Tipping in Spatially Extended Systems: early warnings & control

2026-09-17, Floris Takens seminar, RUG
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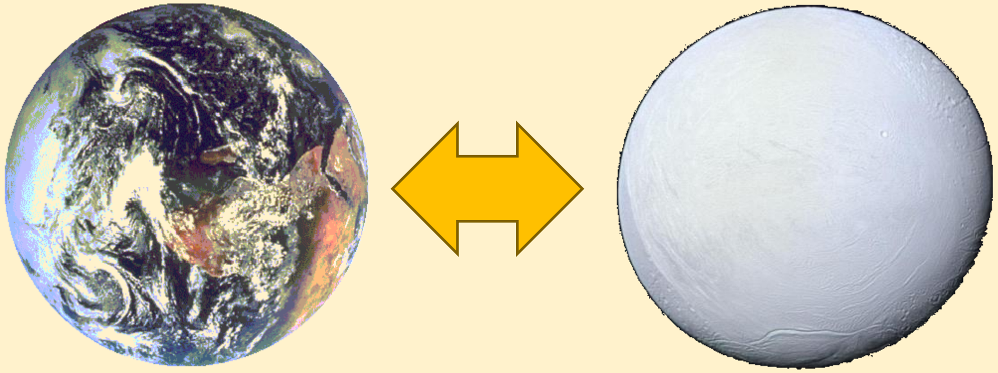
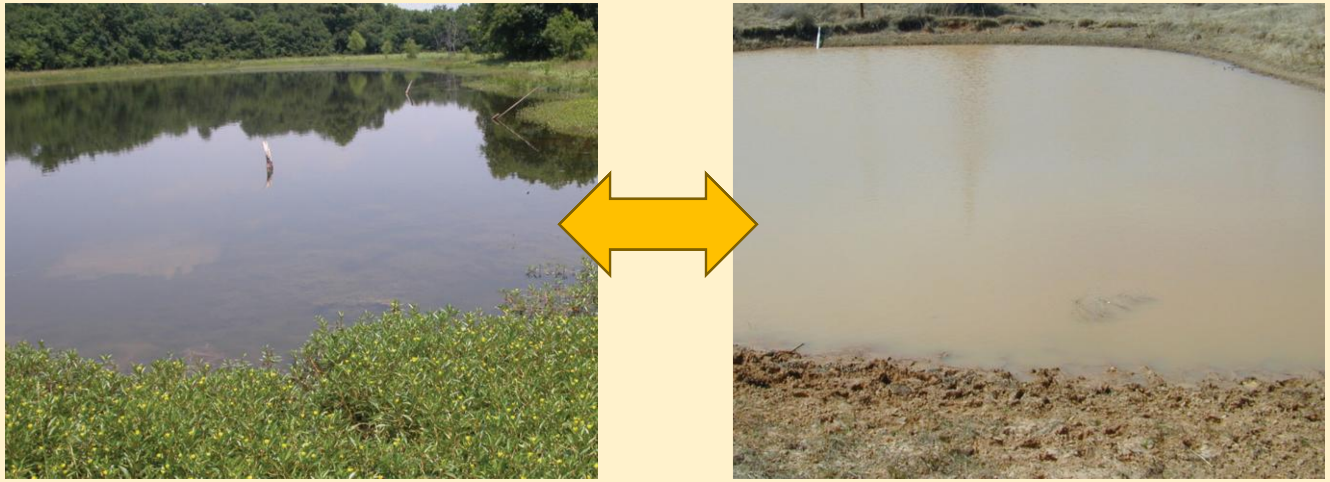
Tipping Points

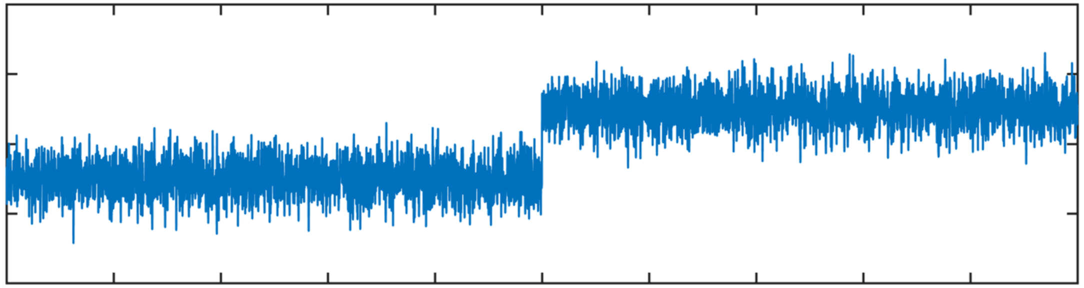
IPCC AR6 (2021) : "a critical threshold beyond which a system reorganizes, often abruptly and/or irreversibly"



Planetary transitions

Ecosystem shifts





Tipping Points

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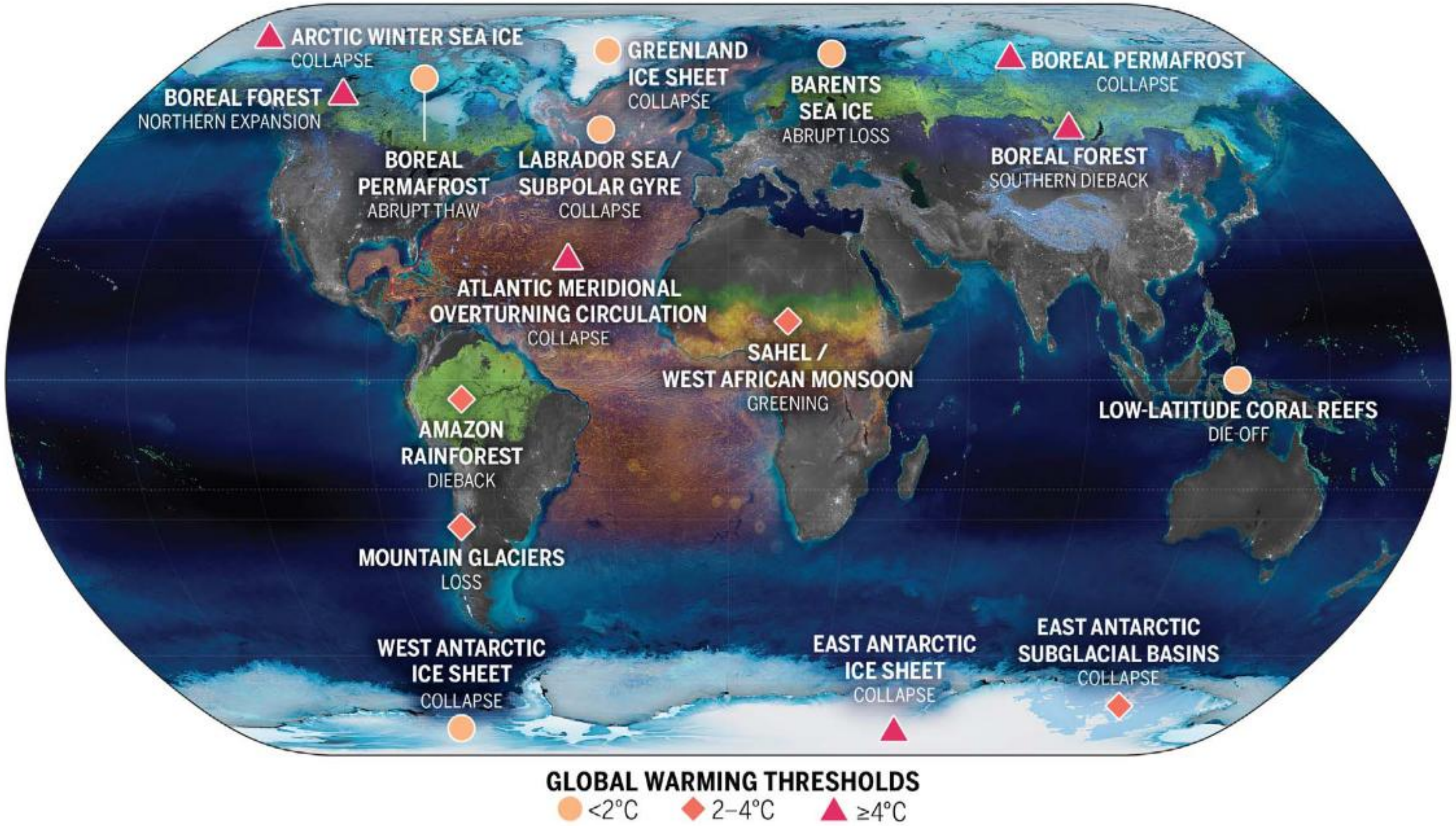
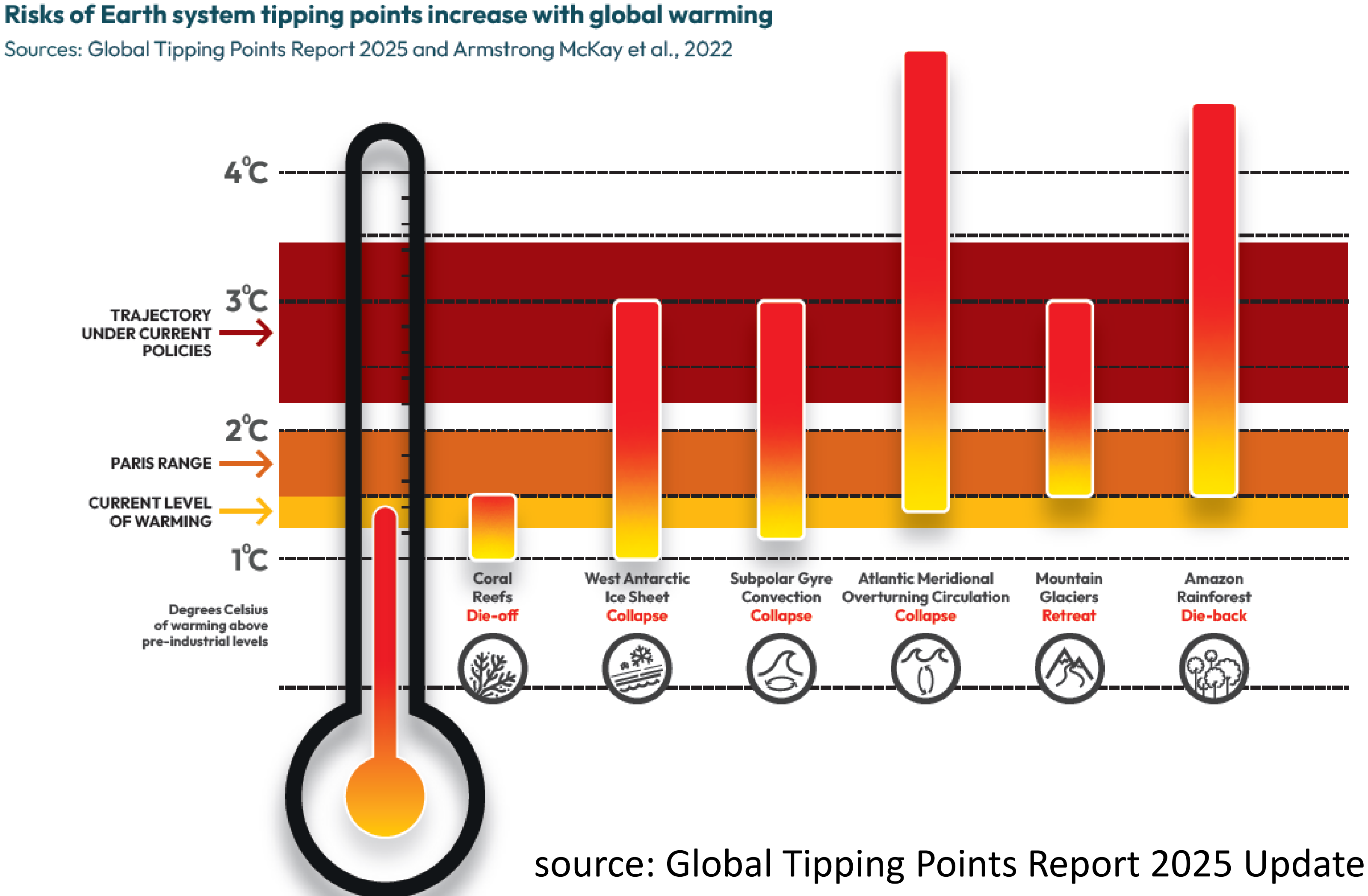


source: McKay et al, 2022

Tipping Points

IPCC AR6 (2021) : "a critical threshold beyond which a system reorganizes, often abruptly and/or irreversibly"

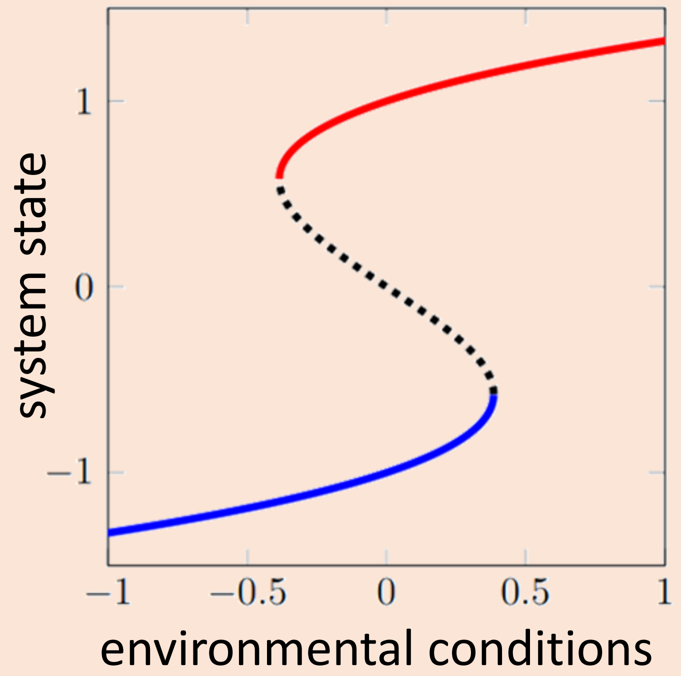


source: Global Tipping Points Report 2025 Update

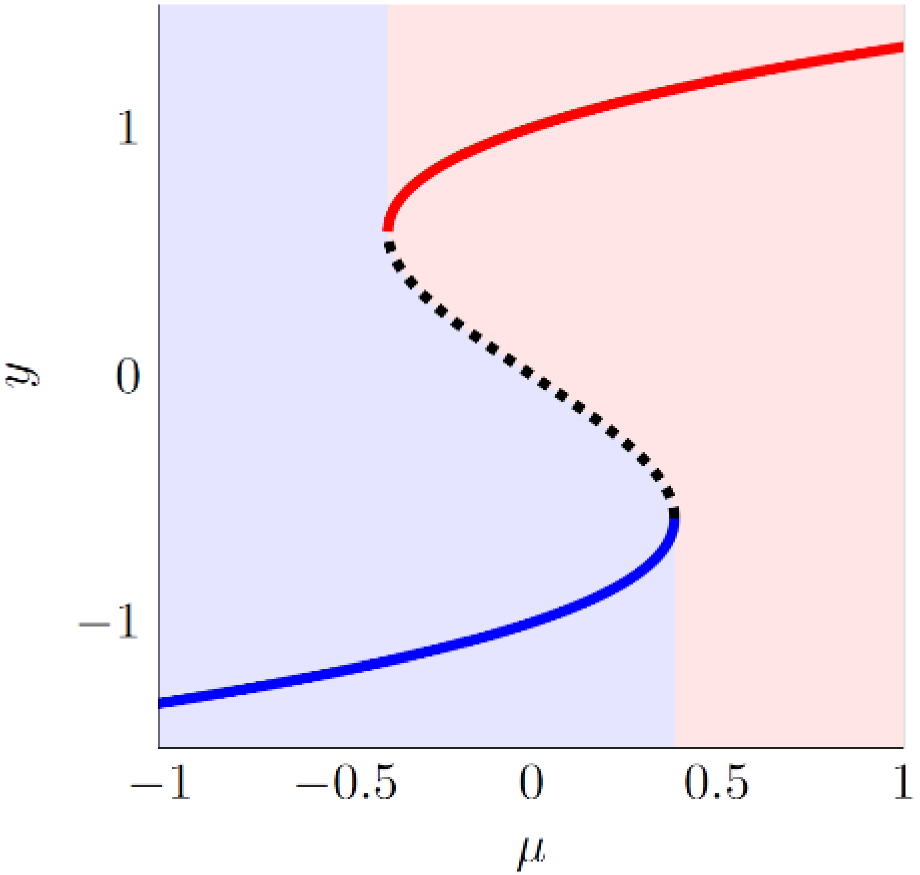
Mathematics

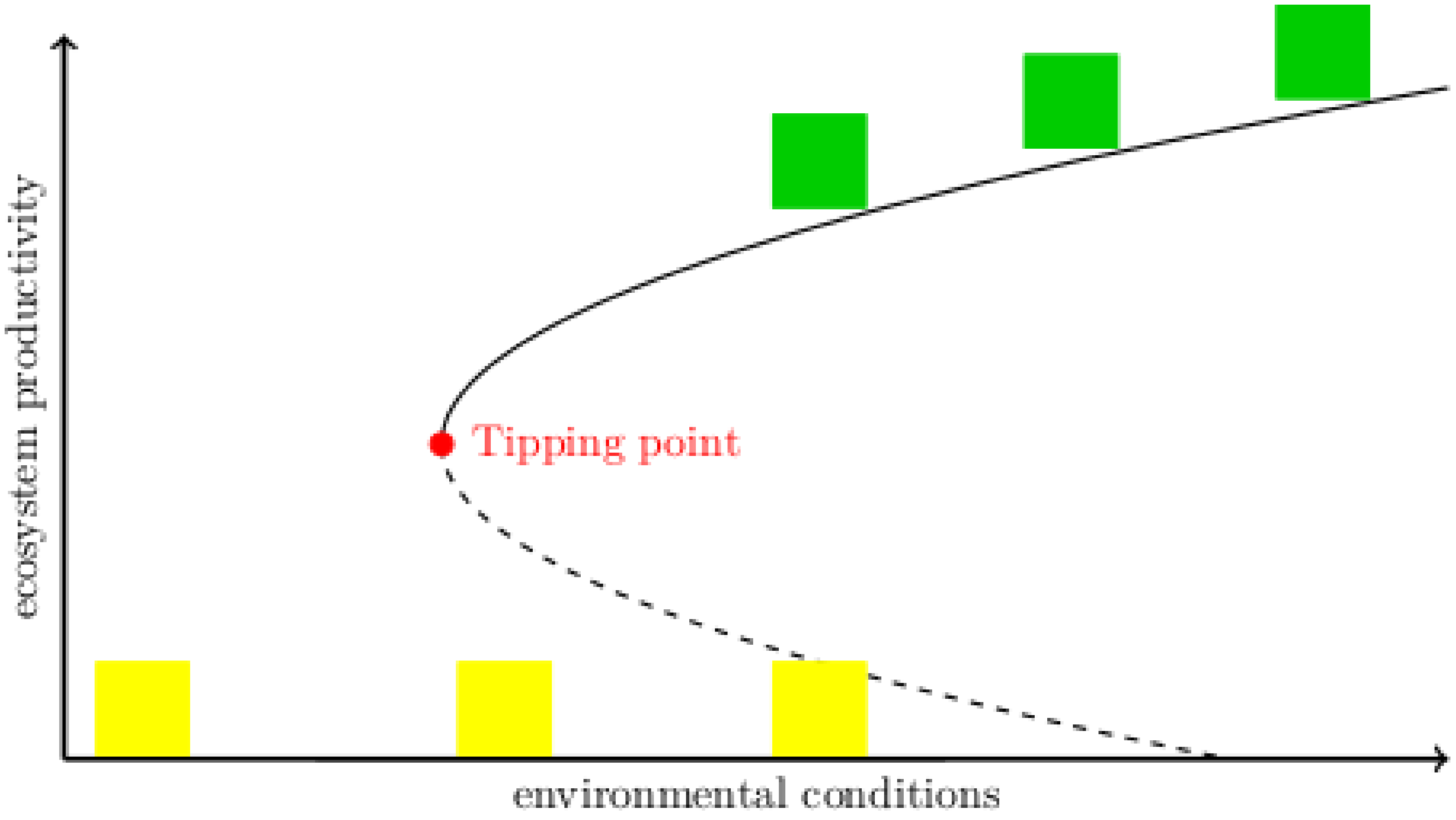
Tipping points ↔ Bifurcations

$$\frac{dy}{dt} = f(y, \mu)$$



Classic Theory of Tipping





Canonical example:

