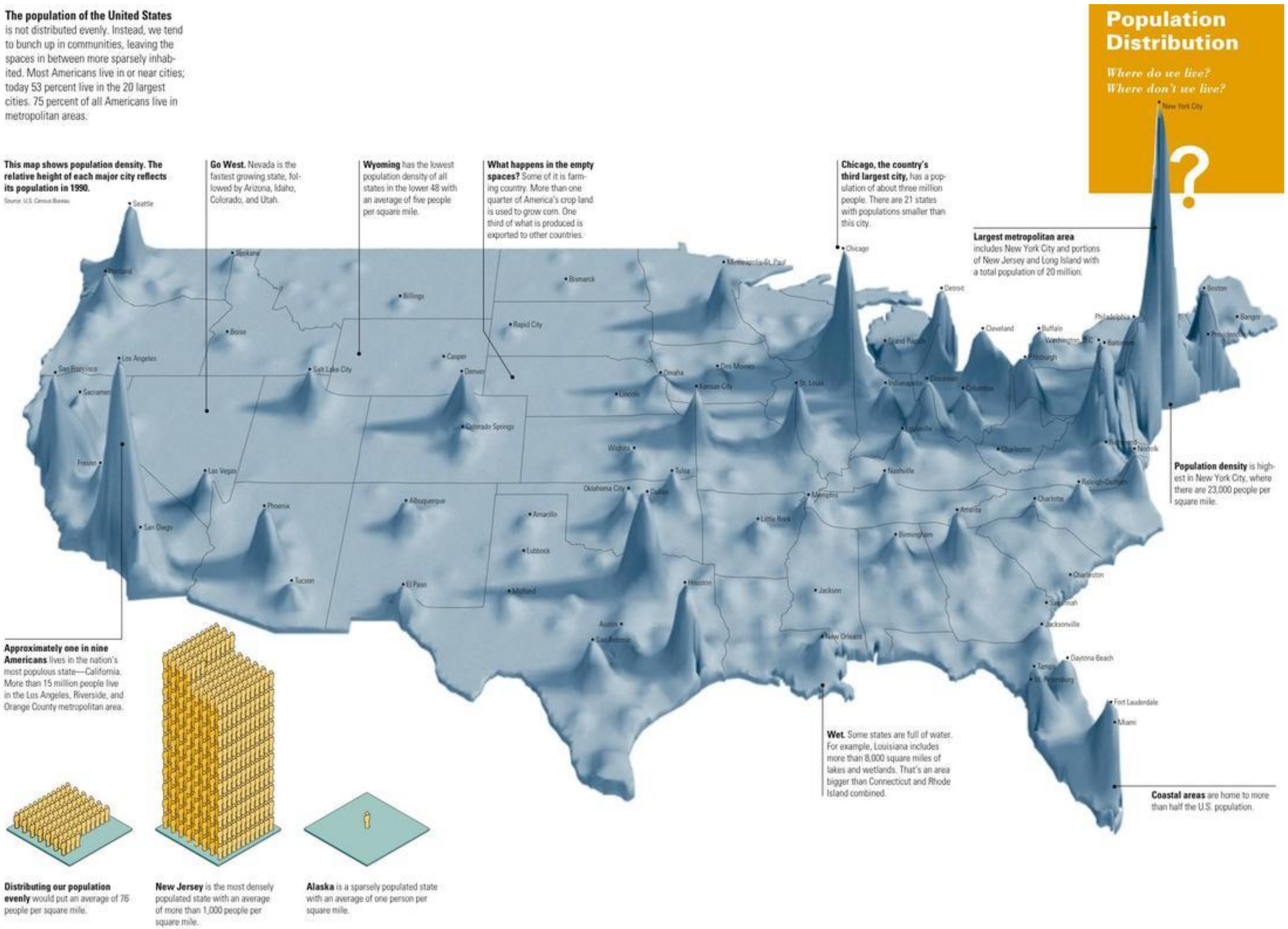


algae bloom in Lake St. Clair
[NASA's Earth observatory]