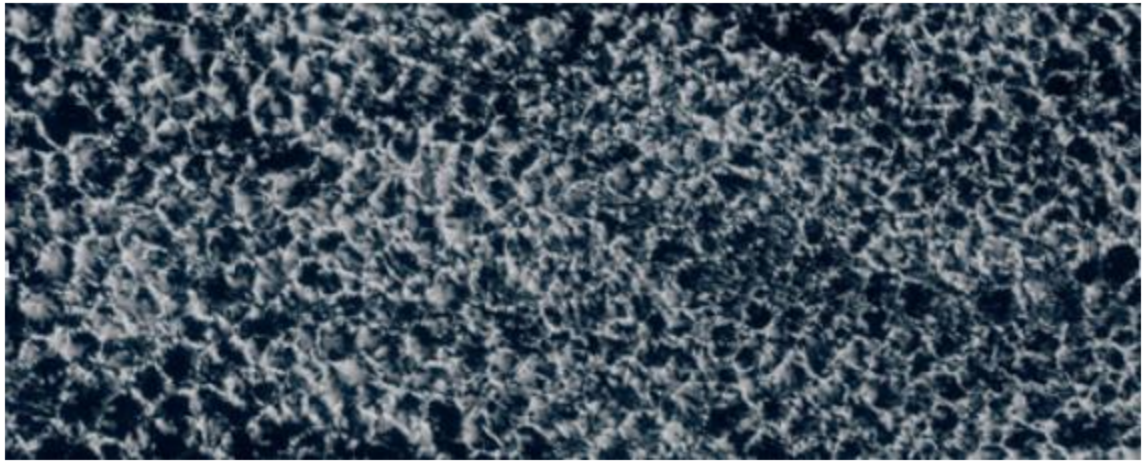
$$\frac{dy}{dt} = y(1 - y^2) + \mu$$

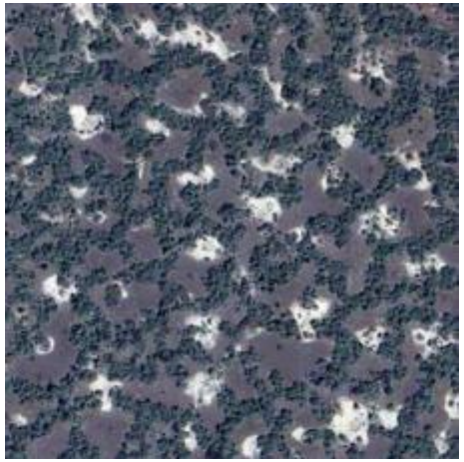
$$\frac{d\vec{y}}{dt} = f(\vec{y}; \mu)$$

$$\begin{cases} \frac{du}{dt} = f(u, v; \mu) \\ \frac{dv}{dt} = g(u, v; \mu) \end{cases}$$

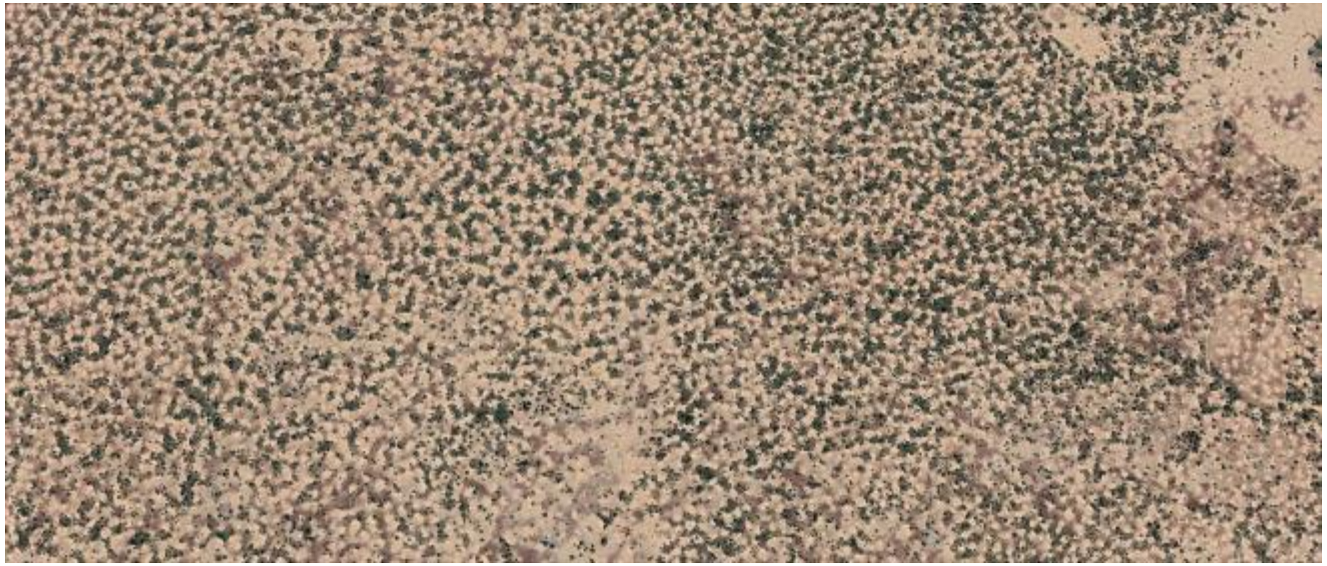
Examples of spatial patterning – regular patterns


mussel beds


clouds

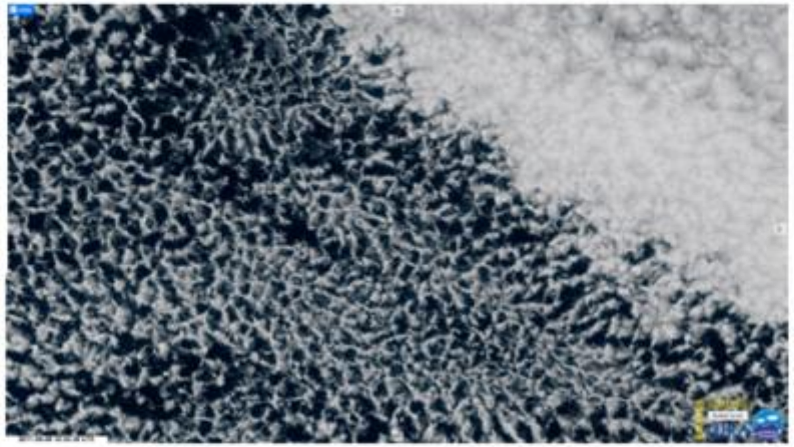

savannas


melt ponds


drylands

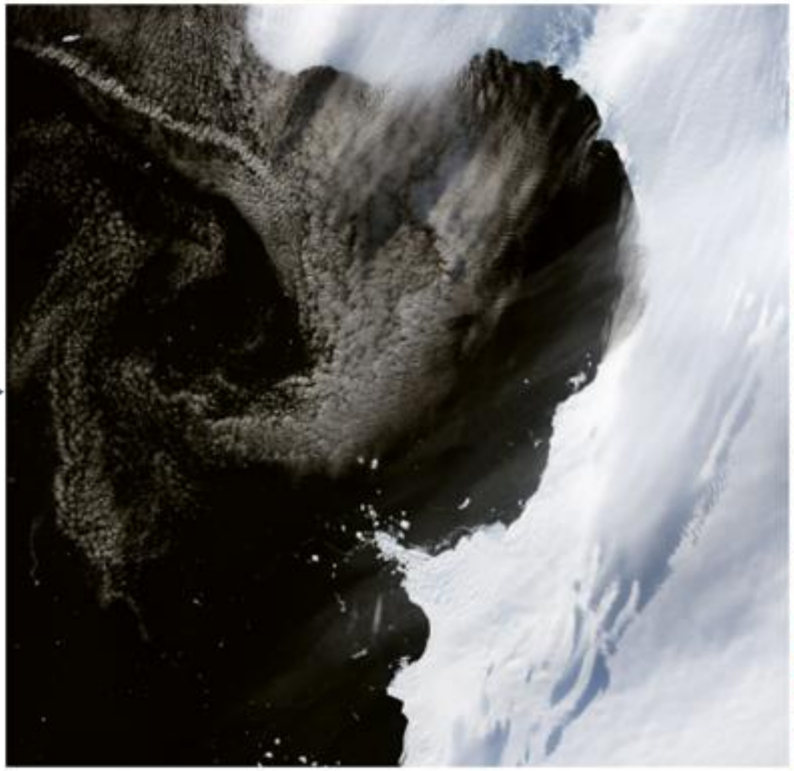
Examples of spatial patterning – spatial interfaces

tropical forest
& savanna
ecosystems
[Google Earth]



types of
stratocumulus
clouds
[RAMMB/CIRA SLIDER]

sea-ice & water
at Eltanin Bay
[NASA's Earth observatory]



algae bloom
in Lake St. Clair
[NASA's Earth observatory]

Tipping
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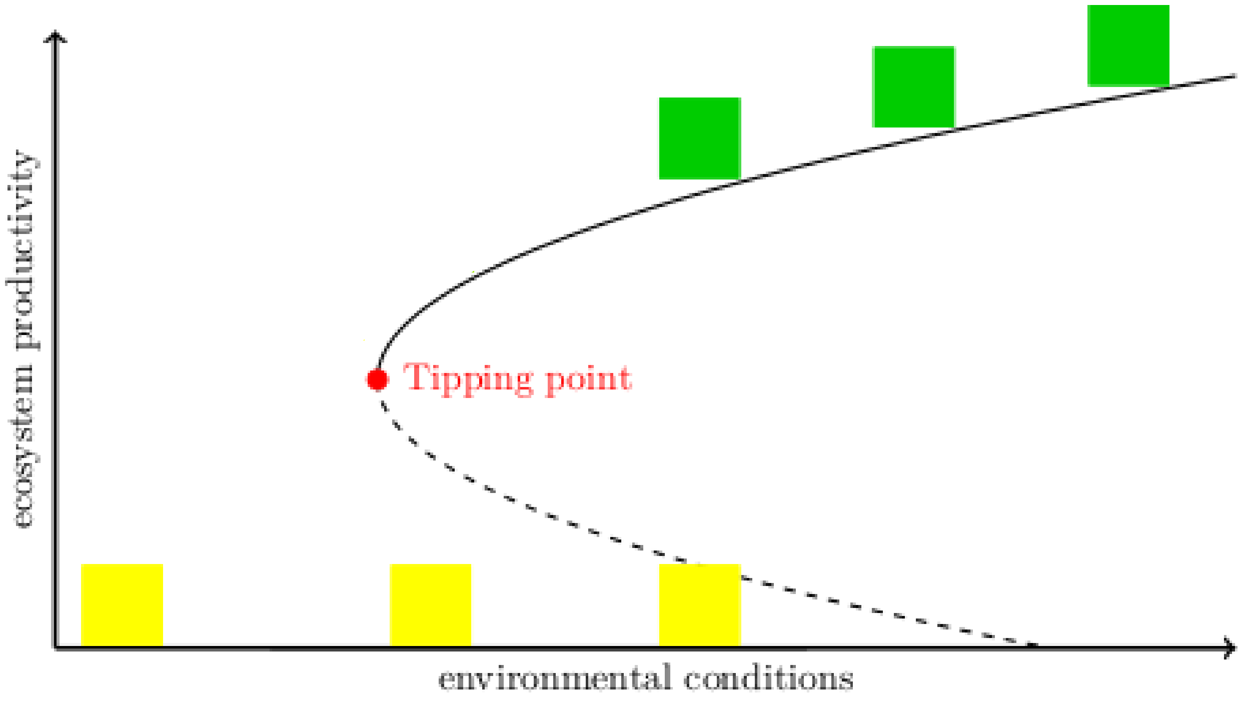
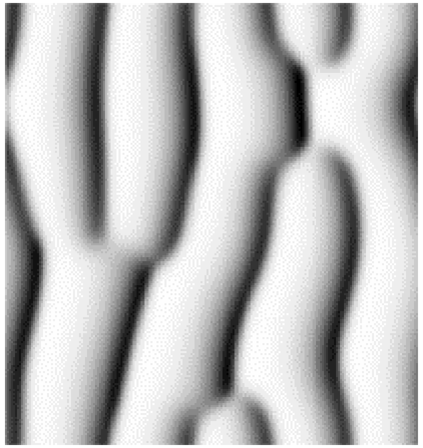
Patterns in models

Add spatial transport:

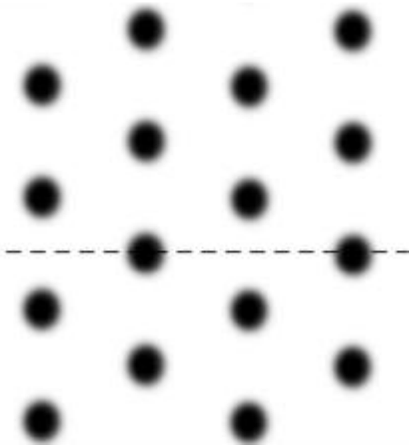
Reaction-Diffusion equations:

$$\begin{cases} \dfrac{du}{dt} = f(u,v) + D_u \, \Delta u \\ \dfrac{dv}{dt} = g(u,v) + D_v \, \Delta v \end{cases}$$



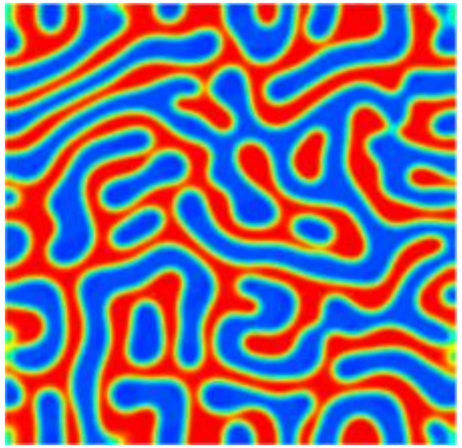


[Klausmeier, 1999]



[Gilad et al, 2004]

[Rietkerk et al, 2002]



[Liu et al, 2013]

Turing patterns

$$\begin{cases} \dfrac{du}{dt} = f(u,v) + D_u\, \Delta u \\ \dfrac{dv}{dt} = g(u,v) + D_v\, \Delta v \end{cases}$$

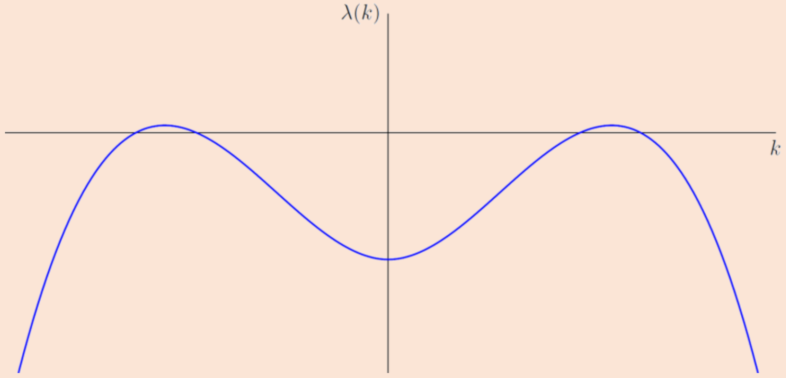
Turing bifurcation

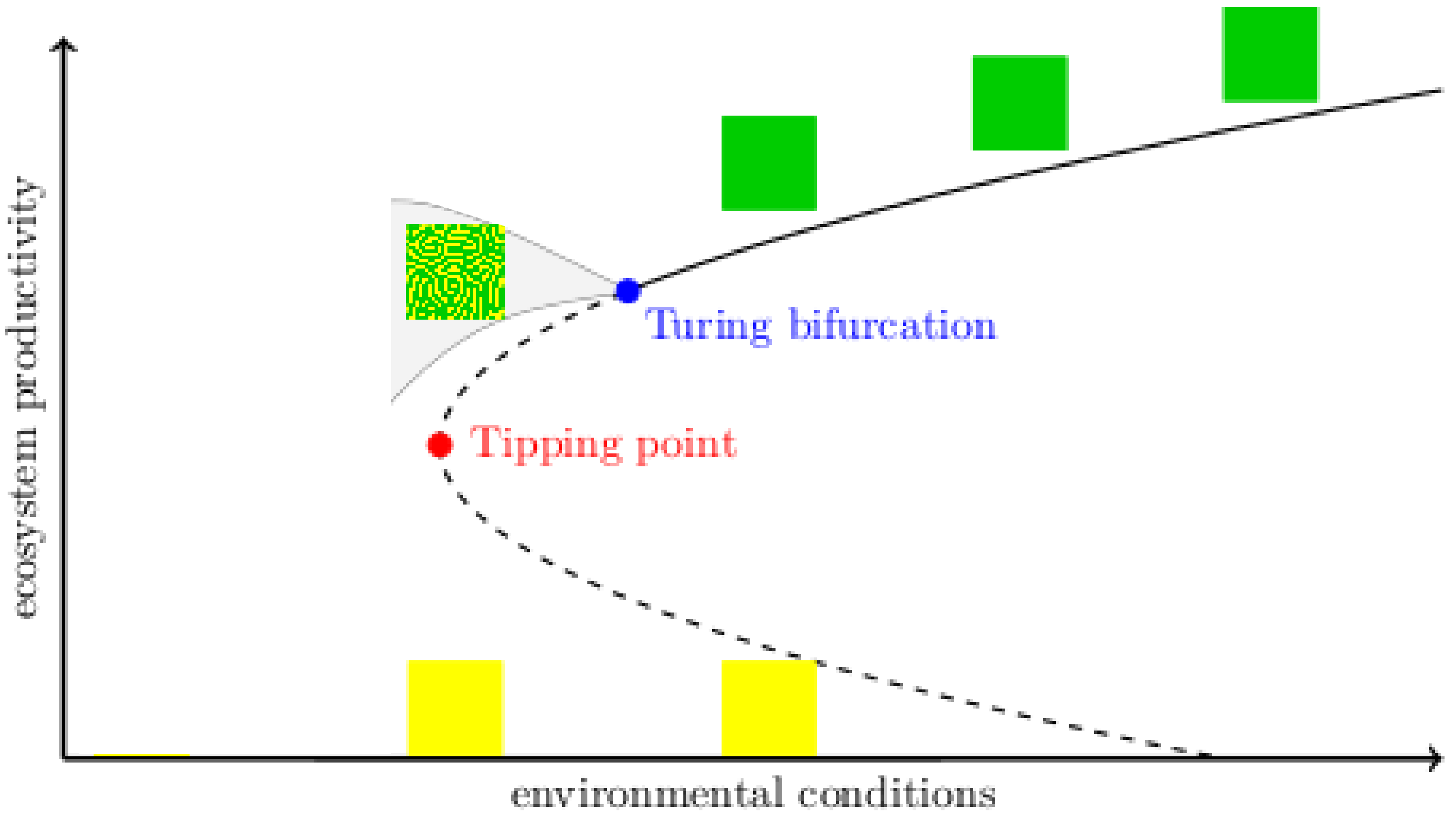
Instability to non-uniform perturbations

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u_* \\ v_* \end{pmatrix} + e^{\lambda t} e^{ikx} \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix}$$

→ Dispersion relation

$$\lambda(k) = \cdots$$





Weakly non-linear analysis

Ginzburg-Landau equation / Amplitude Equation

& Eckhaus/Benjamin-Feir-Newel criterion

[Eckhaus, 1965; Benjamin & Feir, 1967; Newell, 1974]

Busse balloon

$$\begin{cases} \dfrac{du}{dt} = f(u,v) + D_u \, \Delta u \\ \dfrac{dv}{dt} = g(u,v) + D_v \, \Delta v \end{cases}$$

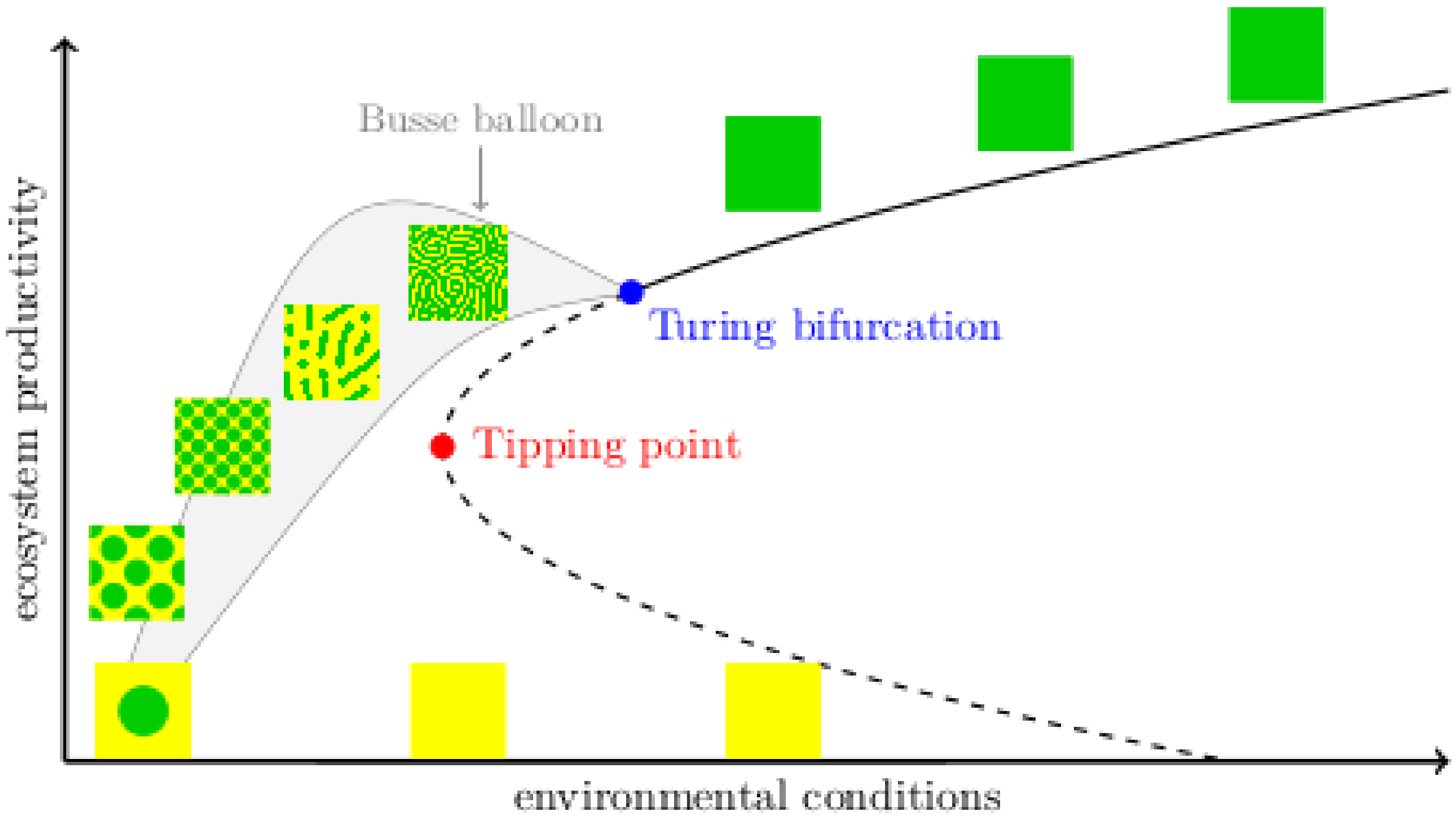
Busse balloon

A model-dependent shape in *(parameter, observable)* space that indicates all stable patterned solutions to the PDE.

Construction Busse balloon

Via numerical continuation

few general results on the shape of Busse balloon

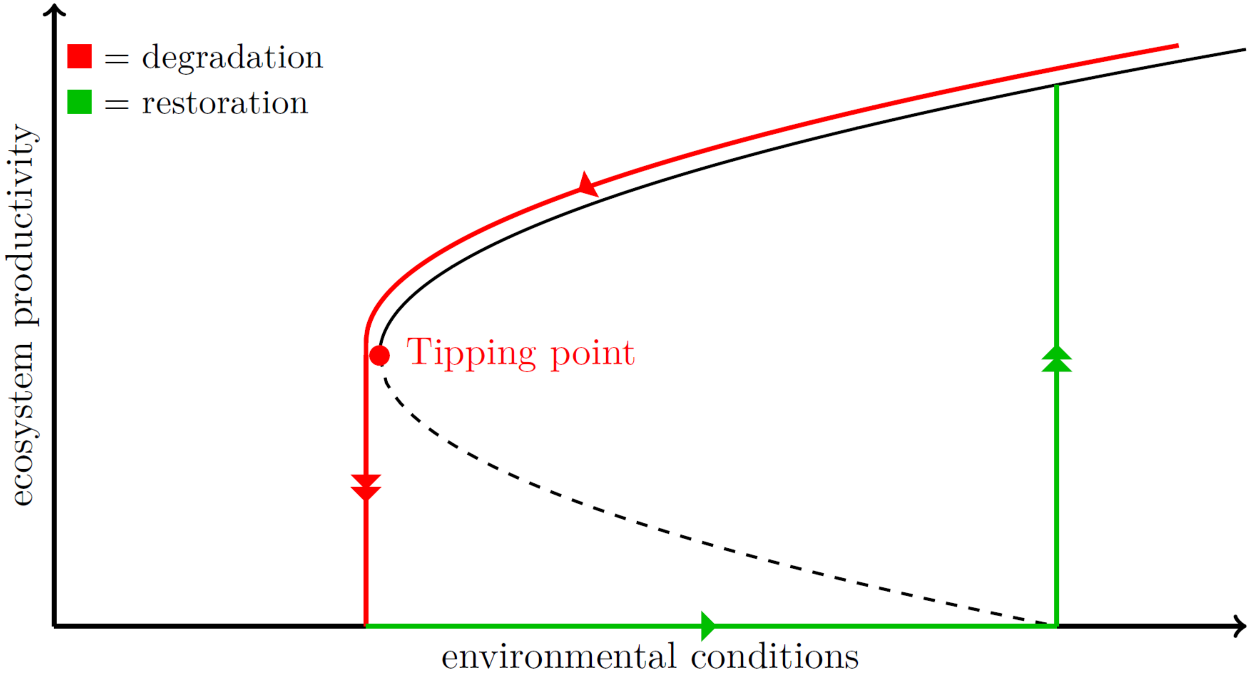


Busse balloon

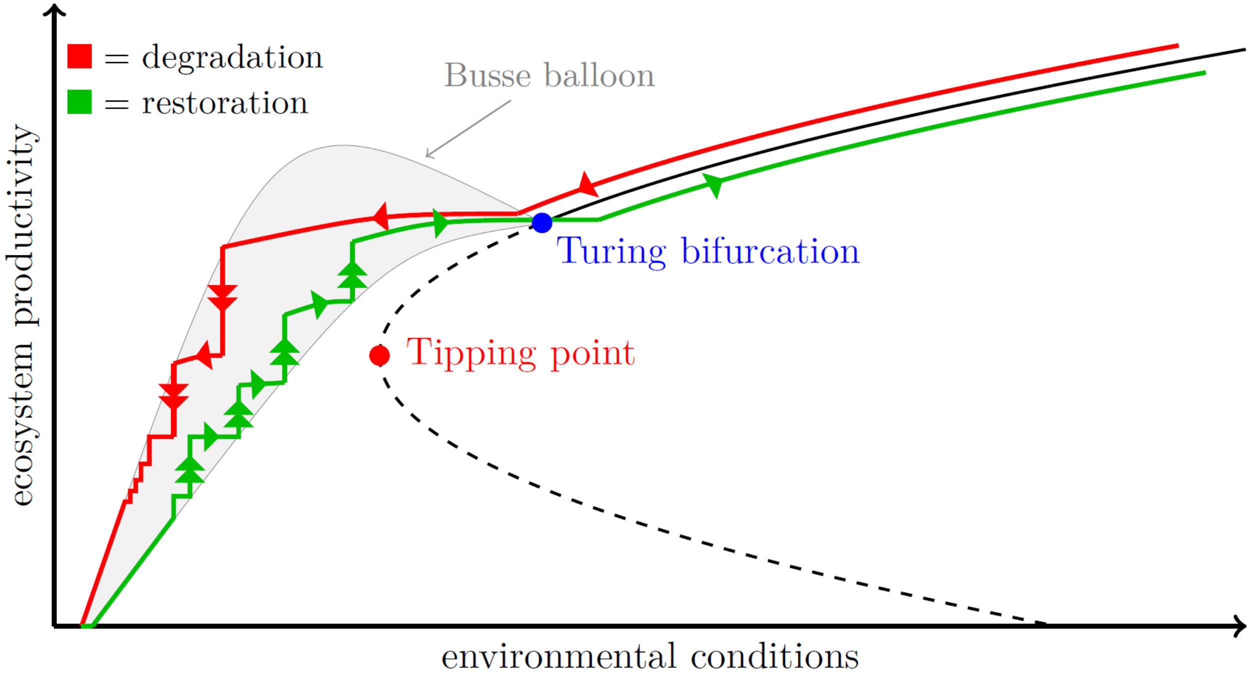
Idea originates from thermal convection

[Busse, 1978]

Tipping of patterned systems



Classic tipping

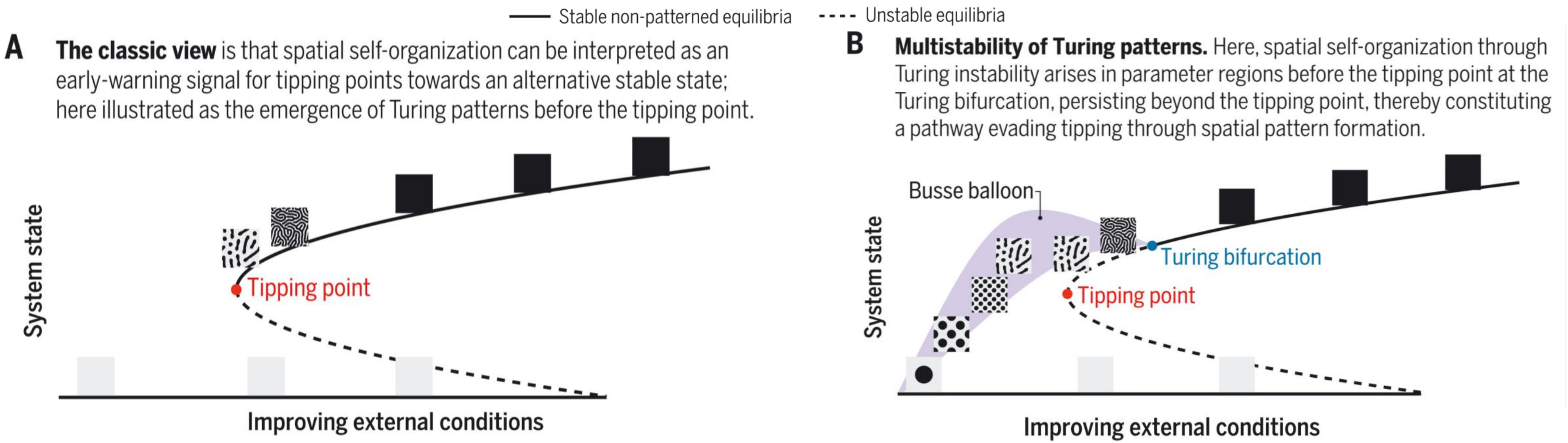


Tipping of patterns

Rietkerk, M., Bastiaansen, R., Banerjee, S., van de Koppel, J., Baudena, M., & Doelman, A. (2021). Evasion of tipping in complex systems through spatial pattern formation. Science, 374(6564), eabj0359.

Tipping
in
Spatially Extended Systems:
early warnings & control

Early warning of critical transitions: Distinguishing tipping points from turing destabilizations



Research by Phd Candidate

Paul Sanders

Paper out:

Sanders, P. A., & Bastiaansen, R. (2026). Early warning of critical transitions: distinguishing tipping points from Turing destabilizations. *Physica D: Nonlinear Phenomena*, 135300.

Recall: disperion relations

Turing patterns

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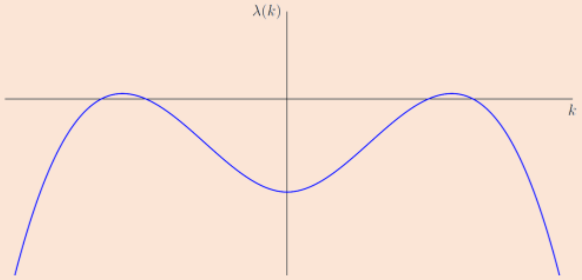
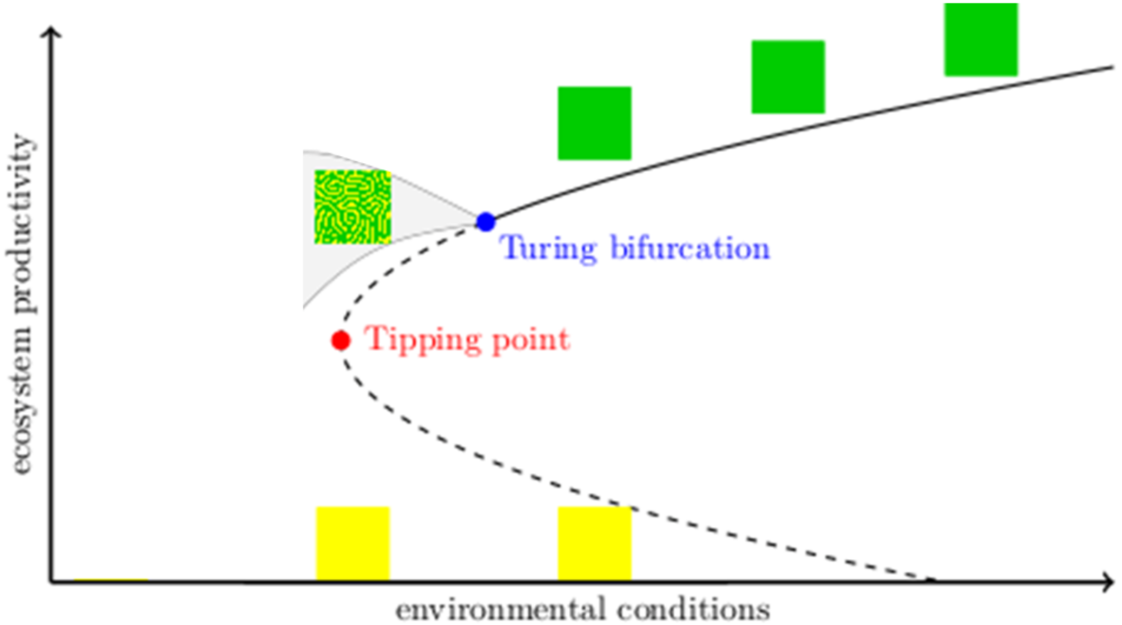




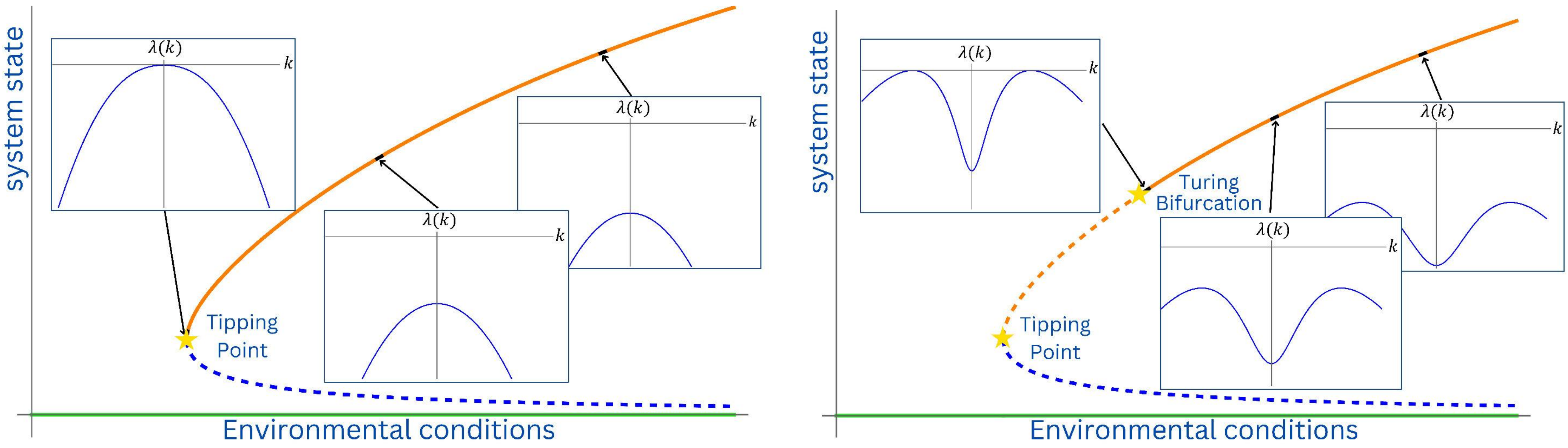
Weakly non-linear analysis

Ginzburg-Landau equation / Amplitude Equation

& Eckhaus/Benjamin-Feir-Newel criterion

[Eckhaus, 1965; Benjamin & Feir, 1967; Newell, 1974]

Use disperion relations for early warning



(a) Destabilization via a 'Tipping' bifurcation

(b) Destabilization via a Turing bifurcation

Determining dispersion relation from noisy data

Data Y(x, t)

Perturbed Data

$$\delta Y(x,t) = Y(x,t) - \bar{Y}(t)$$

fit perturbed data to a **linearized spatial stability model**

$$\delta Y_t = A\delta Y + B\delta Y$$

Spatial differential operator

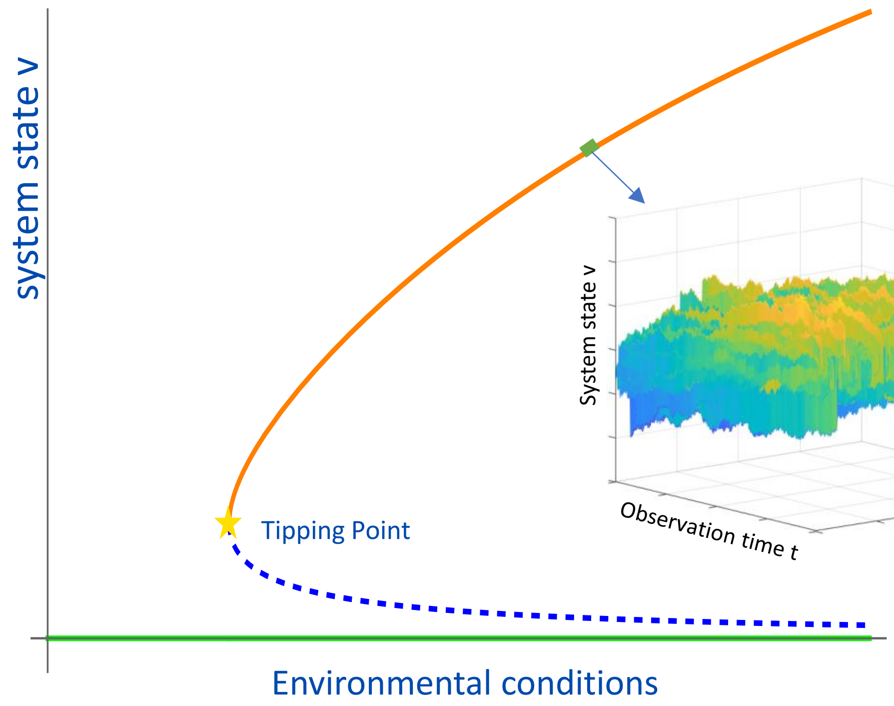
Retrieve **dispersion relation** to understand the **stability of different spatial patterns**

$$\lambda(k)\delta Y = A\delta Y + \hat{B}(k)\delta Y$$

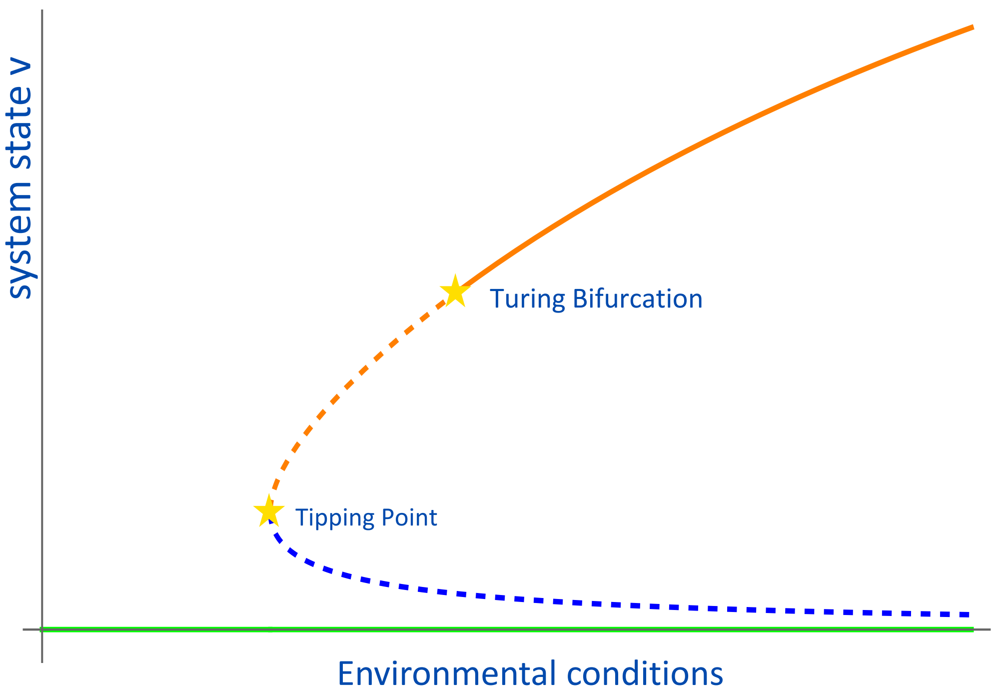
Test case: extended Klausmeier model

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + p - u - uv^2 + Noise_1(x,t)$$
$$\frac{\partial v}{\partial t} = \delta \frac{\partial^2 v}{\partial x^2} + uv^2(1 - hv) - mv + Noise_2(x,t)$$