Examples of spatial patterning – animals



Examples of spatial patterning – sociology



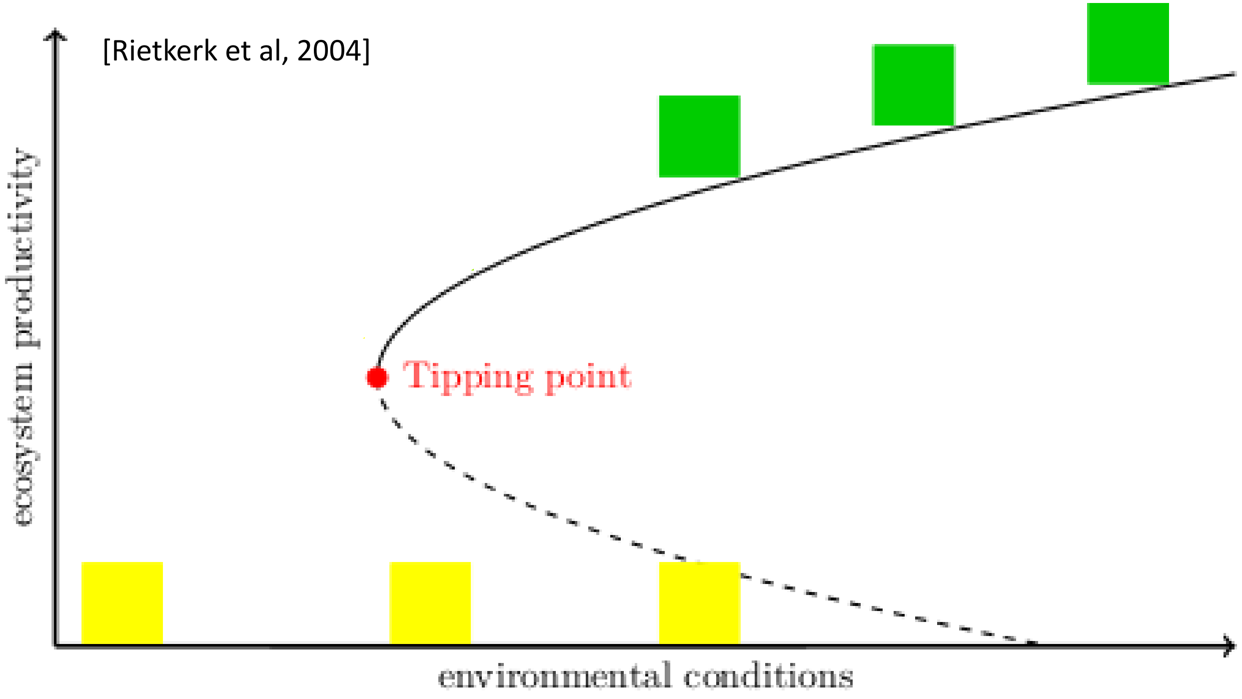
Examples of spatial patterning – physics

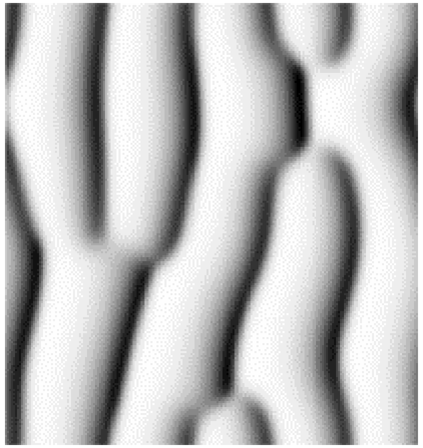
Patterns in models

Add spatial transport:

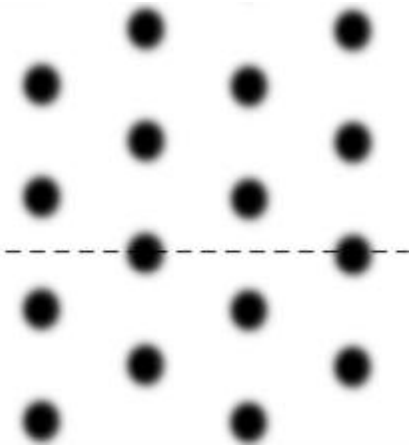
Reaction-Diffusion equations:

$$\begin{cases} \dfrac{du}{dt} = f(u,v) + D_u \, \Delta u \\ \dfrac{dv}{dt} = g(u,v) + D_v \, \Delta v \end{cases}$$



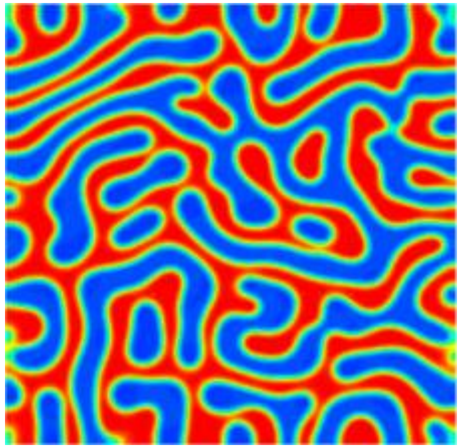


[Klausmeier, 1999]



[Gilad et al, 2004]

[Rietkerk et al, 2002]



[Liu et al, 2013]

Turing patterns

$$\begin{cases} \frac{du}{dt} = f(u,v) + D_u \, \Delta u \\ \frac{dv}{dt} = g(u,v) + D_v \, \Delta v \end{cases}$$

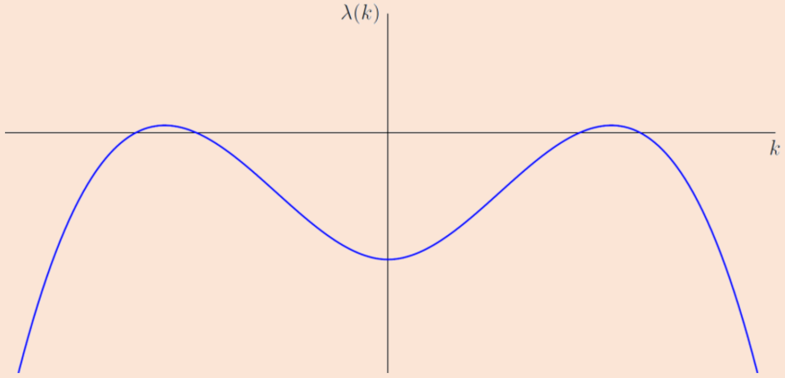
Turing bifurcation

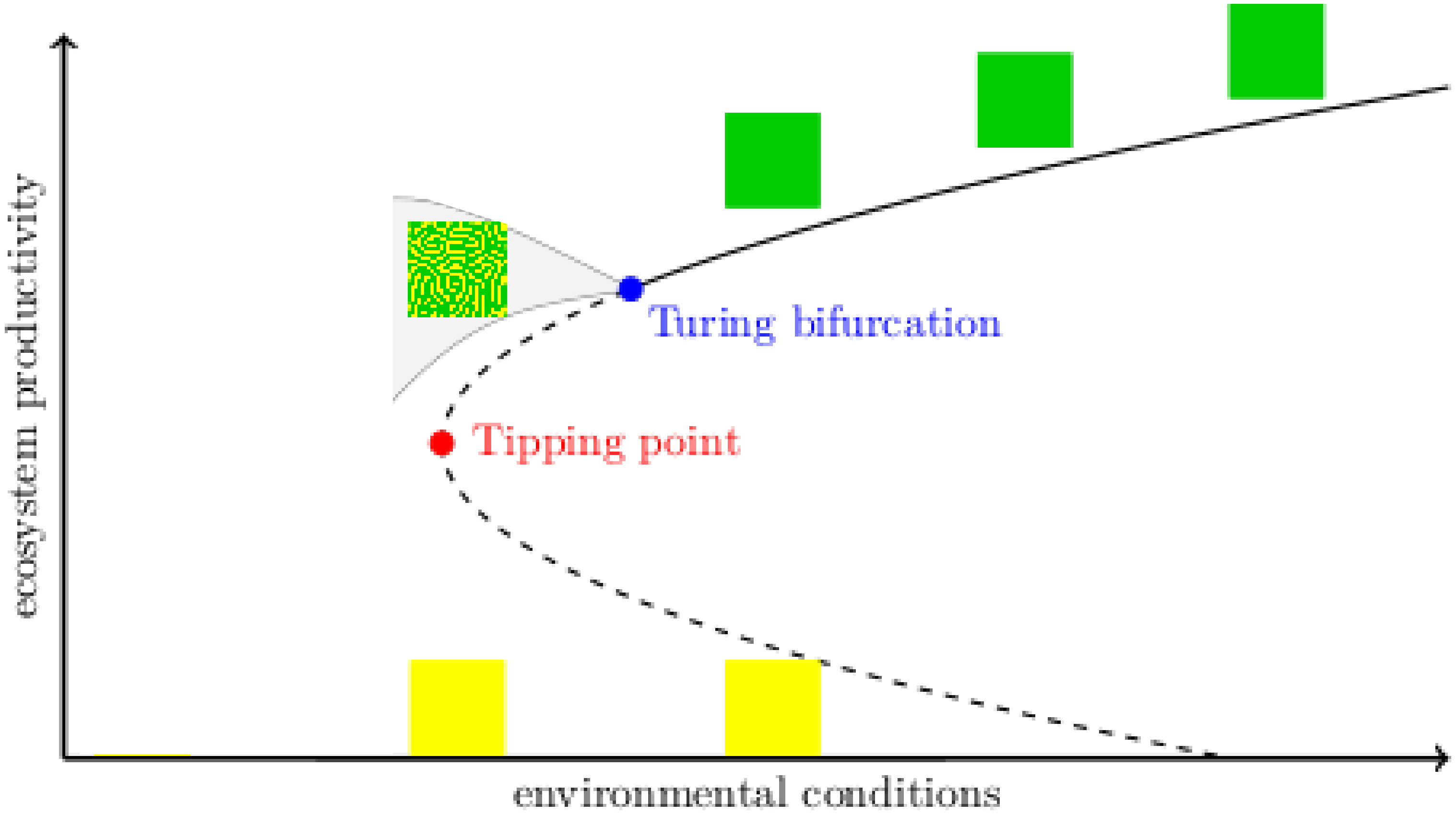
Instability to non-uniform perturbations

$$\binom{u}{v} = \binom{u_*}{v_*} + e^{\lambda t} e^{ikx} \binom{\bar{u}}{\bar{v}}$$