Case 1: Saddle-node bifurcation

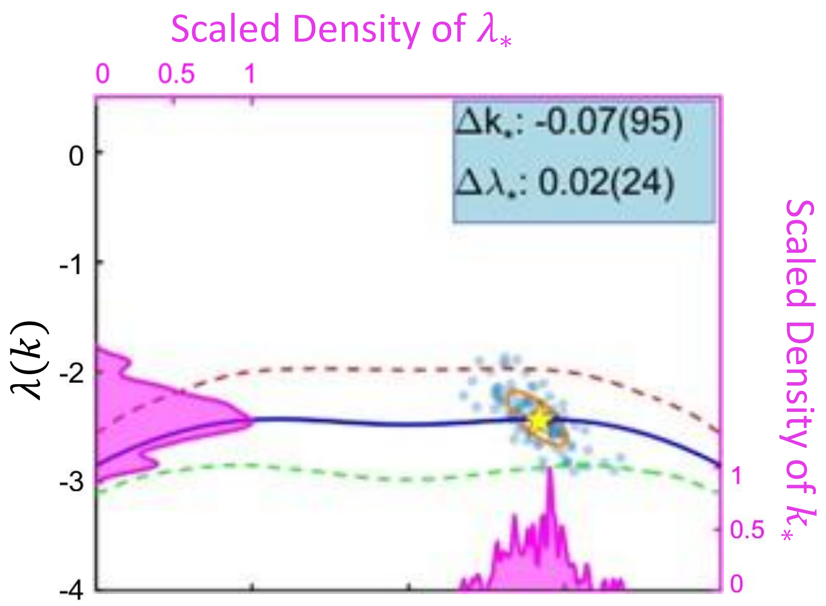


Case 2: Turing bifurcation

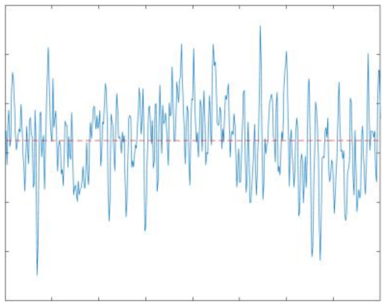


Method testing

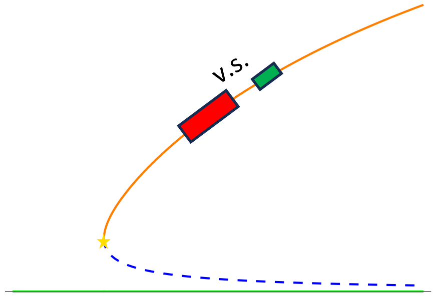
- 100 Distinct datasets
- Spatially Correlated Noise
- Average and extreme dispersion relation outcomes plotted
- Density plotted for found dominant eigenmode k_* and dominant eigenvalue λ_*
- $\Delta k_* = k_{*,real} - k_*$



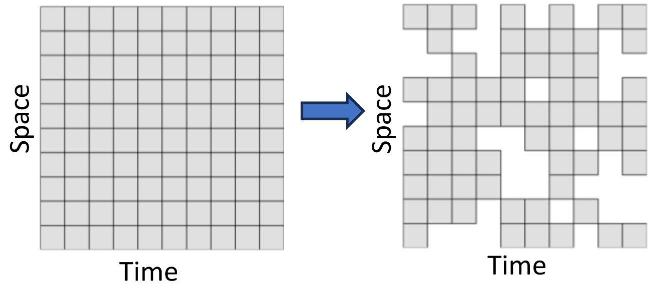
1. Dynamical Noise



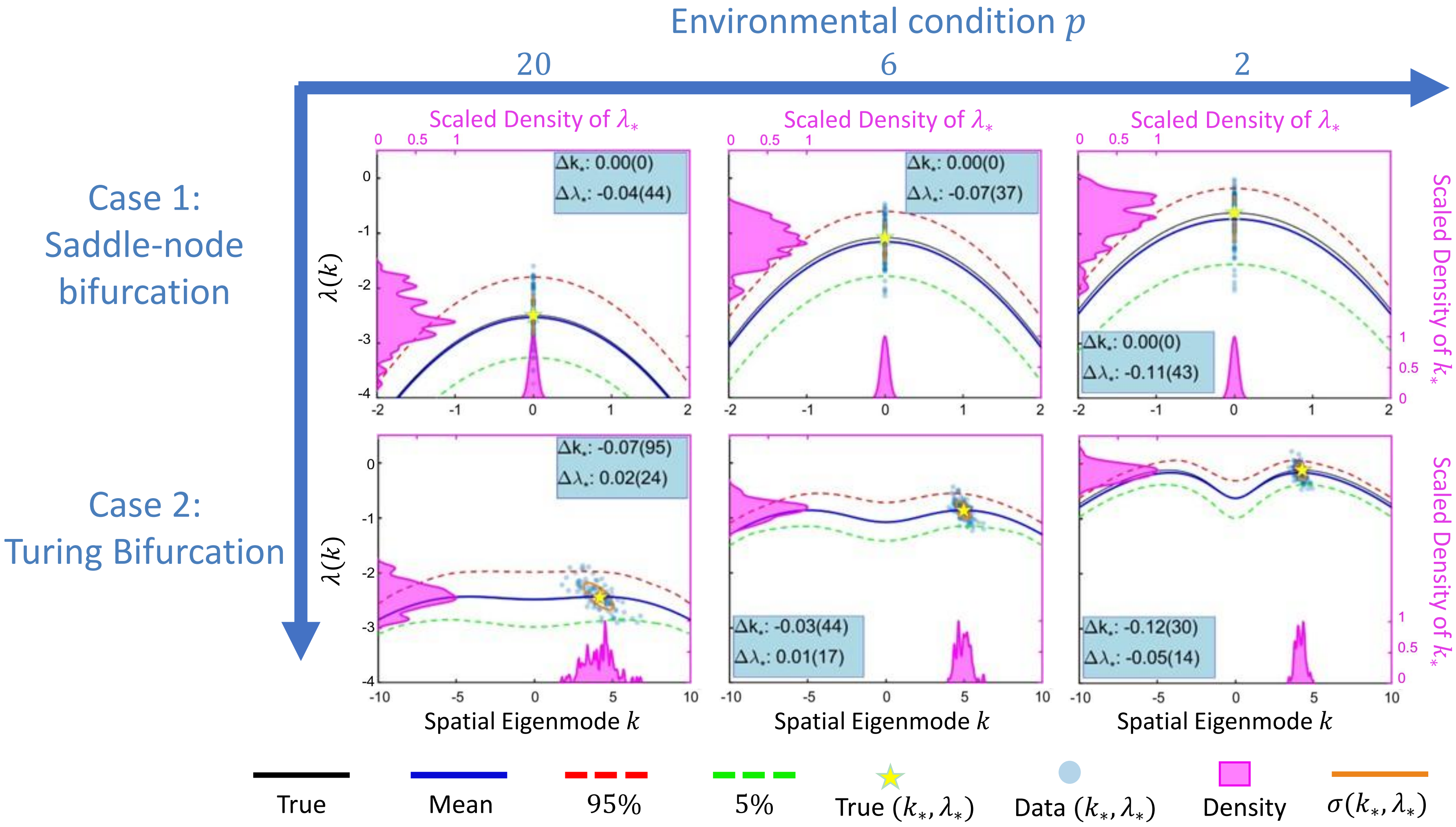
2. Observation time

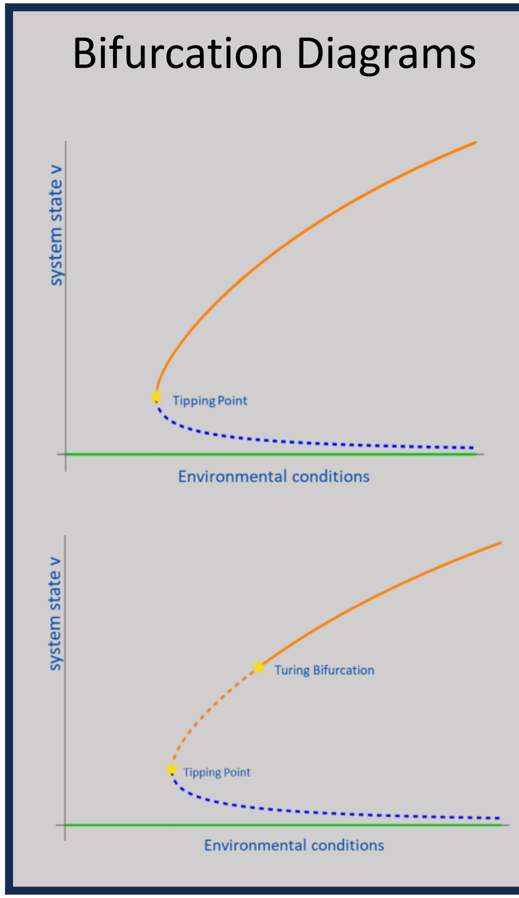


3. Data Sampling

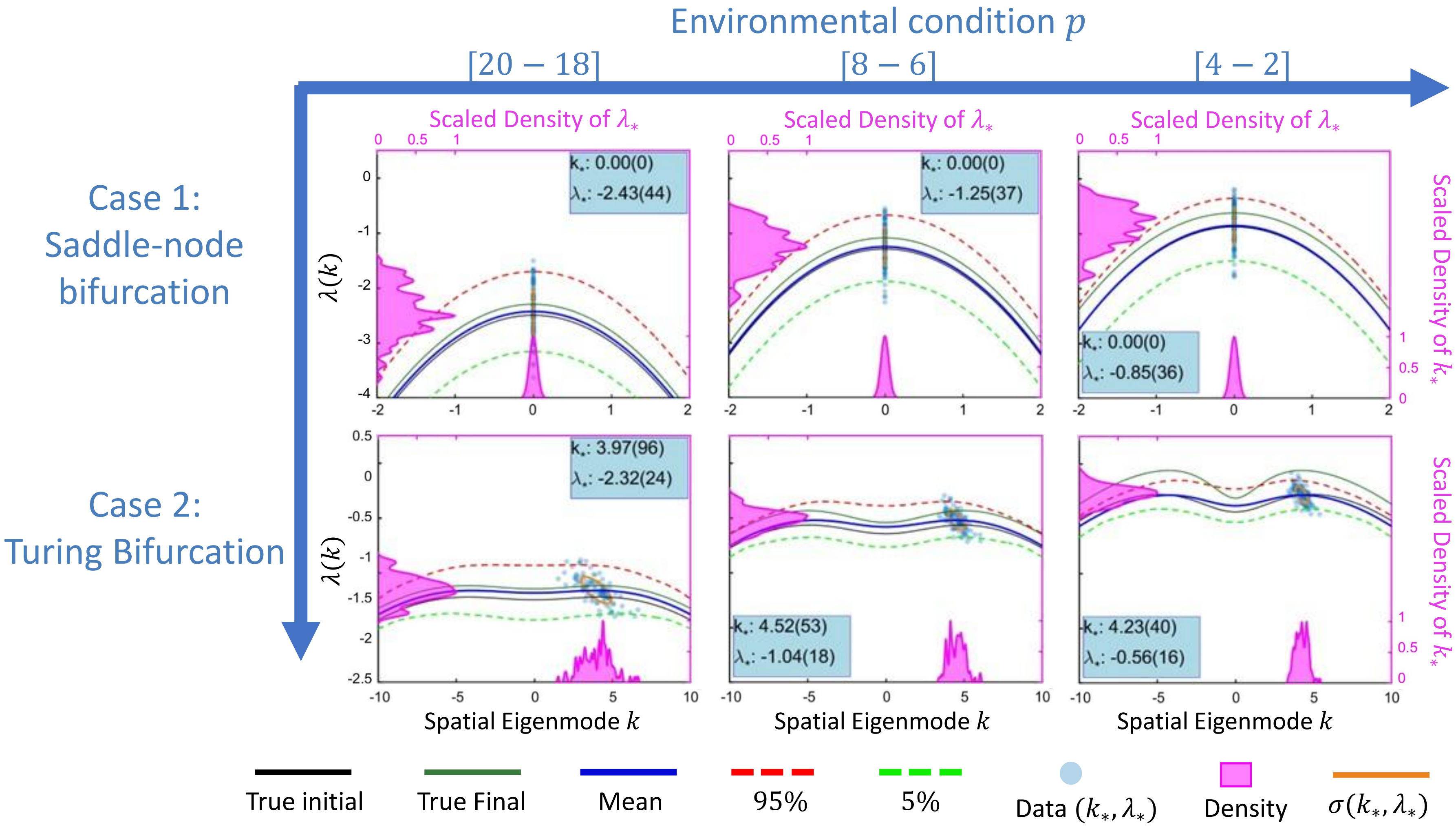


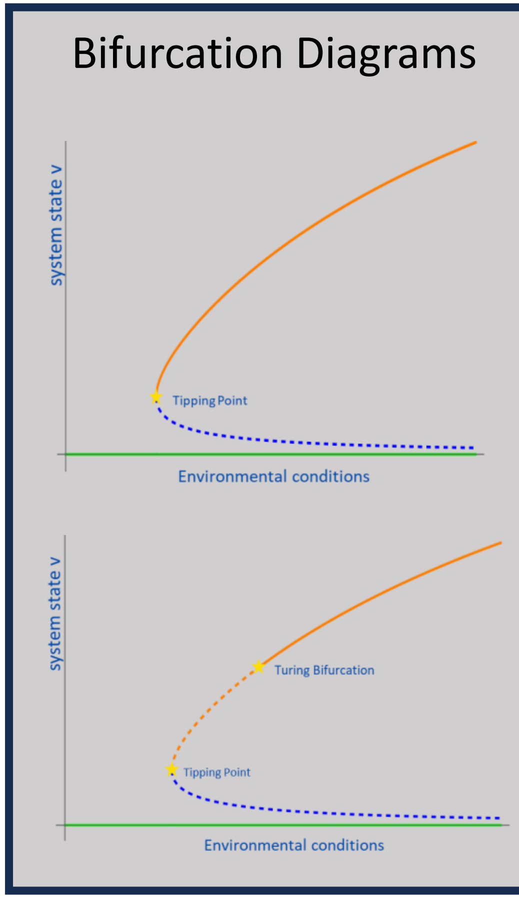
Constant forcing numerical experiment



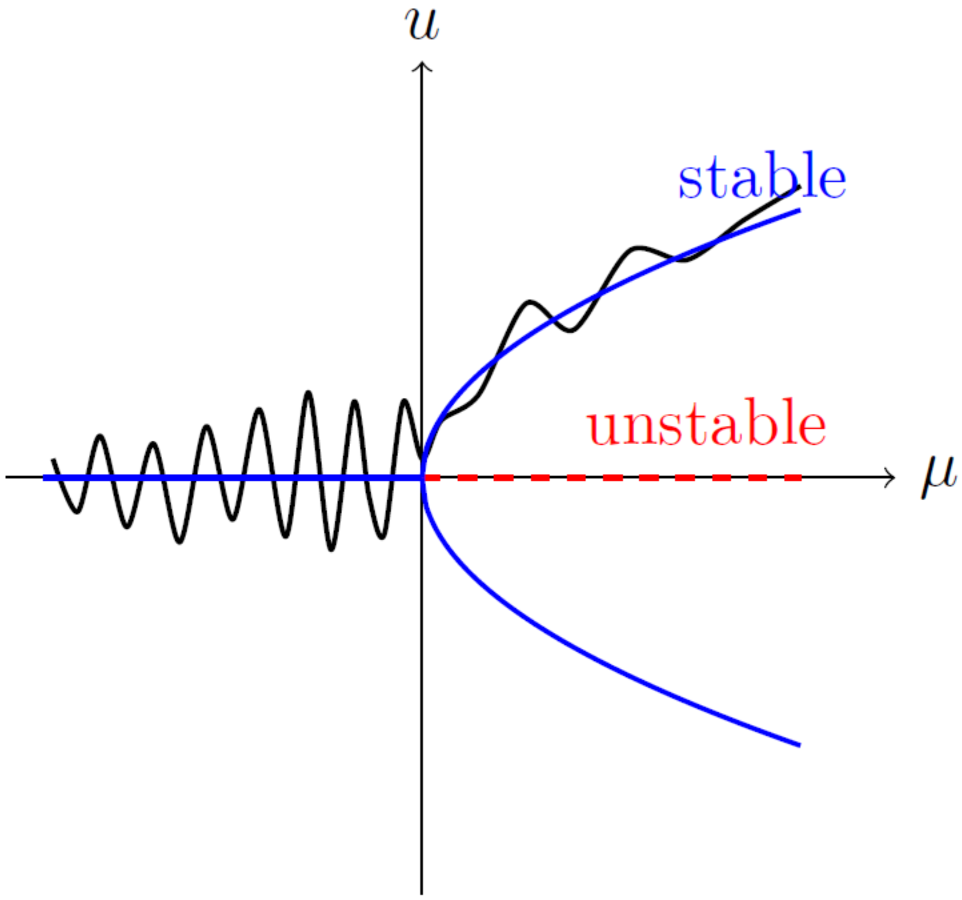


Time-varying forcing numerical experiment

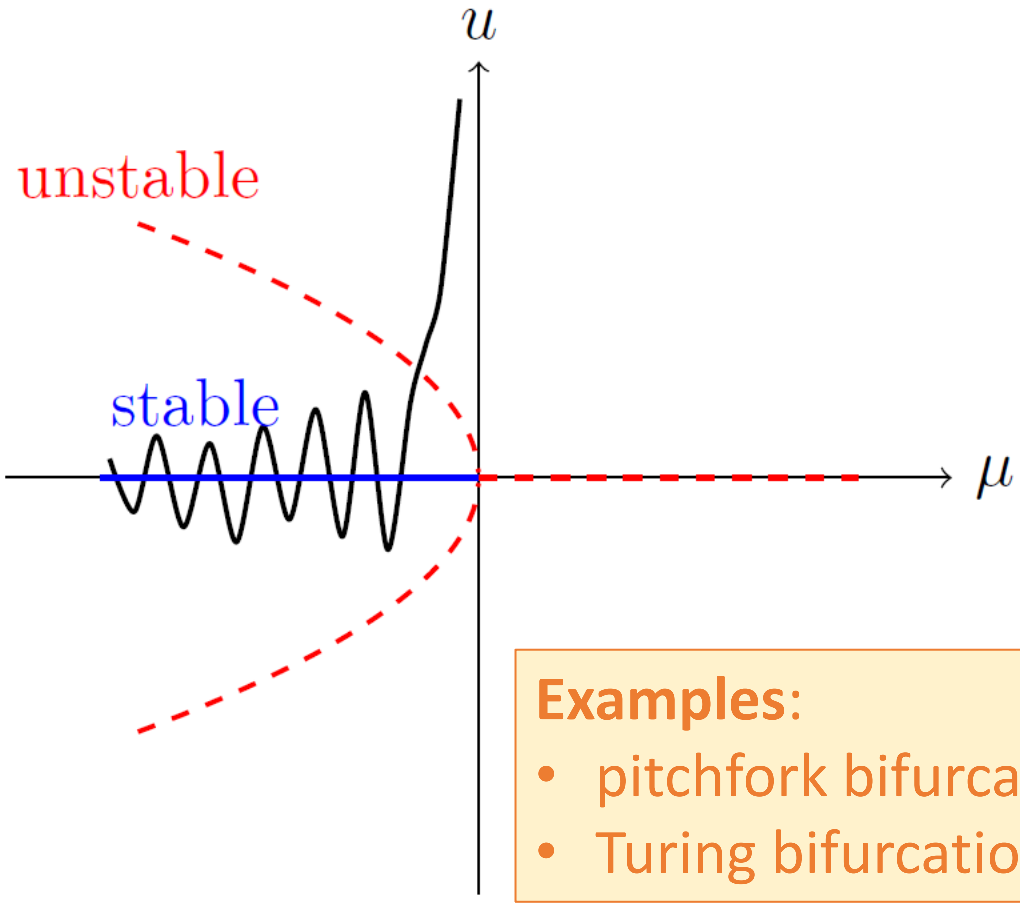




Follow-up: can we determine bifurcation criticality?



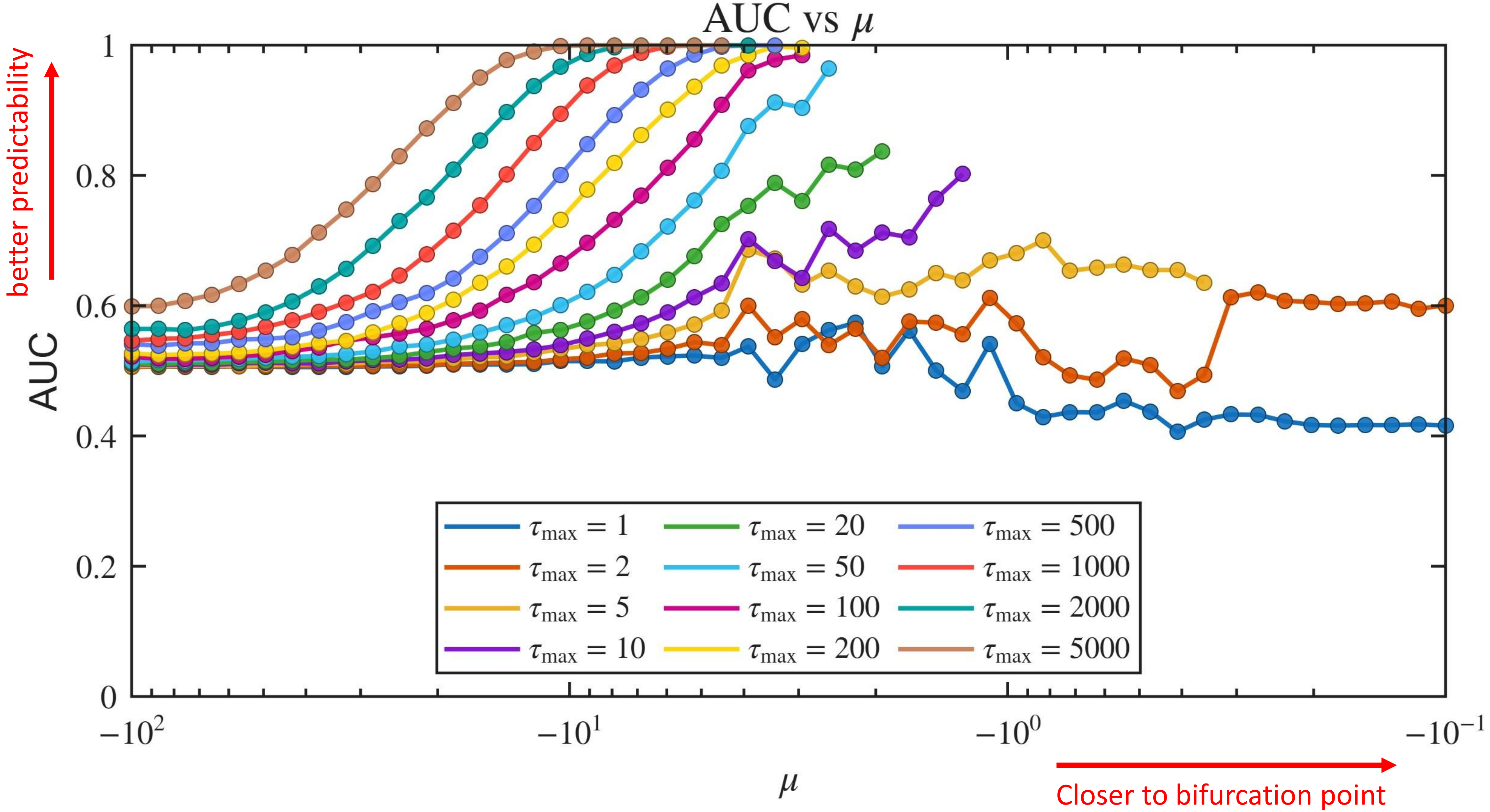
Supercritical bifurcation



Subcritical bifurcation

Examples:
- pitchfork bifurcation
- Turing bifurcation

Follow-up: can we determine bifurcation criticality?



Tipping
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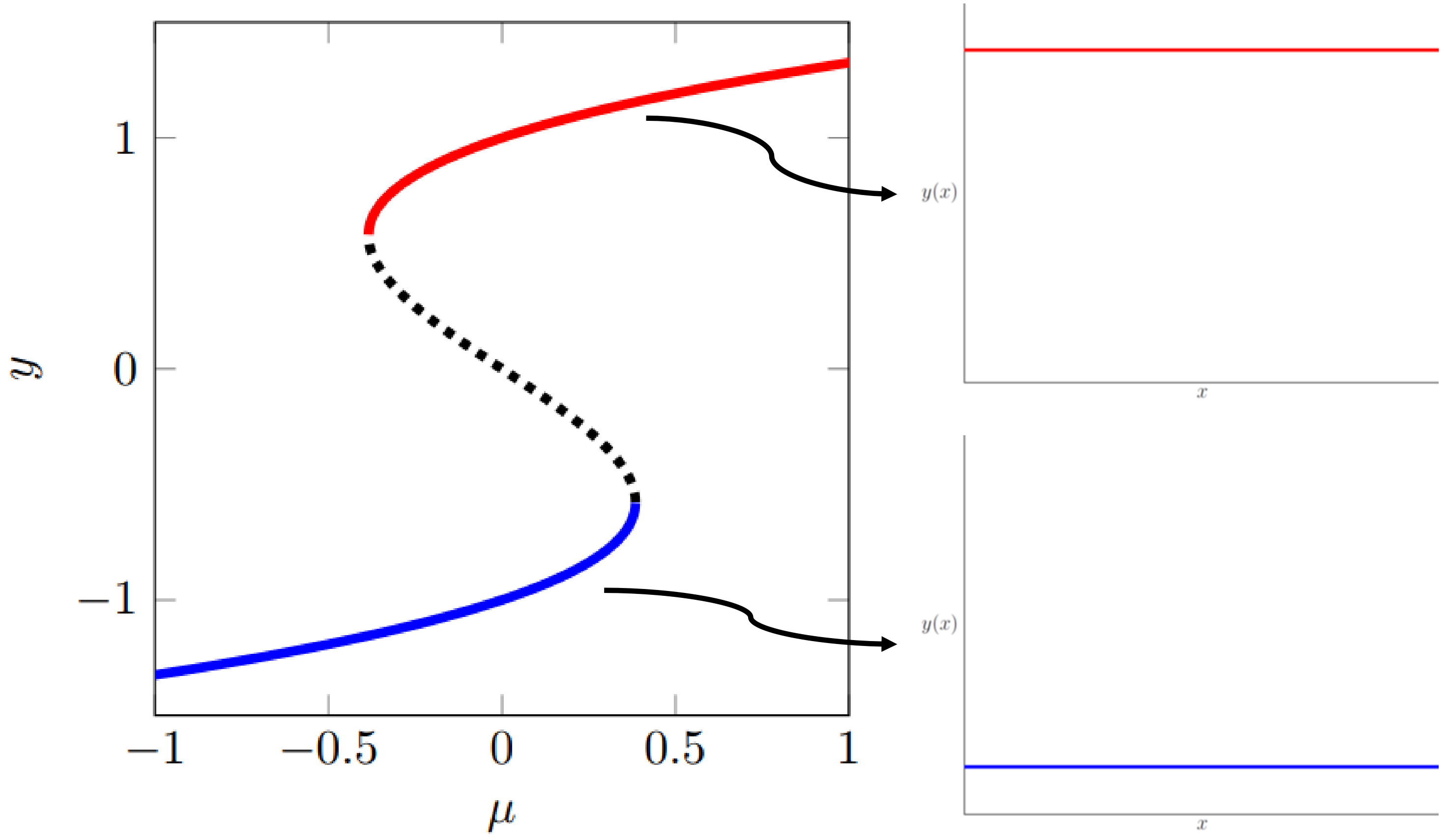
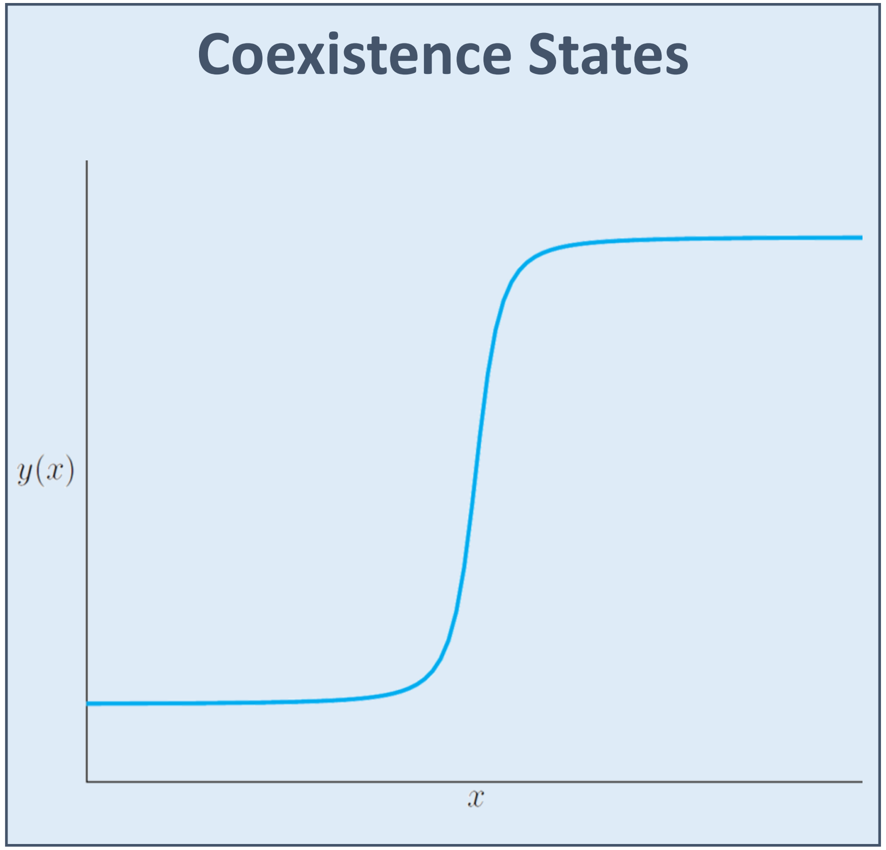
Spatially Extended Systems:
early warnings & control

Coexistence states

Bistable (Allen-Cahn/Nagumo) equation:

$$\frac{\partial y}{\partial t} = y(1 - y^2) + \mu + D\,\frac{\partial^2 y}{\partial x^2}$$



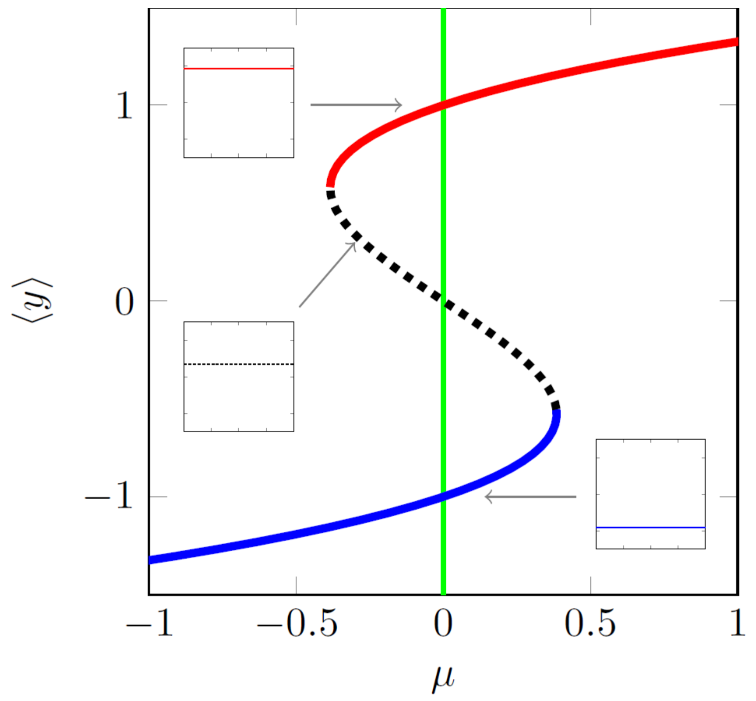


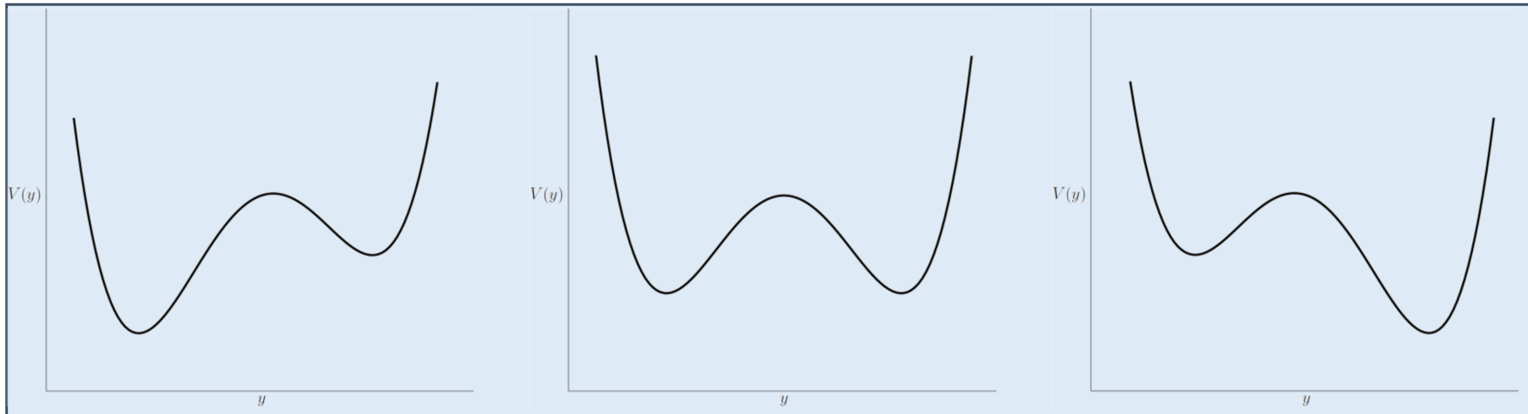
Front Dynamics

$$\frac{\partial y}{\partial t} = D\,\frac{\partial^2 y}{\partial x^2} + f(y;\mu)$$

Potential function $V(y;\mu)$:

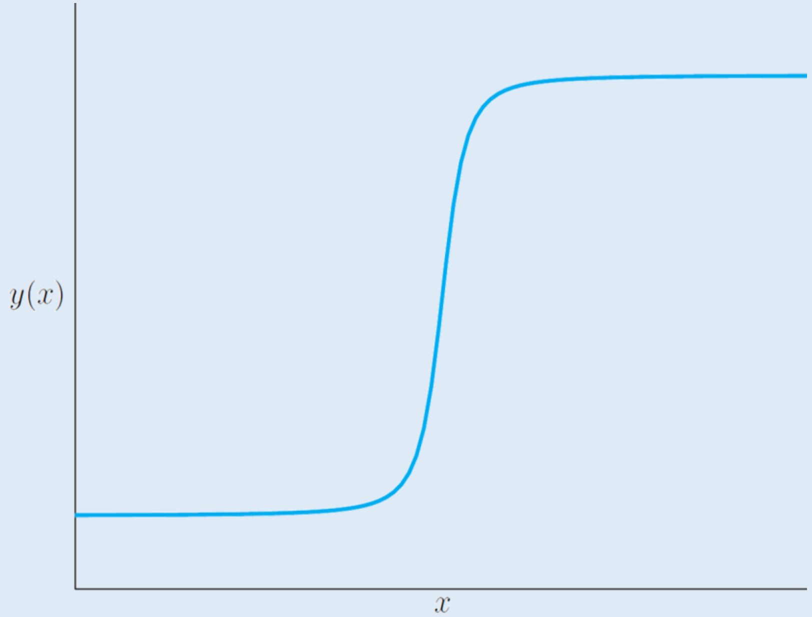
$$\frac{\partial V}{\partial y}(y;\mu) = -f(y;\mu)$$





moves right | stationary | moves left

Maxwell Point $\mu_{maxwell}$



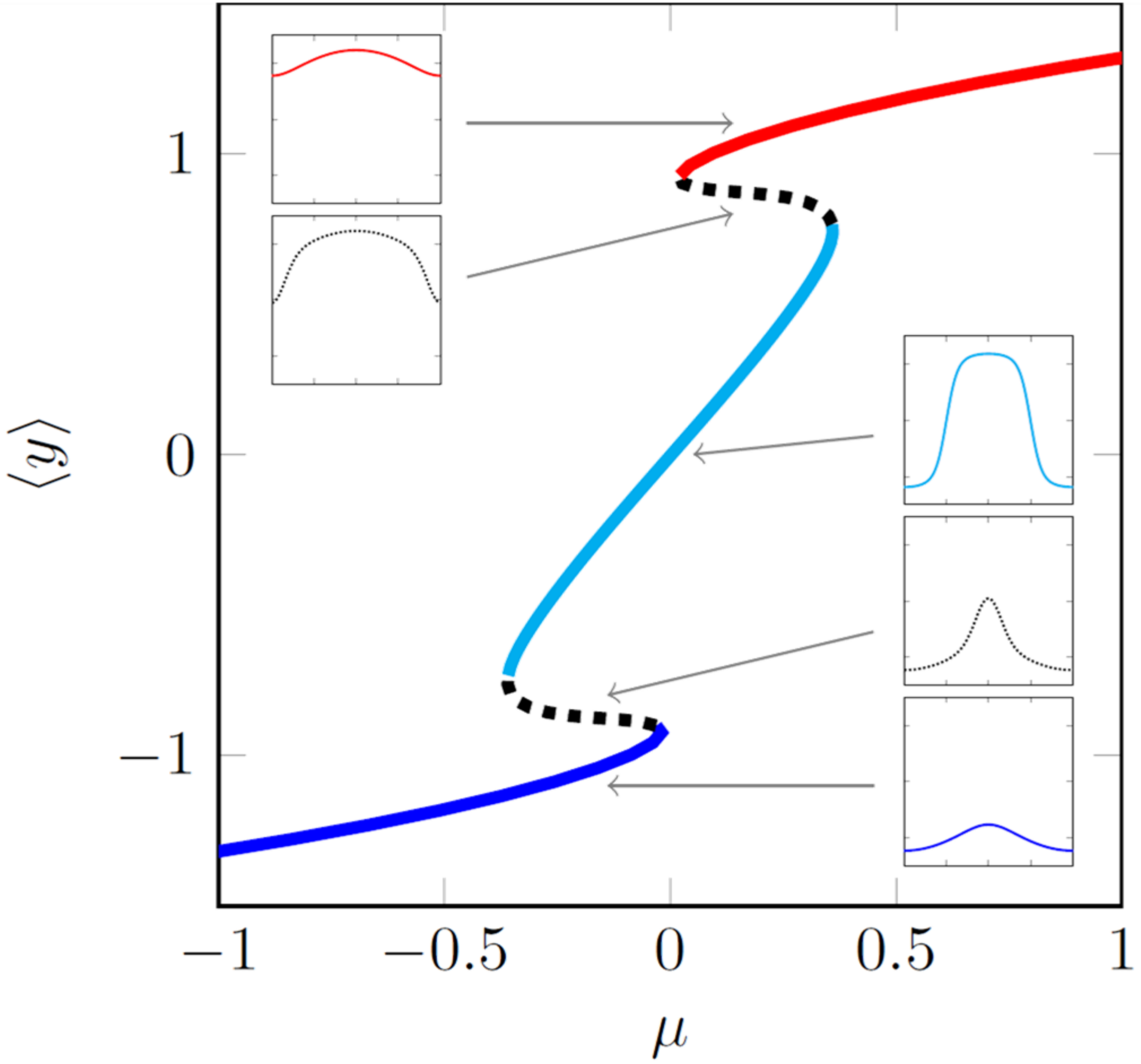
Adding Spatial Heterogeneity

$$\frac{\partial y}{\partial t} = D\frac{\partial^2 y}{\partial x^2} + f(y, {\color{red}x}; \mu)$$