→ Dispersion relation

$$\lambda(k) = \cdots$$





Weakly non-linear analysis

Ginzburg-Landau equation / Amplitude Equation

& Eckhaus/Benjamin-Feir-Newel criterion

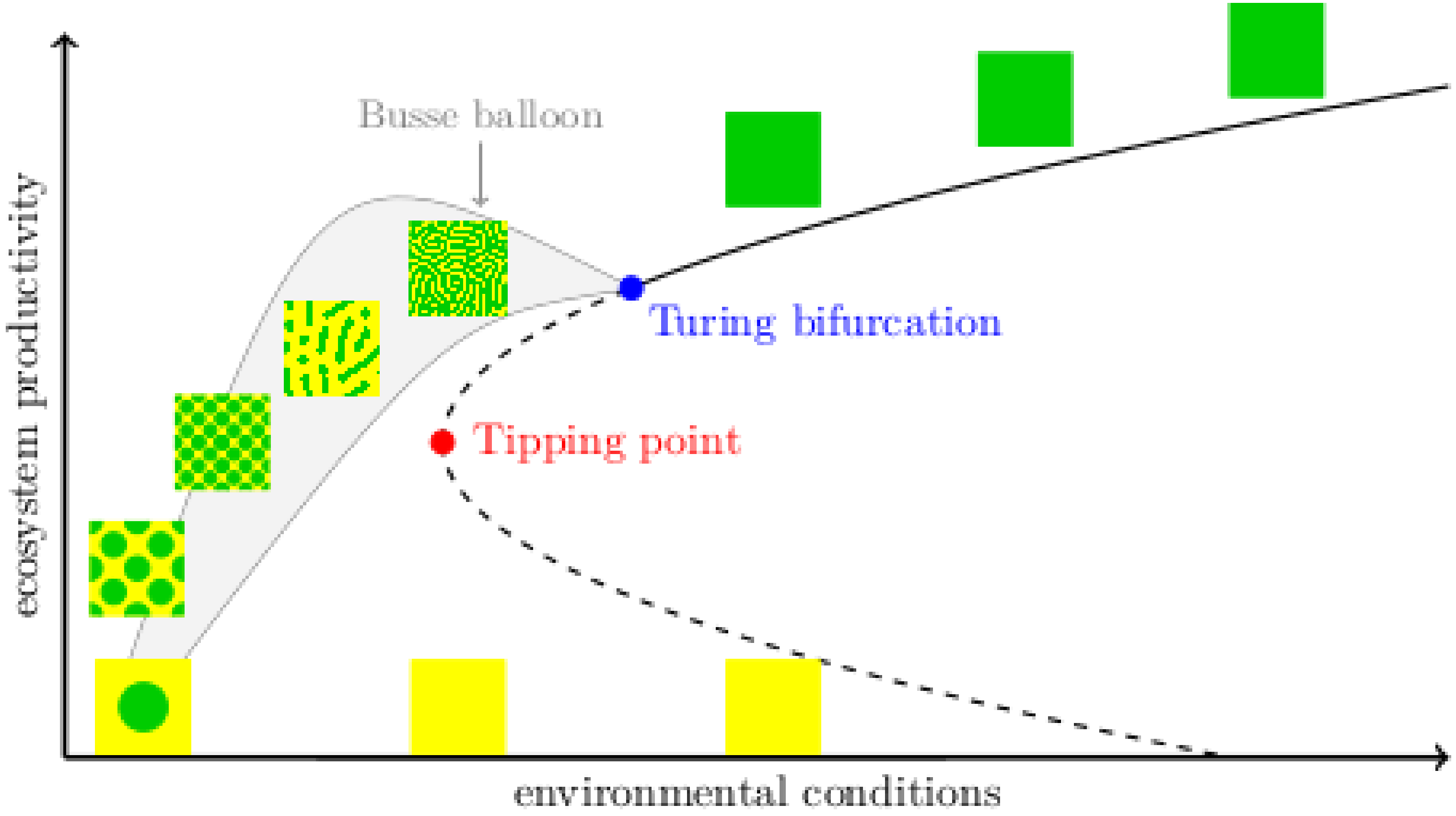
[Eckhaus, 1965; Benjamin & Feir, 1967; Newell, 1974]

Busse balloon

$$\begin{cases} \frac{du}{dt} = f(u,v) + D_u \Delta u \\ \frac{dv}{dt} = g(u,v) + D_v \Delta v \end{cases}$$

Busse balloon

A model-dependent shape in *(parameter, observable)* space that indicates all stable patterned solutions to the PDE.



Construction Busse balloon

Via numerical continuation

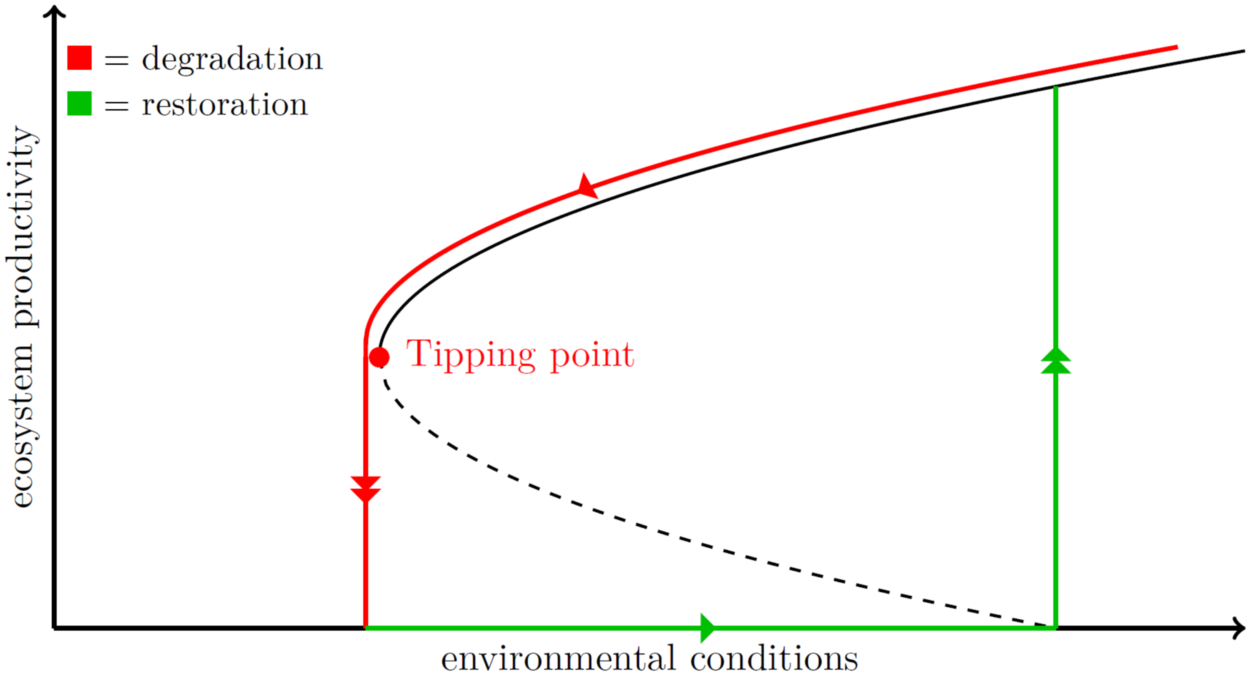
few general results on the shape of Busse balloon

Busse balloon

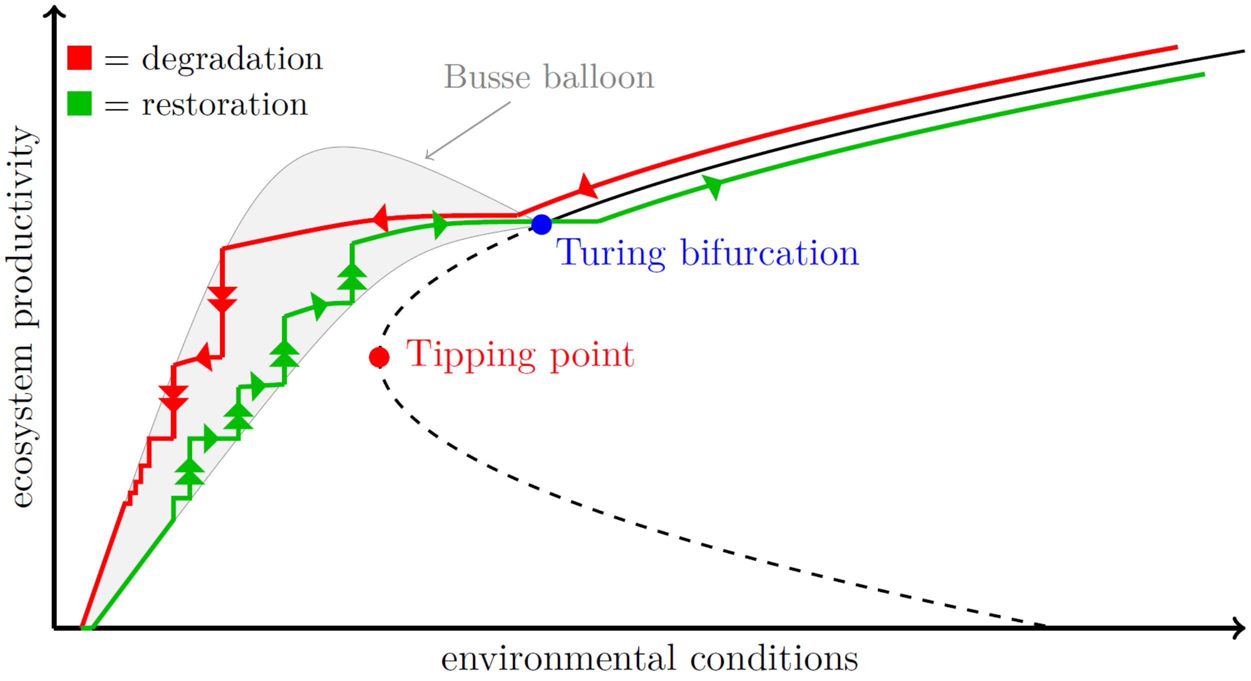
Idea originates from thermal convection

[Busse, 1978]