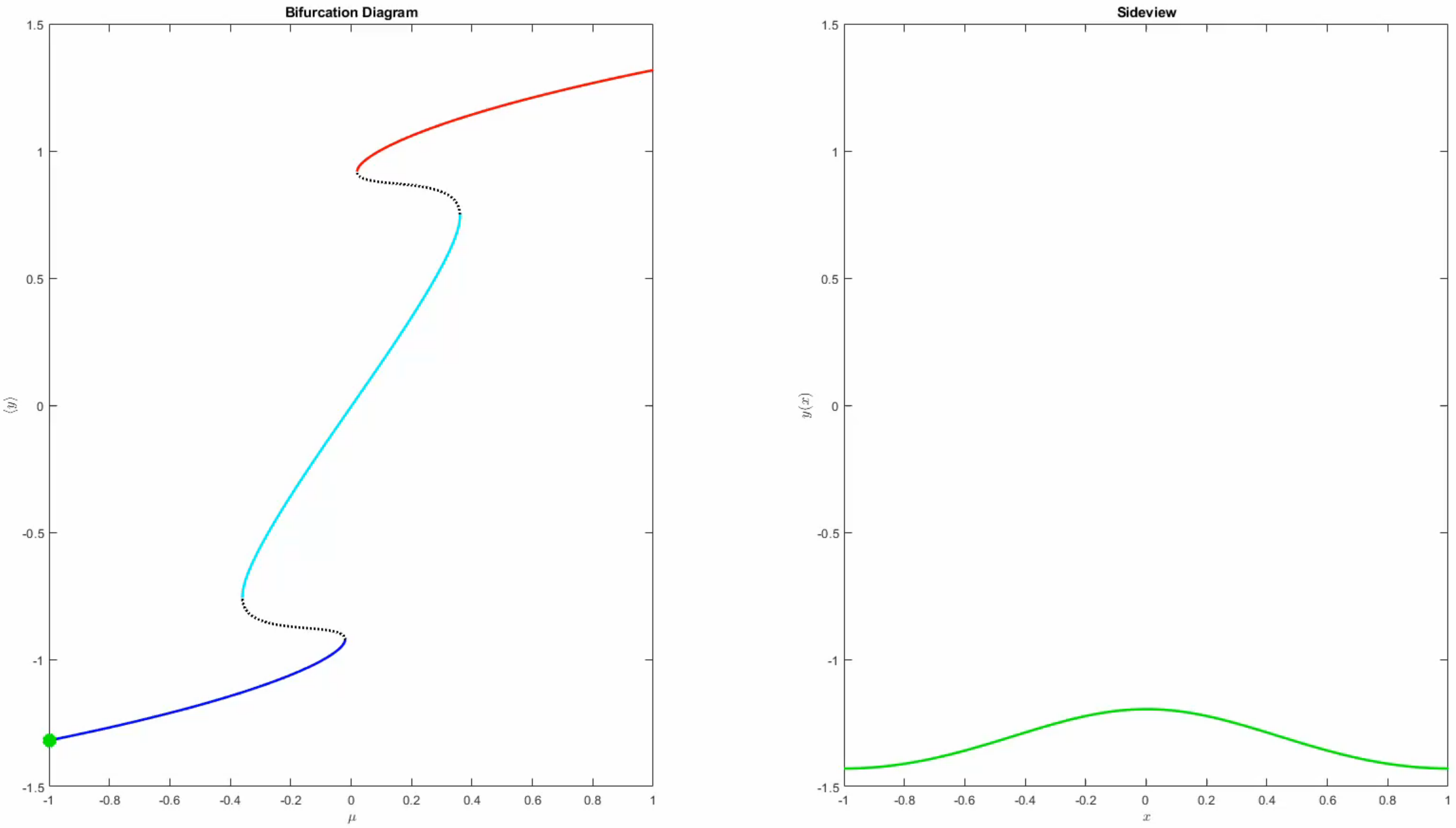
Now, the **local** difference in potentials determines the front movement

New behaviour:

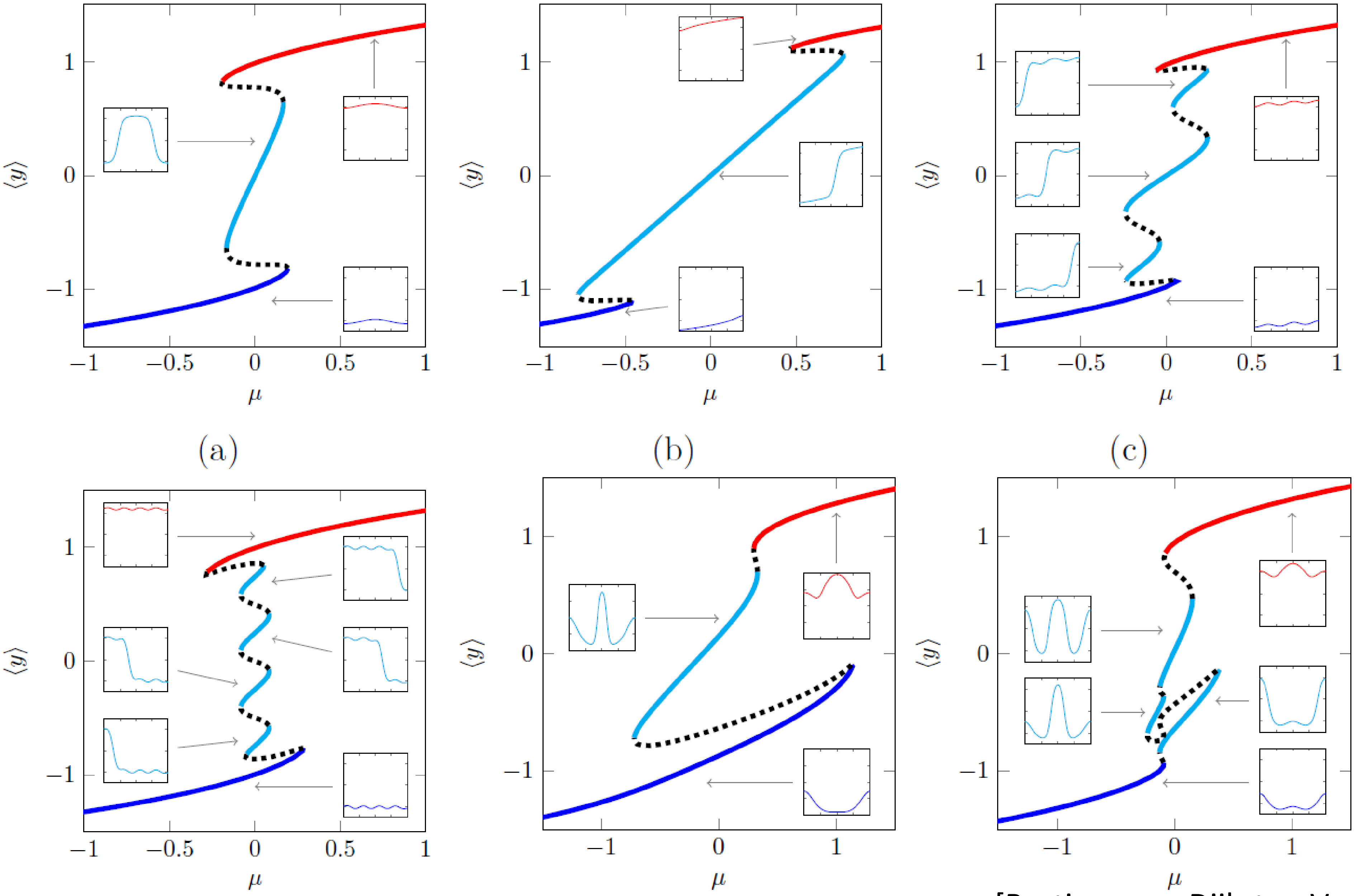
- Multi-fronts can be stationary [Bastiaansen, Doelman, Kaper, 2025, *preprint*]
- Maxwell point is smeared out



Fragmented Tipping



Other Spatial Heterogeneities



[Bastiaansen, Dijkstra, Von der Heydt, 2022]

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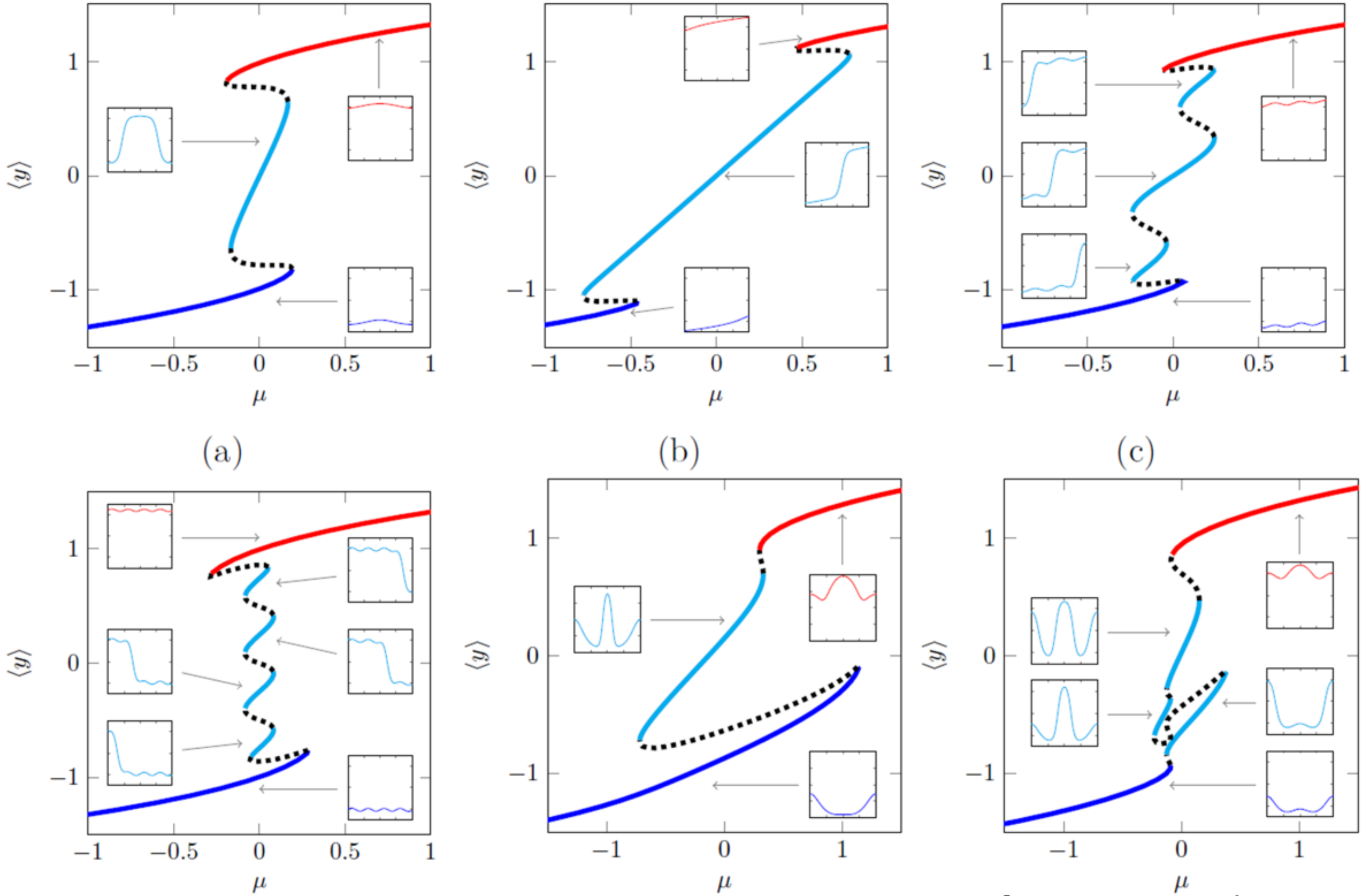
Optimisation of heterogeneity in Allen Cahn equation

Other Spatial Heterogeneities



[Bastiaansen, Dijkstra, Von der Heydt, 2022]

Can’t we just pick and choose our favourite!?

Research by Phd Candidate

Aurora Faure Ragani

Preprint:
Optimisation of tipping pathways in a spatially heterogeneous world. *arXiv preprint arXiv:2606.23012*.

Idea: with local action, prevent larger-scale change

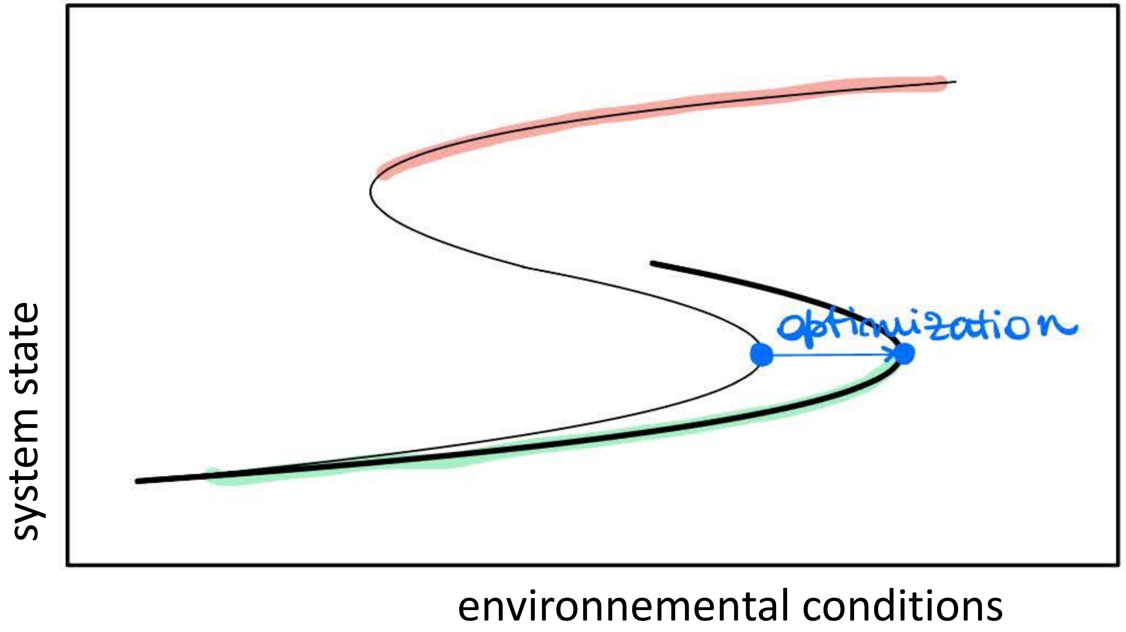
Source: Andrew Millison on Youtube (see his videos on Great Green Wall)

Approach: make it an optimisation problem

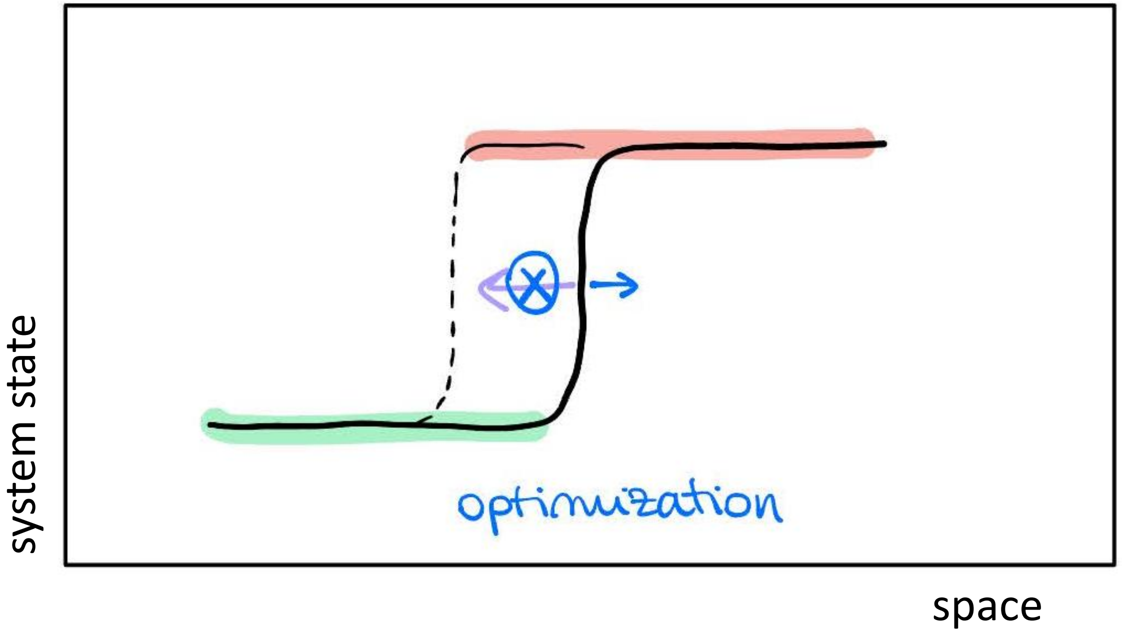
Mathematical Optimisation Problem

Minimize	$f(z)$
subject to	$g(z) = 0$
	$h(z) \leq 0$

Bifurcation diagrams



Front dynamics



Approach 1: “simulation-like”

Mathematical Optimisation Problem

Minimize $f(z)$

subject to $g(z) = 0$

$h(z) \le 0$

COST FUNCTION

For example:

Minimize the globally averaged state at time t_e

$$f(z) := \int y(x, t_e; z)\, dx$$

Model-Constraint

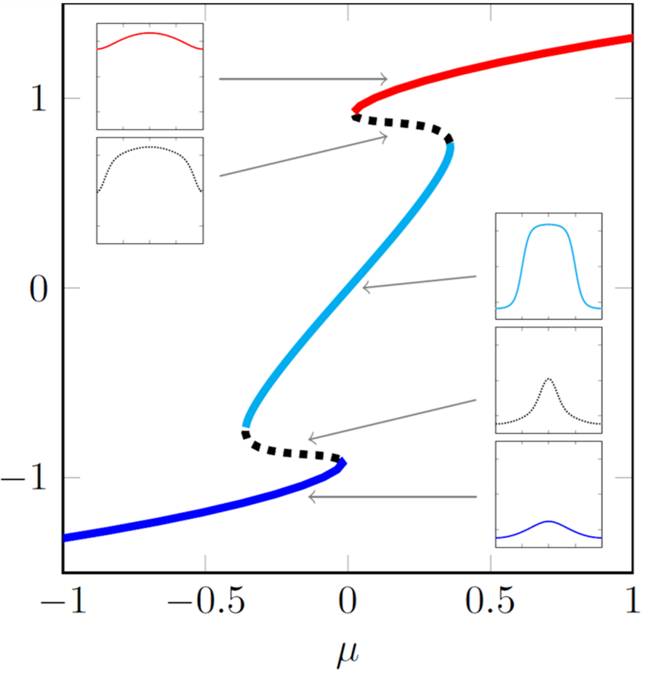
Some model with specified scenario:

$$g(z) := -\frac{\partial y}{\partial t} + D\,\frac{\partial^2 y}{\partial x^2} + y(1 - y^2) + \mu(x,t) + z(x)$$