Tipping of spatial patterns



Classic tipping



Tipping of patterns

System heterogeneity

Heterogeneous systems

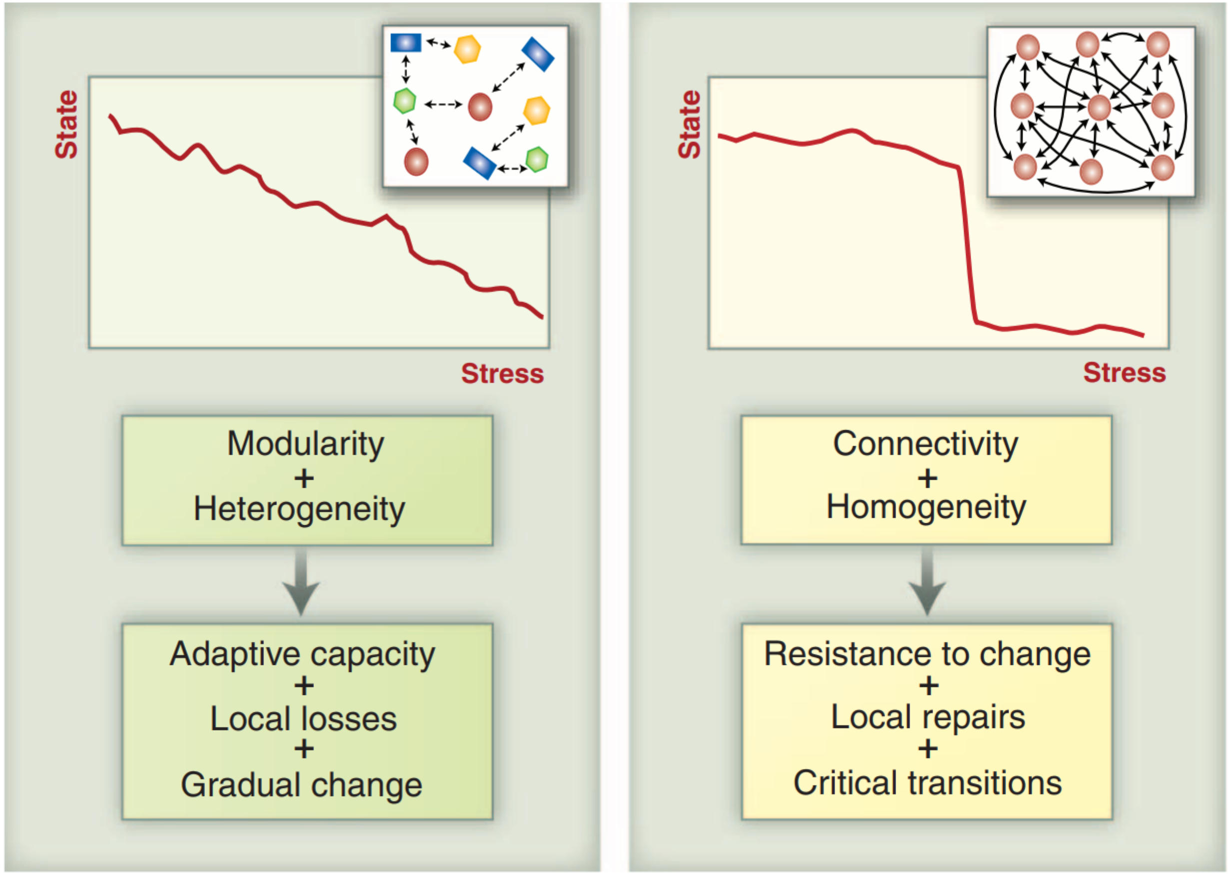
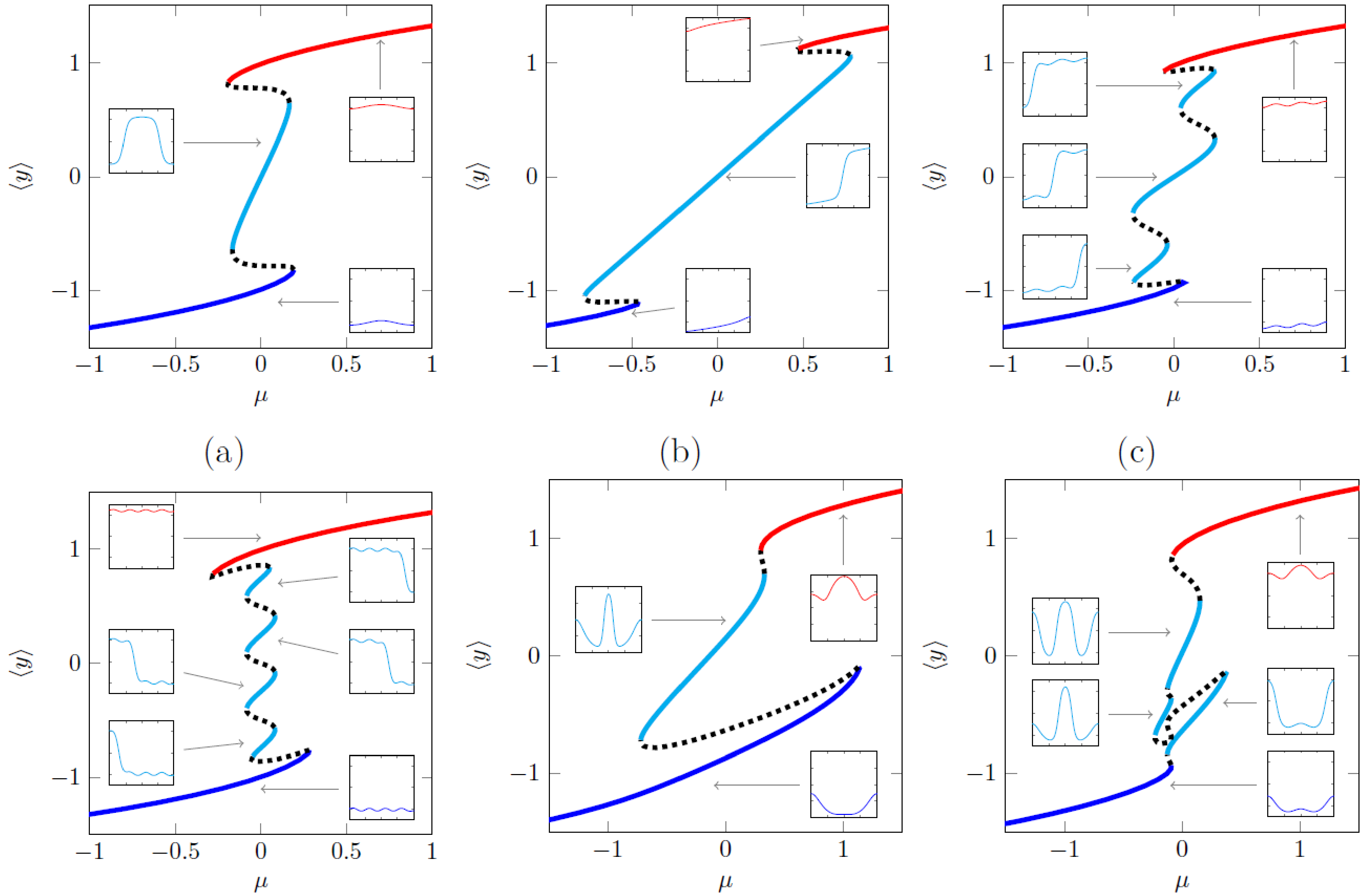


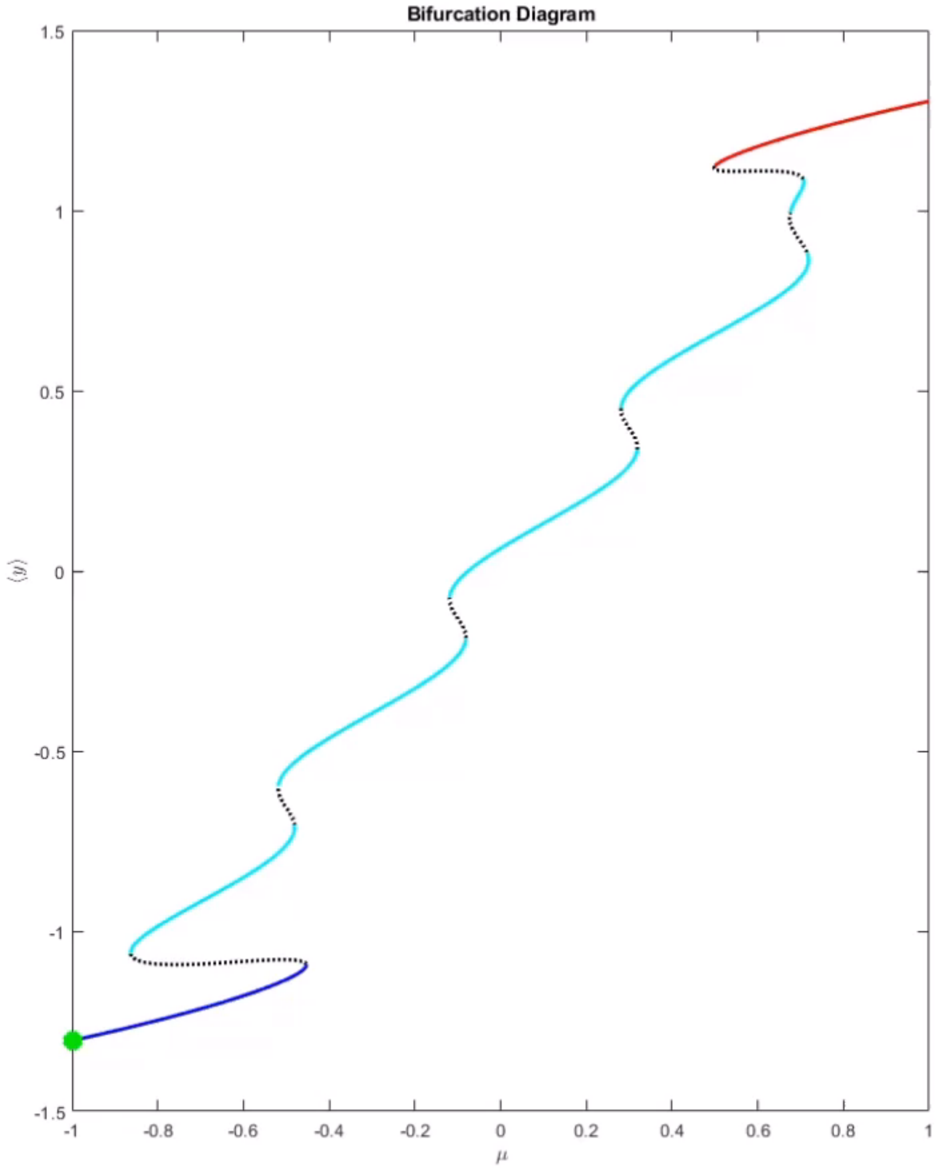
Fig. 1. The connectivity and homogeneity of the units affect the way in which distributed systems with local alternative states respond to changing conditions. Networks in which the components differ (are heterogeneous) and where incomplete connectivity causes modularity tend to have adaptive capacity in that they adjust gradually to change. By contrast, in highly connected networks, local losses tend to be "repaired" by subsidiary inputs from linked units until at a critical stress level the system collapses. The particular structure of connections also has important consequences for the robustness of networks, depending on the kind of interactions between the nodes of the network.