Perturbation size restriction

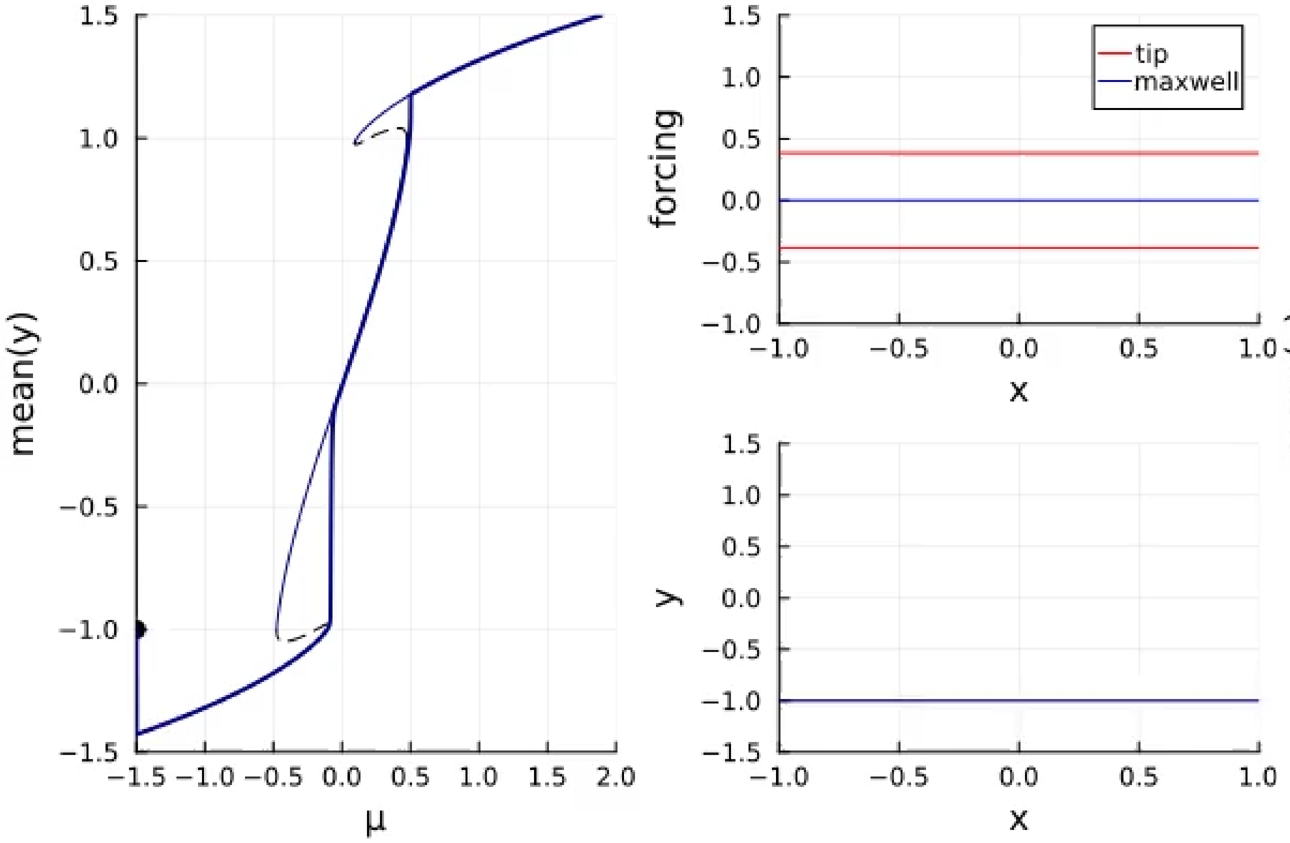
The applied perturbation cannot be too large

$$h(z) := \|z\| - \delta \le 0$$

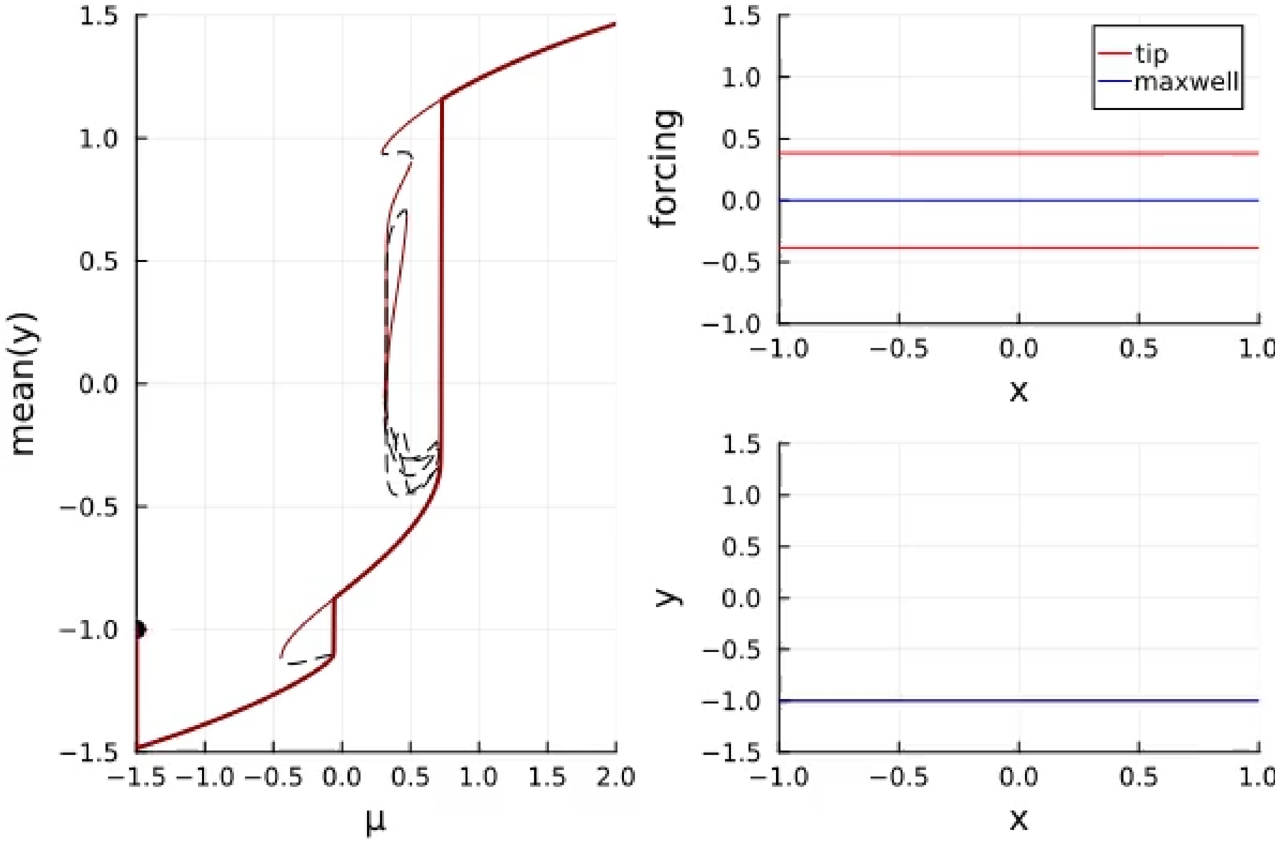


Example optimisation

ORIGINAL

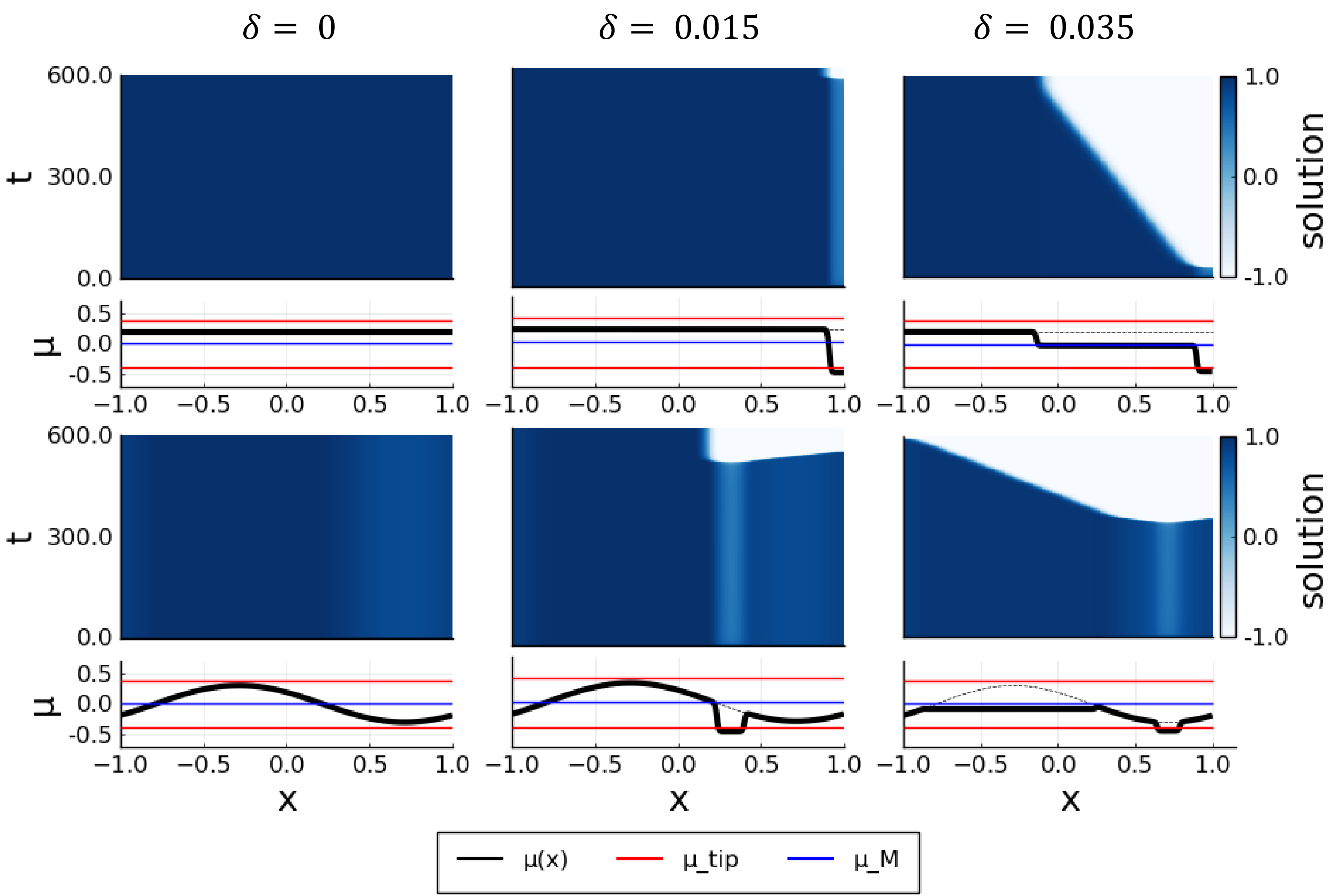


OPTIMISED



Details
Objective: minimise the mean of the final state
$\mu(t_e) = 0.7$
$\delta = 0.3$
Pre-existing heterogeneity: $\frac{1}{2}\cos(\pi x)$

Example optimisation



Approach 2: “continuation-like”

Mathematical Optimisation Problem

$$
\begin{aligned}
\text{Minimize} \quad & f(z) \\
\text{subject to} \quad & g(z) = 0 \\
& h(z) \leq 0
\end{aligned}
$$

COST FUNCTION

For example:

Location of bifurcation as far as possible to the right

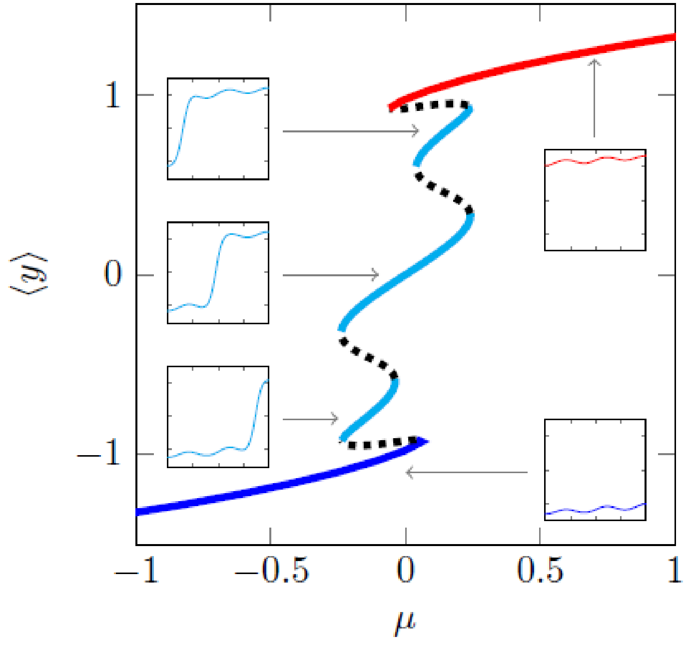
$$f(z) := -\mu_{BIF}(z)$$

Bifurcation constraint

Needs to be a solution at a saddle-node bifurcation

$$g_1(z) := D\,\frac{\partial^2 y}{\partial x^2} + y(1-y^2) + \mu(x) + z(x)$$

$$g_2(z) := \text{“}eigenvalue\ at\ zero\text{”}$$

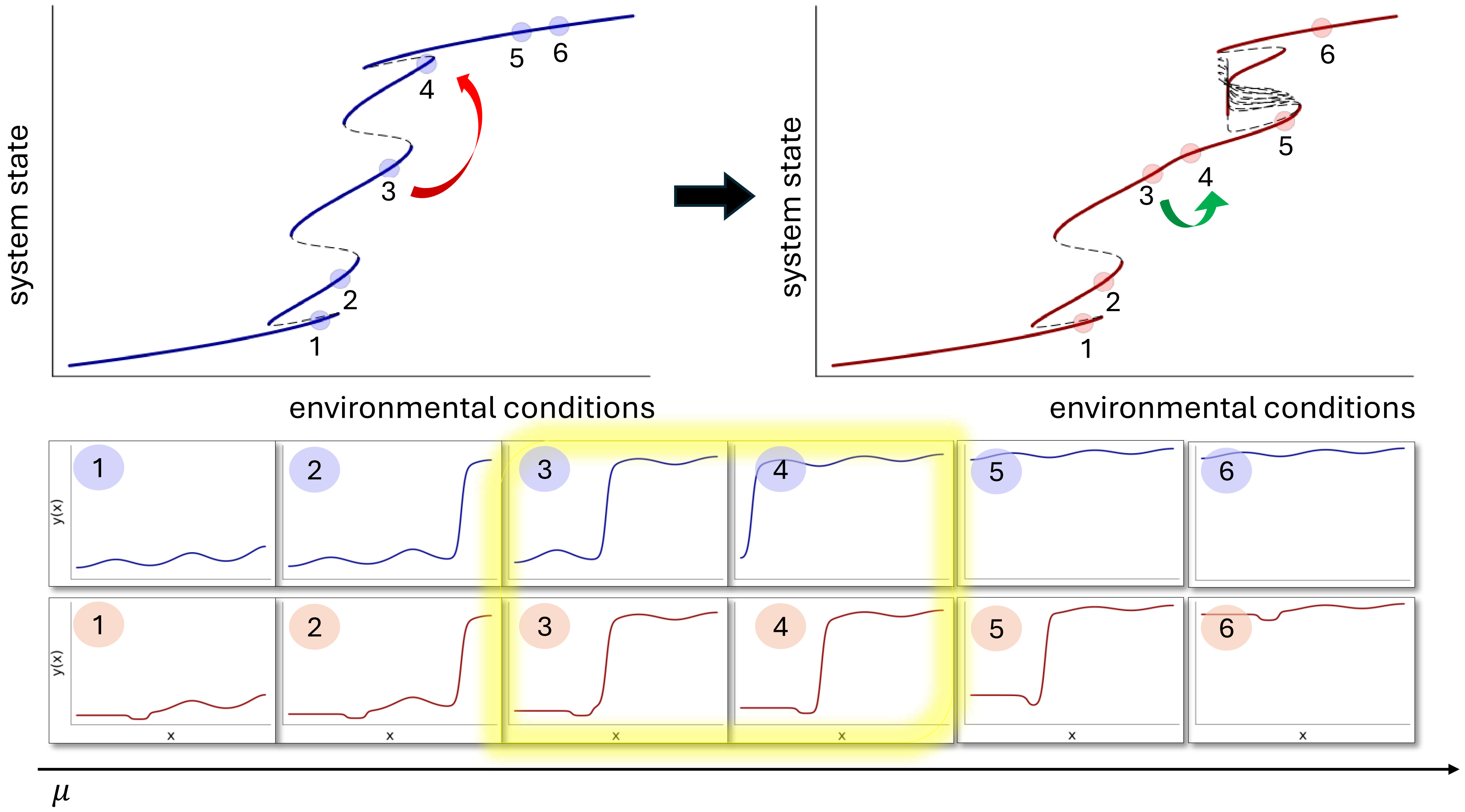


Perturbation size restriction

The applied perturbation cannot be too large

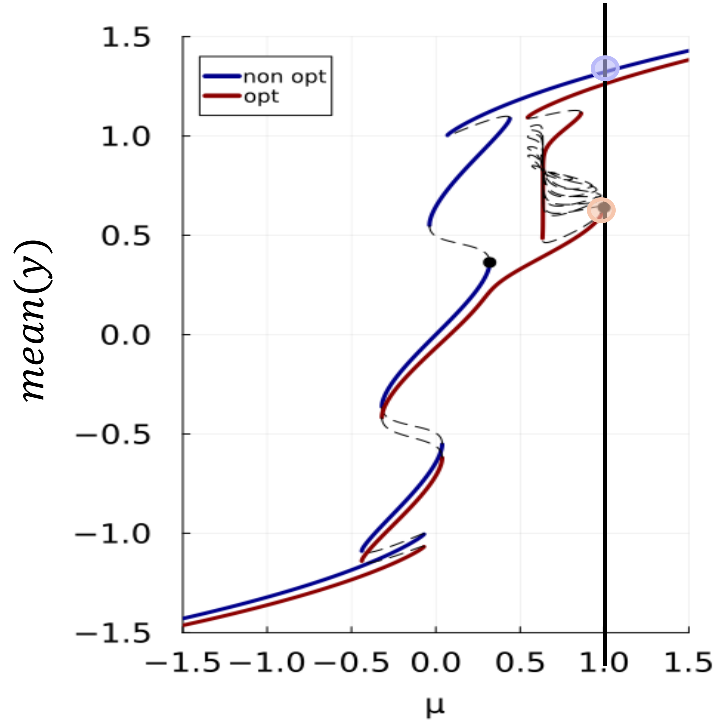
$$h(z) := \|z\| - \delta \leq 0$$

Equilibrium optimisation: example B

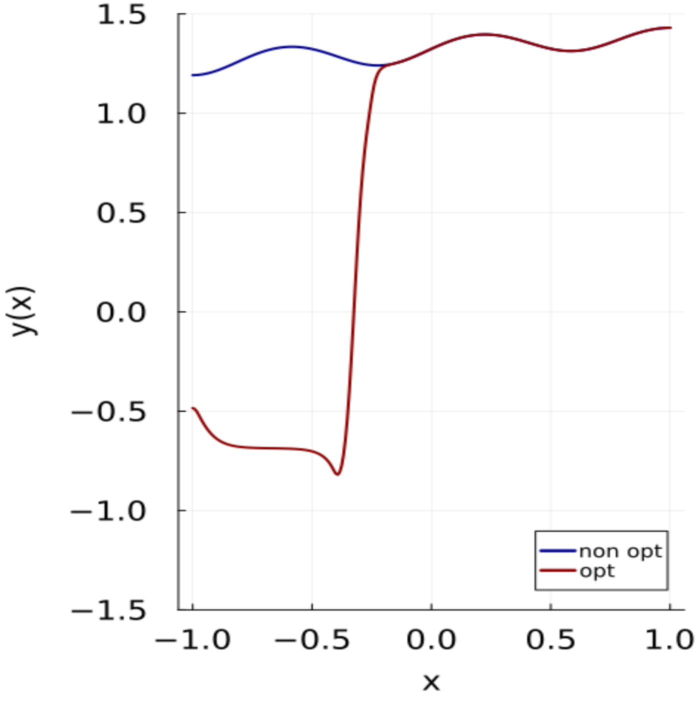


Equilibrium optimisation: example B - details

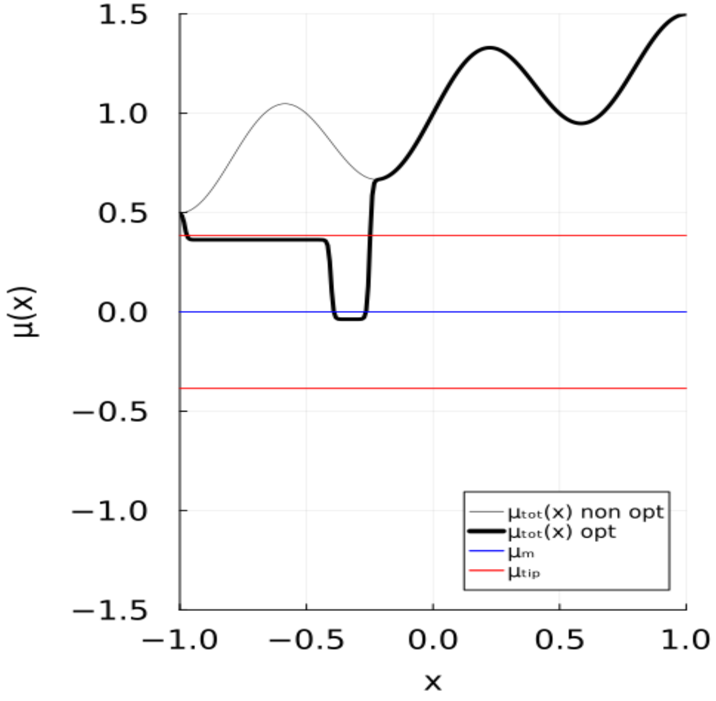
Bifurcation diagram



Solution at optimal μ_{SN1}



$\mu_{tot}(x)$ for $\mu = \mu_{SN1}$

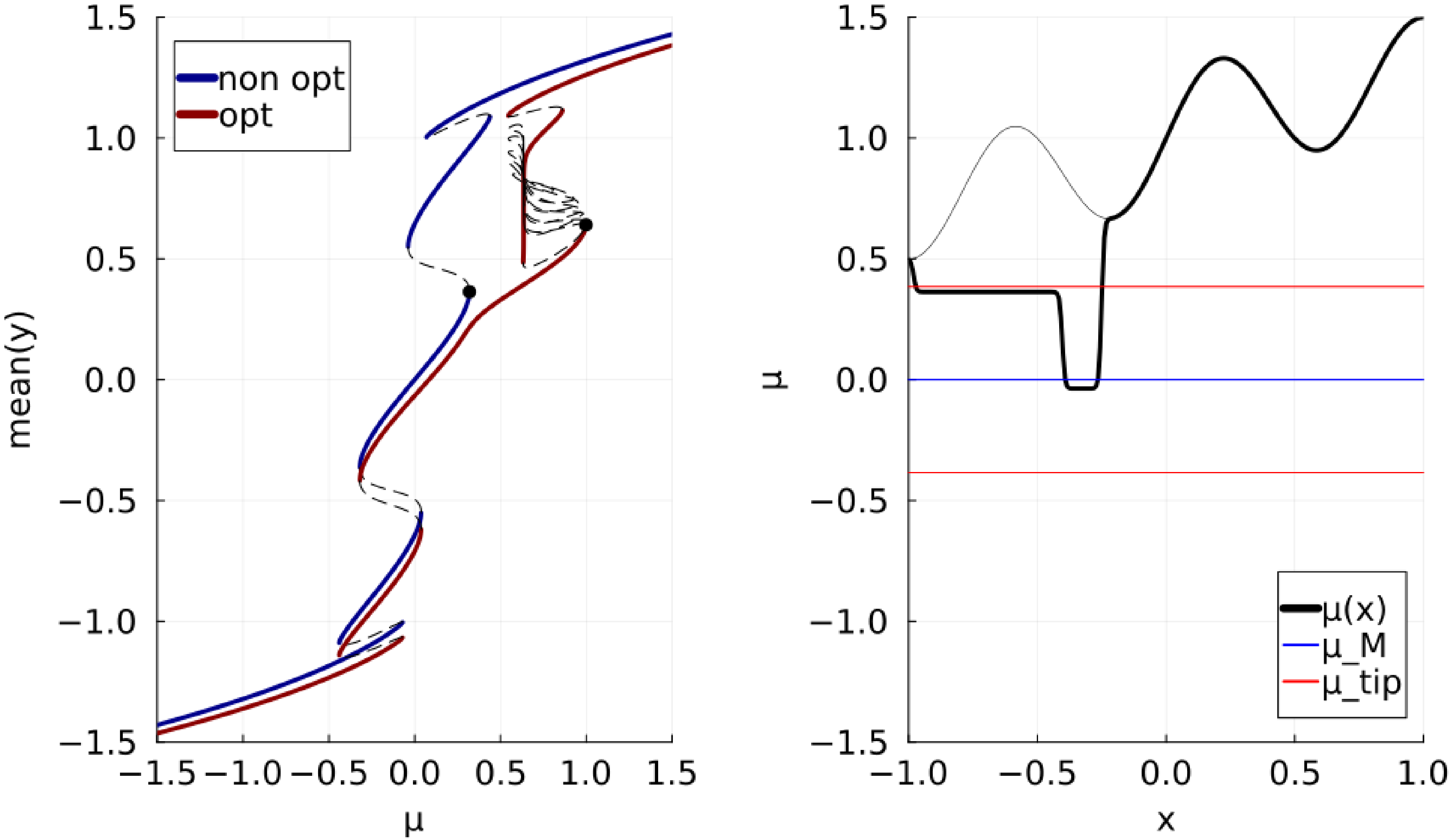


Non optimised system: $y_t = D\, y_{xx} + y(1-y^2) + \mu + \frac{1}{2}\cos(\pi x)\sin(2\pi x)$

Objective function: $-\mu_{SN5}$

Optimised system: $y_t = D\, y_{xx} + y(1-y^2) \underbrace{+ \mu + \frac{1}{2}\cos(\pi x)\sin(2\pi x) + \mu_{pert}(x)}_{\mu_{tot}(x)}$

Example optimisation



Details

Objective: move bifurcation to the right as far as possible

$\delta = 0.3$

Pre-existing heterogeneity: $\frac{1}{2}\cos(\pi x)\sin(2\pi x)$

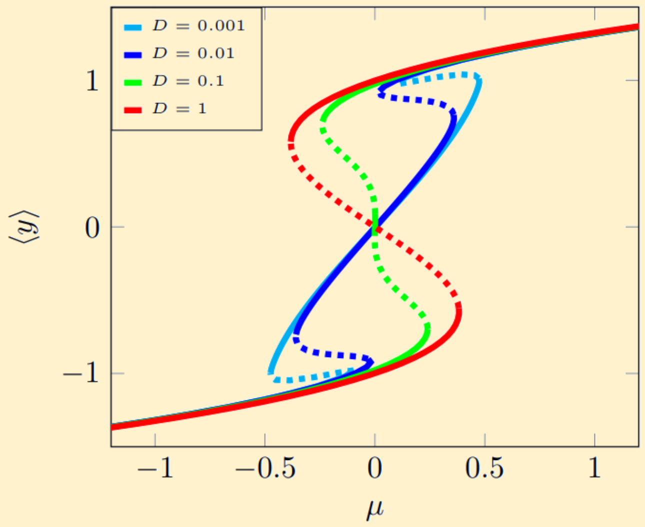
Tipping in Spatially Extended Systems: early warnings & control

Do systems always behave like this? (a.k.a. the small print)

No.

Well-mixed systems

Spatially confined systems



→ Such systems (again) behave like ODEs ←

But even in other systems terms & conditions apply:
System-specific knowledge is required!

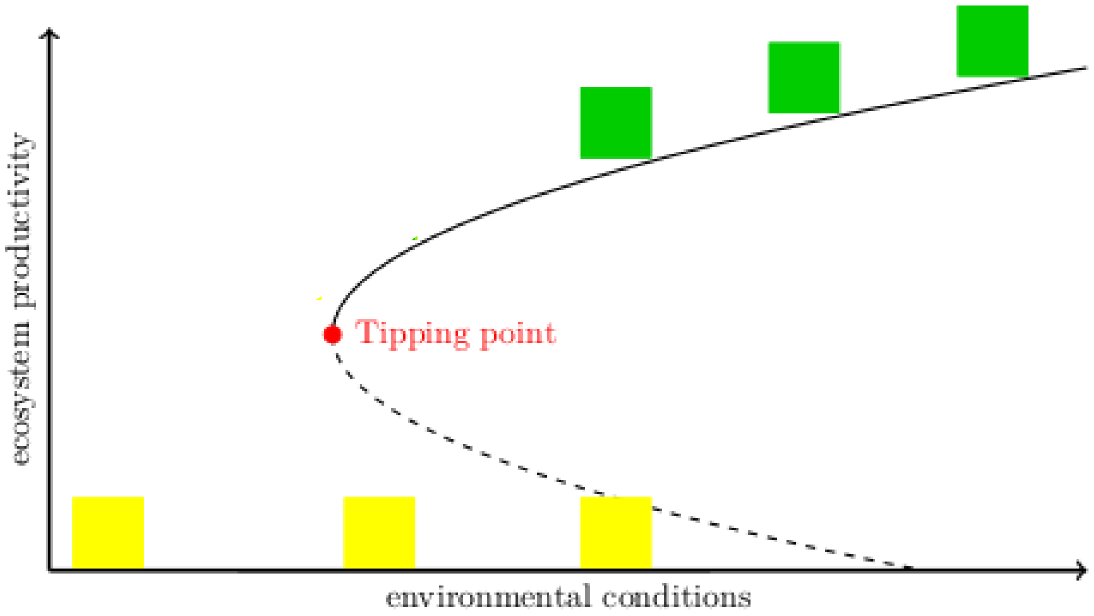
Time scale of feedback

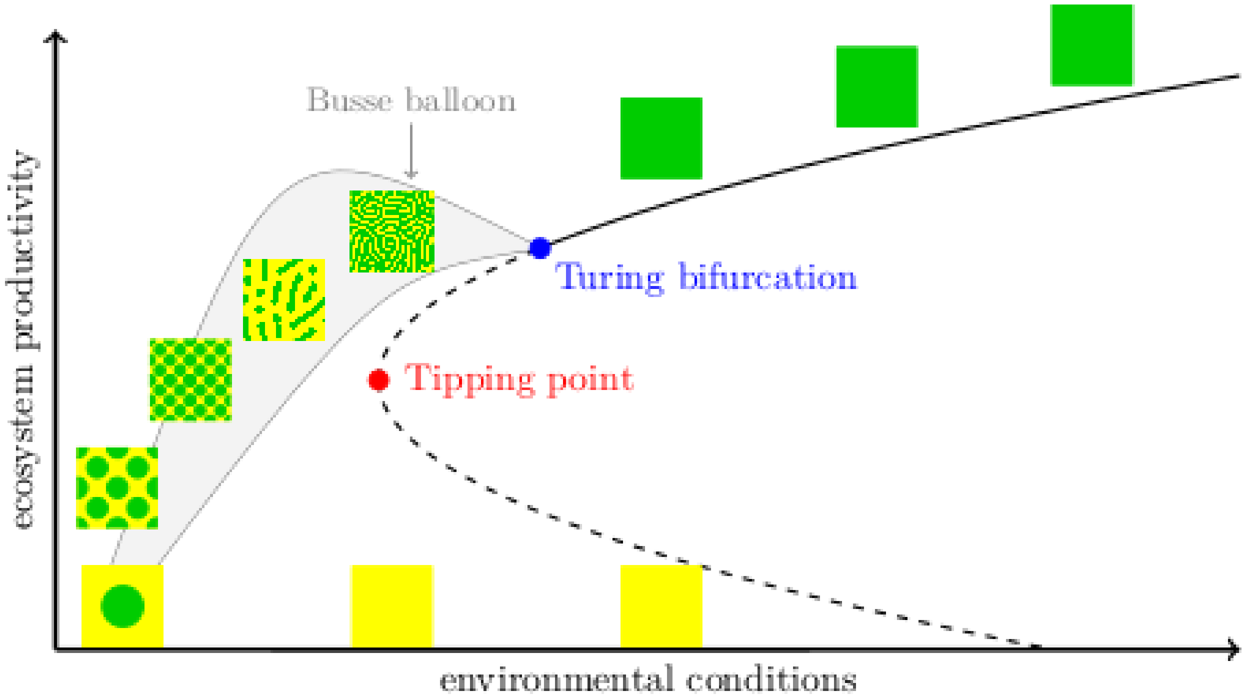
resource

consumer

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + a - u - uv^2$$

$$\frac{\partial v}{\partial t} = D\frac{\partial^2 v}{\partial x^2} + uv^2(1 - hv) - mv$$





Time scale of feedback

resource

consumer

engineered construct

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + a - u - uv\mathbf{s}$$

$$\frac{\partial v}{\partial t} = D\frac{\partial^2 v}{\partial x^2} + uv\mathbf{s}(1 - hv)$$

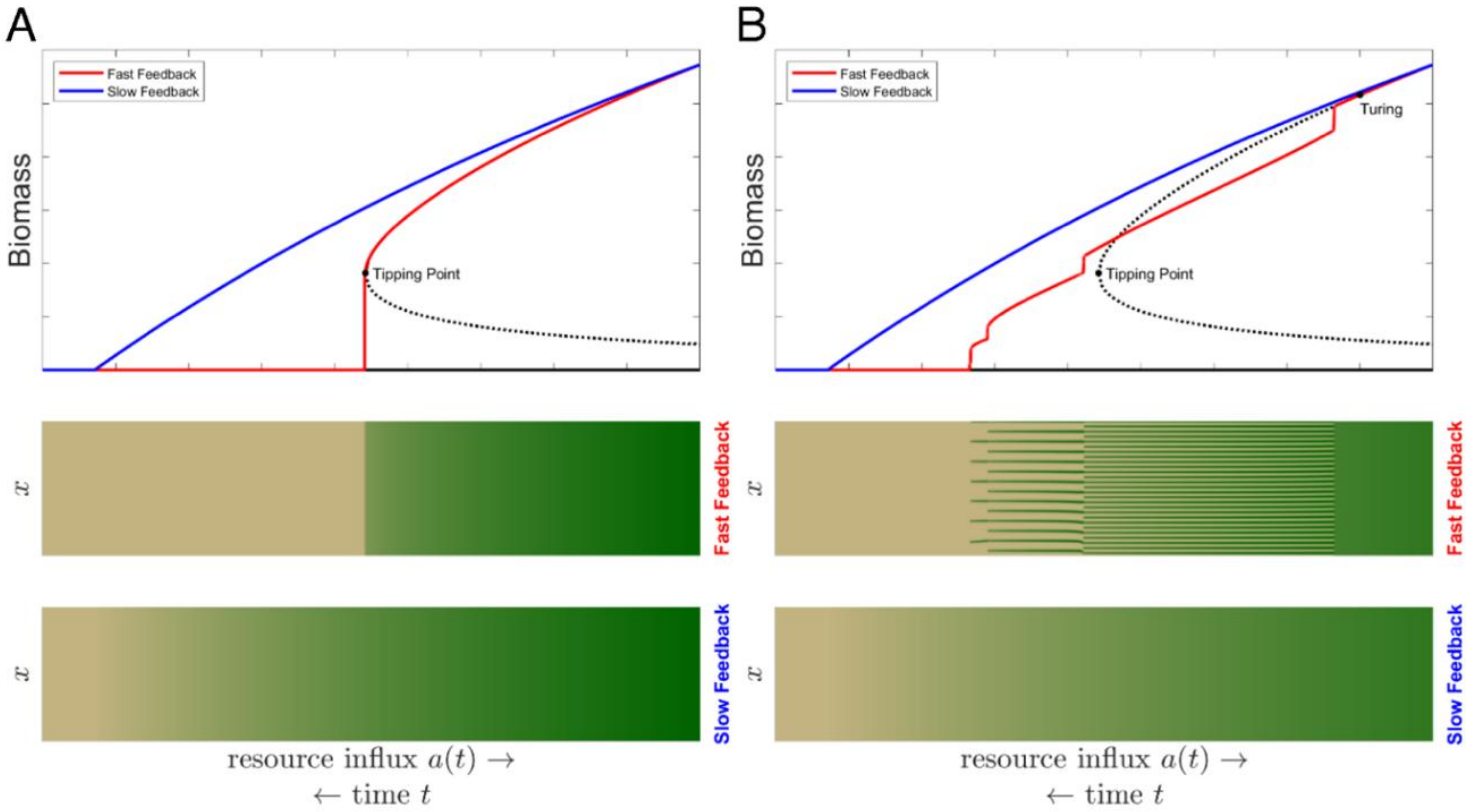
$$\frac{\partial \mathbf{s}}{\partial t} = (v - \mathbf{s})/\tau_s$$

Research by MSc student

Rita Mak

Now PhD candidate @ Osnabrück

Time scale of feedback



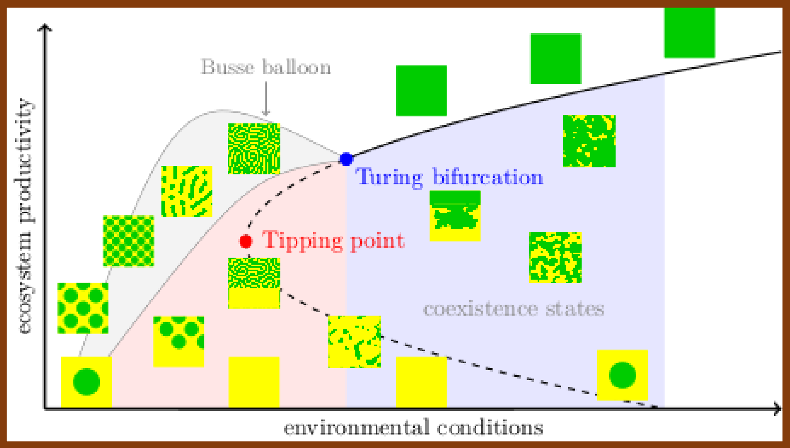
Tipping in Spatially Extended Systems: early warnings & control

Summary

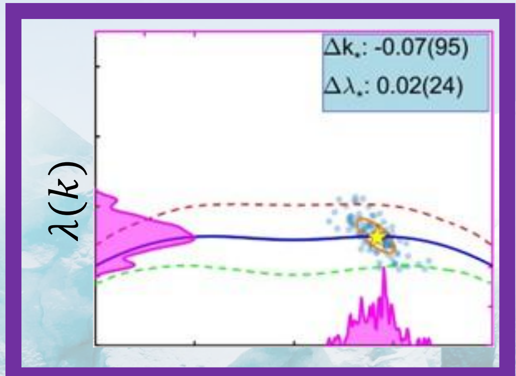
Summary

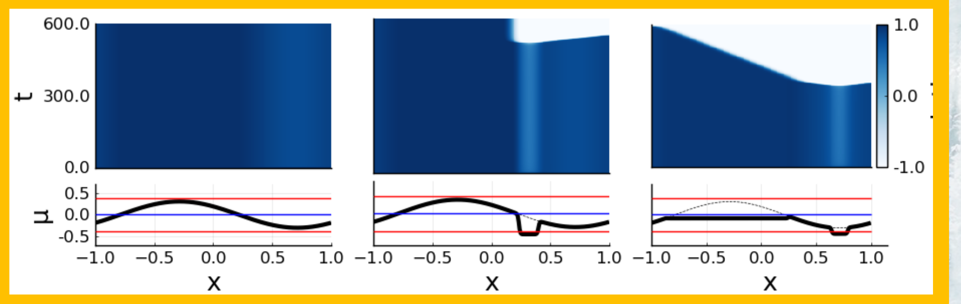
1. Tipping can be more subtle:

- Spatial reorganization
- Fragmented Tipping



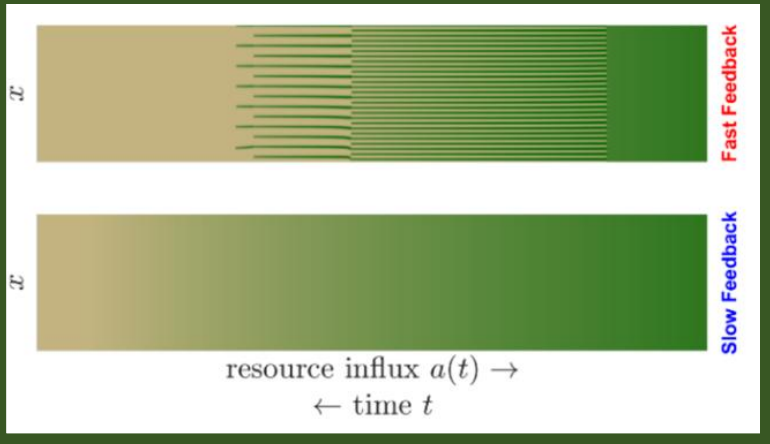
2. Refined spatial early warning signs can signal not only for *when* but also *what* transition happens





3. Optimising heterogeneity can postpone, halt, prevent or revert tipping

4. Time scales be important!



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Slides at bastiaansen.github.io