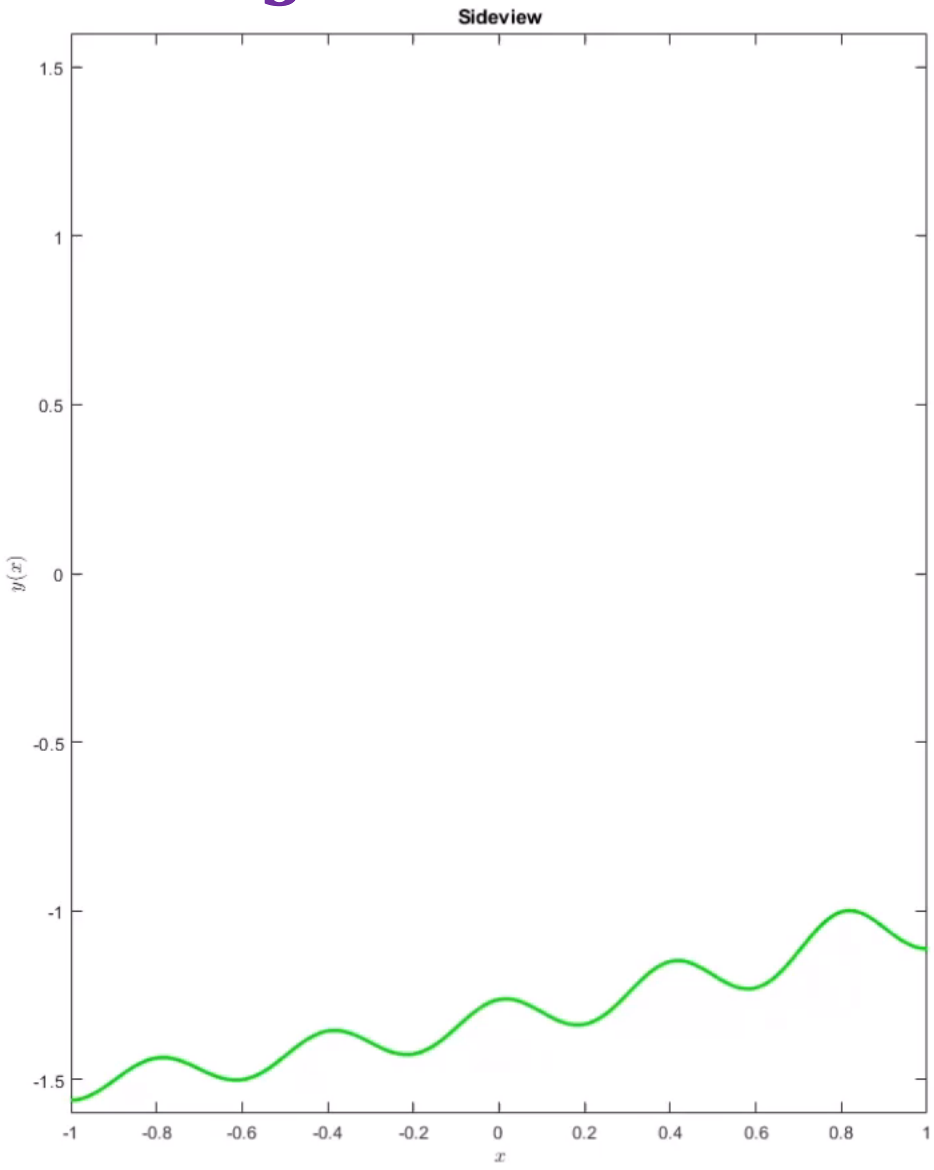
[Scheffer et al, 2012]

Other Spatial Heterogeneities



$$y_t = D\, y_{xx} + y(1 - y^2) + \mu + x + \frac{2}{5}\cos(5\pi x)$$





TIPPING BEHAVIOUR IN COMPLEX SYSTEMS – AN INTRODUCTION

ROBBIN BASTIAANSEN
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CCSS, 2025-09-11

Summary

Tipping can occur in many complex systems

<u>Types of tipping</u>:

- B-tipping: due to incremental parameter shift
- N-tipping: due to noise and state change
- R-tipping: due to fastness of change
- …

Typically, tipping …

- … are **fast** abrupt changes
- … lead to **system-wide** transition
- … lead to **alternative** system **state**
- … is **irreversible** (hysteresis)

However, in complex systems things might be more complex;
e.g. due to

- multiple time scales of dynamics
- spatial effects
- system heterogeneity

slides available at bastiaansen.github.io